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CONTINUOUS DERIVATIONS OF THE RING OF WITT-VECTORS

Peter RUSSELL
Department of Mathematics, McGill University Montreal, P.Q. H3C 3G1, Canada

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Let k be a commutative ring, $W = W(k)$ the ring of Witt-vectors over k and $W_n = W_n(k)$ the ring obtained from W by truncation at $\{1, ..., n\} \subset \mathbf{N}$ (see [1]). Attempts to compute higher K-groups of $k[t]/t^n$ have shown the need to learn something about the W-modules $\Omega(W_n)$ (the Kähler differentials of W_n). It is the purpose of this note to construct continuous derivations of W (if the target module is discrete these are precisely the derivations factoring through W_n for some n) powerful enough to answer some of the questions arising in K-theory (see [2]). The construction is based on higher derivations of k.

1. Let n be a positive integer, p a prime and $\gamma = (\gamma_0, \gamma_1, ...)$ with γ_i a non-negative integer and $\gamma_i = 0$ for all but finitely many i. We put

1.1. (i) n_p = largest power of p dividing n, $n_p = p^{r(n)}$;
(ii) $|\gamma| = \Sigma_i \gamma_i$, $\|\gamma\| = \Sigma_i i\gamma_i$, $\{\gamma\} = \mathrm{GCD}(\gamma_0, \gamma_1, ...)$;
(iii) $\binom{n}{\gamma} = n!/\Pi_i \gamma_i!$ (this notation will be used only if $|\gamma| = n$);
(iv) $x^\gamma = \Pi_i x_i^{\gamma_i}$ if $x = (x_0, x_1, ...)$ is a sequence of elements in some (commutative) ring; $x^p = (x_0^p, x_1^p, ...)$.

Lemma 1.2. (i) *If* $p \mid \{\gamma\}$, *then* $\binom{n}{\gamma} \equiv \binom{n/p}{\gamma/p} \bmod n_p$.
(ii) *If* $p \nmid \{\gamma\}$, *then* $\binom{n}{\gamma} \equiv 0 \bmod n_p$.

Proof. Suppose $\gamma_i = 0$ for $i > s$ and let $X_0, X_1, ..., X_s$ be indeterminates over $\mathbf{Z}$. Now $(X_0 + ... + X_s)^p \equiv X_0^p + ... + X_s^p \bmod p$ implies $(X_0 + ... + X_s)^n \equiv (X_0^p + ... + X_s^p)^{n/p} \bmod n_p$. Hence $\Sigma_{|\gamma|=n} \binom{n}{\gamma} X^\gamma \equiv \Sigma_{|\delta|=n/p} \binom{n/p}{\delta} X^{p\delta} \bmod n_p$ and we deduce the lemma comparing coefficients.

Corollary 1.2. $\binom{n}{\gamma} \equiv 0 \bmod n_p/\{\gamma\}_p$.

Corollary 1.3. *Let d, e be positive integers such that $e \mid nd$. Suppose $|\gamma| = nd/e$ and $\{\gamma\} \mid n$.*

(i) $e\binom{nd/e}{\gamma} \equiv 0 \bmod d$.
(ii) *If* $p \mid \{\gamma\}$, *then* $e/d\binom{nd/e}{\gamma} \equiv e/d\binom{nd/pe}{\gamma/p} \bmod n_p$.
(iii) *If* $p \nmid \{\gamma\}$, *then* $e/d\binom{nd/e}{\gamma} \equiv 0 \bmod n_p$.

Definition 1.4. Let $X = (X_0, X_1, \ldots)$ be a sequence of variables and m, n non-negative integers. We put

$$\Phi_{n,m}(X) = \sum_I \binom{m}{\gamma} X^\gamma ,$$

where $I = \{\gamma \mid |\gamma| = m, \|\gamma\| = n\}$.

Lemma 1.5. *Let n, d, e be positive integers such that $e \mid nd$.*
(i) $e/d\ \Phi_{n,nd/e}(X) \in \mathbf{Z}[X]$.
(ii) *Let p be a prime such that $p \mid n$ and $pe \mid nd$. Then*

$$e/d\ \Phi_{n,nd/e}(X) \equiv e/d\ \Phi_{n/p,\, nd/pe}(X^p) \bmod n_p .$$

Proof. (i) If $\|\gamma\| = n$, then $\{\gamma\} \mid n$ and $e/d\binom{nd/e}{\gamma} \in \mathbf{Z}$ by 1.3(i).
(ii) $e/d\ \Phi_{n,nd/e}(X) = \Sigma_I\, e/d\binom{nd/e}{\gamma} X^\gamma + \Sigma_J\, e/d\binom{nd/e}{p\delta} X^{p\delta}$ where $I = \{\gamma \mid p \nmid \{\gamma\}, |\gamma| = nd/e, \|\gamma\| = n\}$ and $J = \{\delta \mid |\delta| = nd/pe, \|\delta\| = n/p\}$. The result follows from 1.3 (ii) and (iii).

Proposition 1.6. *Let T_{ij}, $i = 0, 1, 2, \ldots$, $j = 1, 2, \ldots$ be variables. Put $T_j = (T_{0j}, T_{1j}, \ldots)$ and $T = (T_{ij} \mid i = 0, 1, \ldots, j = 1, 2, \ldots)$. Let d be a positive integer. There exist unique polynomials $\varphi_j^{(d)}(T) \in \mathbf{Z}[T]$, $j = 1, 2, \ldots$, such that for all n*

$$\sum_{e \mid nd} e/d\ \Phi_{n,nd/e}(T_e) = \sum_{j \mid n} j\varphi_j^{(d)}(T)^{n/j} .$$

Proof. Let $\Psi_1 = \Sigma_{e \mid nd}\, e/d\ \Phi_{n,nd/e}(T_e)$, $\Psi_2 = \Sigma_{j \mid n, j<n}\, j\varphi_j^{(d)}(T)^{n/j}$. Since we can solve uniquely for $n\varphi_n^{(d)}(T)$ once the $\varphi_j^{(d)}(T)$ have been determined for $j < n$, it will be enough to show that $\Psi_1 \equiv \Psi_2 \bmod n_p$ for all prime divisors p of n.

Fix p. Note that if $j \mid n$, then either $n_p \mid j$ or $j \mid n/p$. Hence $\Psi_2 \equiv \Sigma_{j \mid n/p}\, j\varphi_j^{(d)}(T)^{n/j} \bmod n_p$. Also $\varphi_j^{(d)}(T)^p \equiv \varphi_j^{(d)}(T^p) \bmod p$ implies $j\varphi_j^{(d)}(T)^{n/j} \equiv j\varphi_j^{(d)}(T^p)^{n/pj} \bmod n_p$ if $j \mid n/p$. Hence $\Psi_2 \equiv \Sigma_{j \mid n/p}\, j\varphi_j^{(d)}(T^p)^{n/j} \bmod n_p$ and by induction on n (the statement of the proposition is trivial for $n = 1$)

$$\Psi_2 \equiv \sum_{e \mid nd/p} e/d\ \Phi_{n/p,\, nd/pe}(T_e^p) \bmod n_p .$$

Again, if $e \mid nd$, either $e_p = n_p d_p$ and $n_p \mid e/d$ (i.e. e/dn_p is a rational number without p in the denominator) or $e \mid nd/p$. Hence $\Psi_1 \equiv \Sigma_{e \mid nd/p}\, e/d\ \Phi_{n,nd/e}(T_e) \bmod n_p$, and by 1.5 (ii) $\Psi_1 \equiv \Sigma_{e \mid nd/p}\, e/d\ \Phi_{n/p,\, nd/pe}(T_e^p) \bmod n_p$.

Remark 1.7. $\varphi_n^{(d)}$ depends on the T_{ij} with $i \leqslant n$ and $j \mid nd$ only.

2. Let k and K be commutative rings. Let $D: k \to K$ be a *higher derivation* of k into K, that is a sequence $D = (D_0, D_1, \ldots)$ of additive maps from k to K such that for $n = 0, 1, \ldots$ and all $x, y \in k$

$$D_n(xy) = \sum_{i+j=n} D_i(x) D_j(y) .$$

Lemma 2.1. *Let m, n be non-negative integers and* $x \in k$. *Then*

$$D_n(x^m) = \Phi_{n,m}(D_0(x), D_1(x), \ldots) .$$

Proof. $D_n(x^m) = \Sigma_A (D_{\alpha_1}(x)(D_{\alpha_2}(x)) \ldots (D_{\alpha_m}(x))$ where the α_i are non-negative integers and $A = \{(\alpha_1, \ldots, \alpha_m) \mid \Sigma_i \alpha_i = n\}$. Let $\alpha = (\alpha_1, \ldots, \alpha_m)$. Suppose i appears γ_i times among the α_j, $i = 0, 1, 2, \ldots$, and put $\gamma = (\gamma_0, \gamma_1, \ldots)$. Then $|\gamma| = m$ and $\|\gamma\| = n$. Conversely, a γ with this property determines α up to order of the α_j.

For any commutative ring R let $W(R)$ be the ring of Witt-vectors over R and

$$w = (w_n) : W(R) \to R^{\mathbf{N}}$$

the Witt homomorphism defined by $w_n(x) = \Sigma_{j|n}\, j x_j^{n/j}$ for $x = (x_1, x_2, \ldots) \in W(R)$ and $n = 1, 2, \ldots$ (see [1]). The following now is a straightforward translation of 2.1 and 1.6.

Definition-Proposition 2.2. Let d be a positive integer and $\varphi_n^{(d)}$, $n = 1, 2, \ldots$, as defined in 1.6. Let

$$\delta_d : W(k) \to W(K)$$

be given by

$$\delta_d(x)_n = \varphi_n^{(d)}(D_i(x_j))$$

for $x = (x_1, x_2, \ldots) \in W(k)$. Then

$$w_n \delta_d = 1/d\, D_n w_{nd}$$

for $n = 1, 2, \ldots$.

Lemma 2.3. *Let* $k = \mathbf{Q}[t]$, *let* Δ *be the usual derivation on k and put* $D_i = \Delta^i / i!$. *Then* $D = (D_0, D_1, \ldots)$ *is a higher derivation from k to k and the maps* $D_i : k \to k$ *are algebraically independent over* $\mathbf{Q}$.

Proof. This is all obvious except, maybe, for the last statement, for which, though, we may appeal to well known facts from the theory of differential equations.

Proposition 2.4. *δ_d is additive.*

Proof. Let $A_j(X,Y) \in \mathbb{Z}[X,Y]$, $j=1,2,\ldots$, $X=(X_1,X_2,\ldots)$, $Y=(Y_1,Y_2,\ldots)$ be the universal polynomials defining addition of Witt-vectors. There exist polynomials $\psi_{ij}(T,S) \in \mathbb{Z}[T,S]$, $T=(T_{lm})$, $S=(S_{l'm'})$, $i,l,l'=0,1,\ldots$, $j,m,m'=1,2,\ldots$ such that $D_i(A_j(x,y)) = \psi_{ij}((D_l(x_m)),(D_{l'}(y_{m'})))$ whenever R is a ring, $x=(x_1,x_2,\ldots)$ and $y=(y_1,y_2,\ldots)$ are sequences of elements in R, and $D=(D_0,D_1,\ldots)$ is a higher derivation on R. Now additivity for δ_d means

$$\delta_d(A_j(x,y))_n = A_n(\delta_d(x),\delta_d(y))$$

or, equivalently,

$$\varphi_n^{(d)}(\psi_{ij}((D_l(x_m)),(D_{l'}(y_{m'})))) = A_n((\varphi_s^{(d)}(D_l(x_m))),(\varphi_{s'}^{(d)}(D_{l'}(y_{m'}))))$$

for $n=1,2,\ldots$ and $x,y \in W(k)$. Now if $\operatorname{char} k = 0 = \operatorname{char} K$, then w is injective on $W(k)$ and $W(K)$ and clearly δ_d is additive. In view of 2.3 we deduce that there is an identity

$$\varphi_n^{(d)}(\psi_{ij}(T,S)) = A_n((\varphi_s^{(d)}(T)),(\varphi_{s'}^{(d)}(S))),$$

and this in turn implies additivity for δ_d in general.

Now let $F_d : W(R) \to W(R)$ (R a commutative ring) be the functorial endomorphism satisfying

$$w_n F_d = w_{nd}$$

for $n=1,2,\ldots$ (see [1]).

Proposition 2.5. *Let $\psi : W(k) \times W(k) \to W(K)$ be given by*

$$\psi(x,y) = \delta_d(xy) - F_d D_0(x)\,\delta_d(y) - F_d D_0(y)\,\delta_d(x)\,.$$

(Here we use the same symbol for the homomorphism $D_0 : k \to K$ and the homomorphism $W(k) \to W(K)$ obtained by applying D_0 componentwise.) Let p be a prime. Then for $n \leqslant d_p$ and $n = pd_p$ we have

$$\psi(x,y)_n \equiv 0 \bmod p$$

for any $x,y \in W(k)$.

Proof. We will show that in fact the coefficients of the polynomials used to define $\psi(x,y)_n$ are divisble by p if n is as in the statement of the proposition. We claim first that $w_n(\psi(x,y)) \equiv 0 \bmod pn_p$. Now

$$w_n(\psi(x,y)) = 1/d \sum_{i+j=n} D_i(w_{nd}(x))D_j(w_{nd}(y))$$
$$- 1/d\ w_{nd}D_0(x)D_n(w_{nd}(y)) - 1/d\ w_{nd}D_0(y)D_n(w_{nd}(x))$$

$$= 1/d \sum_{i+j=n,0<i,j} D_i(w_{nd}(x))D_j(w_{nd}(y))$$

$$= \sum_I ef/d \binom{nd/e}{\gamma}\binom{nd/f}{\delta} u_e^{\gamma} v_f^{\delta} ,$$

where $I = \{(e,f,\gamma,\delta) |\ e|nd, f|nd,\ |\gamma| = nd/e,\ |\delta| = nd/f,\ 0 < \|\gamma\|,\ 0 < \|\delta\|,$ $\|\gamma\| + \|\delta\| = n\}$ and $u_e = (D_0(x_e), D_1(x_e), ...)$, $v_f = (D_0(y_f), D_1(y_f), ...)$. By 1.2 (ii) we have

$$ef/d \binom{nd/e}{\gamma}\binom{nd/f}{\delta} \equiv 0 \mod p^a ,$$

where $a = r(e) + r(f) - r(d) + 2r(n) + 2r(d) - r(e) - r(f) - r(\{\gamma\}) - r(\{\delta\})$. Now if $n \leqslant d_p = p^{r(d)}$, then $r(d) > \max\{r(\{\gamma\}), r(\{\delta\})\}$ (we use $0 < \|\gamma\|, \|\delta\|$ and $n = \|\gamma\| + \|\delta\|$). Also, $r(n) \geqslant \min\{r(\{\gamma\}), r(\{\delta\})\}$. Hence $a \geqslant r(n) + 1$. If $n = pd_p = p^{r(d)+1}$, then $r(n) = r(d) + 1$ and $r(\{\gamma\}) \leqslant r(d)$, $r(\{\delta\}) \leqslant r(d)$. Hence $a \geqslant r(d) + 2 = r(n) + 1$. This establishes the claim. Now $w_n(\psi(x,y)) = n\psi_n(x,y) + U$, where U is a sum of terms $e\psi_e(x,y)^{n/e}$ with $e|n$ and $e < n$. We may assume by induction that $\psi_e(x,y) \equiv 0 \mod p$ for $e|n, e < n$. Since $n_p = e_p(n/e)_p < e_p p^{n/e}$, we have $U \equiv 0 \mod pn_p$. Hence $n\psi_n(x,y) \equiv 0 \mod pn_p$ and $\psi_n(x,y) \equiv 0 \mod p$.

Summarizing we obtain

Theorem 2.6. *Let k and K be commutative rings and suppose* char $K = p$, *p a prime. Let $D = (D_0, D_1, ...)$ be a higher derivation from k to K, d a positive integer and $d_p = p^r$. Consider $W(K)$ as a $W(k)$-module via $F_d D_0$. Then the map*

$$\delta_d : W(k) \to W(K)$$

of 2.2 induces via truncation (see [1]) continuous derivations $W(k) \to W_{(1,2,...,n)}(K)$ for $n \leqslant p^r$ and $W(k) \to W_{(1,p,...,p^s)}(K)$ for $s \leqslant r + 1$.

Proof. Additivity and the derivation property follow from 2.4 and 2.5, the continuity from 1.7.

References

[1] G.M. Bergman, Lecture 26 in D. Mumford, Lectures on curves on an algebraic surface, Annals of Mathematics Studies 59 (Princeton, 1966).

[2] J. Labute and P. Russell, On K_2 of truncated polynomial rings, J. Pure and Applied Algebra 6 (1975) 239–251.