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A hinged plate equation and iterated Dirichlet Laplace operator on domains with concave corners

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Abstract

Fourth order hinged plate type problems are usually solved via a system of two second order equations. For smooth domains such an approach can be justified. However, when the domain has a concave corner the bi-Laplace problem with Navier boundary conditions may have two different types of solutions, namely u_1 with $u_1, \Delta u_1 \in \mathring{H}^1$ and $u_2 \in H^2 \cap \mathring{H}^1$. We will compare these two solutions. A striking difference is that in general only the first solution, obtained by decoupling into a system, preserves positivity, that is, a positive source implies that the solution is positive. The other type of solution is more relevant in the context of the hinged plate. We will also address the higher-dimensional case. Our main analytical tools will be the weighted Sobolev spaces that originate from Kondratiev. In two dimensions we will show an alternative that uses conformal transformation. Next to rigorous proofs the results are illustrated by some numerical experiments for planar domains.

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1. Introduction

Let Ω a bounded domain in $\mathbb{R}^n$ and consider the boundary value problem

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

If Ω is a polygonal domain in $\mathbb{R}^2$, these equations form the linear model for a hinged plate. We are interested in the question if (1.1) is positivity preserving, meaning $f \geqslant 0$ implies that $u \geqslant 0$. For the plate it can be rephrased as:

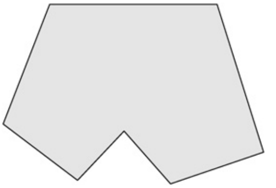
Does a one-sided force moves the plate in that same direction in each point?

This positivity question is rather trivial for (1.1) if $\partial\Omega$ is smooth since in that case the result is a direct consequence of the maximum principle. Writing $u = w$ and $-\Delta u = v$ the boundary conditions uncouple into a system

$$\begin{cases} -\Delta v = f & \text{in } \Omega \text{ with } v = 0 \text{ on } \partial\Omega, \\ -\Delta w = v & \text{in } \Omega \text{ with } w = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.2}$$

and standard arguments yield that for any $f \in L^2(\Omega)$ and $f \geqslant 0$ one finds a unique positive solution $v \in \mathring{H}^1(\Omega)$ and even that $v \in H^2(\Omega)$. Repeating that argument results in the existence of a unique positive solution $w \in H^2(\Omega) \cap \mathring{H}^1(\Omega)$. Moreover, for $\partial\Omega \in C^4$ regularity arguments even show that $w \in H^4(\Omega)$. The advantage in a numerical approach is that the solution of the system can be approximated with piecewise linear finite elements which are readily available in the standard packages. A direct numerical approach to the fourth order problem would need piecewise quadratic finite elements.

The positivity question becomes more interesting if $\partial\Omega$ is not smooth, for example if $\Omega \subset \mathbb{R}^2$ is like here below. Without smoothness assumption on the boundary one may still solve this system for $f \in L^2(\Omega)$ to find $v = -\Delta w \in \mathring{H}^1(\Omega)$ respectively $w \in \mathring{H}^1(\Omega)$ and by the maximum principle that $f \geqslant 0$ implies $v \geqslant 0$ and hence $w \geqslant 0$. In general however one does not obtain $w \in H^2(\Omega)$.



Alternatively one may look for a possible minimizer of $\int_\Omega ((\Delta u)^2 - fu)\,dx$ in $H^2(\Omega) \cap \mathring{H}^1(\Omega)$. If this functional has a minimizer u it does not have to be equal to w. Indeed this difference of the appropriate solutions for the single equation and for the system has been discussed by Maz'ya and coauthors in [25] (see also [26, Section 5.8] and [30, Section 6.6.2]).

The second question we will address is:

How do these two types of solutions compare?

We will show that for two-dimensional domains a solution $u \in H^2(\Omega) \cap \mathring{H}^1(\Omega)$, the one which is physically more relevant, might change sign for $f \geqslant 0$ if the domain has a 'concave' corner. We will also show that for some Ω such a minimizer may not exist.

The main part of this paper is concerned with the analytical treatment of the problem and after explaining the general setting we will do so by considering dimension $\geqslant 4$, 3 and 2 separately. We will end by showing some numerical results that will illustrate the analytical results and by stating some open problems concerning the positivity question on this type of domains.

Remark 1.0.1. It is known that for systems of second order equations on domains with reentrant corners several distinct types of solutions may exist. For the dynamic Lamé-system $-\mu\Delta\mathbf{u} - (\lambda+\mu)\nabla(\nabla.\mathbf{u}) = \rho\omega^2 u$ this has been studied in [16,29]. For the study of corner singularities for the (time-harmonic) Maxwell equations, that is, for $\nabla \times \mathbf{E} - i\omega\mu\mathbf{H} = 0$ and $\nabla \times \mathbf{H} + i\omega\varepsilon\mathbf{E} = \mathbf{J}$, we refer to [5,9] and also to [1,7,8]. A paper by Birman [6], discussed the Stokes system $-\Delta\mathbf{u} + \nabla p = \mathbf{f}$ with $\nabla.\mathbf{u} = 0$ and compares this result with those for the Lamé-system and for the Maxwell equations.

1.1. Physical background

If $n = 2$ and if the boundary of Ω is a polygon the problem in (1.1) is the linear model for a clamped plate with hinged boundary conditions. For such a problem the energy should be finite and that is guaranteed by $u \in H^2(\Omega)$. Indeed the elastic energy for such a model is defined by

$$E(u;\Omega) = \int_\Omega \left(\frac{1}{2}(\Delta u)^2 + (1-\sigma)\big(u_{xy}^2 - u_{xx}u_{yy}\big) + fu\right)dx \tag{1.3}$$

where f is the exterior force and u the bending of this plate; σ is the Poisson ratio.[2] See for example [33, Chapter VI]. The zero boundary condition of the plate is taken care of by the zero in $u \in \mathring{H}^1(\Omega)$. The hinged boundary condition $\Delta u = 0$ comes as a natural boundary condition. So the appropriate space to be considered for this model is $H^2(\Omega) \cap \mathring{H}^1(\Omega)$. For a minimizer u we find

$$\int_\Omega \big(\Delta u \Delta v + (1-\sigma)(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx}) + fv\big)\,dx = 0$$

for all $v \in H^2(\Omega) \cap \mathring{H}^1(\Omega)$. Assuming $u \in H^4(\Omega)$ we may integrate by part and find, writing $n = (\nu_1, \nu_2)$ for the outside normal, that

$$0 = \int_\Omega \big(\Delta^2 u - f\big)v\,dx + \int_{\partial\Omega} \big(\Delta u + (1-\sigma)\big(2u_{xy}\nu_1\nu_2 - u_{xx}\nu_2^2 - u_{yy}\nu_1^2\big)\big)\frac{\partial v}{\partial n}\,ds.$$

Note that the term $(1-\sigma)(u_{xy}^2 - u_{xx}u_{yy})$ in (1.3) has no influence on the differential equation but does change one of the boundary conditions on none-straight boundary parts. Indeed, instead

[2] The Poisson ratio is defined by $\sigma = \frac{\lambda}{2(\lambda+\mu)}$ with material depending constants λ, μ, the so-called Lamé constants. Usually $\lambda \geqslant 0$ and $\mu > 0$ hold true and hence $0 \leqslant \sigma < \frac{1}{2}$. Some exotic materials have a negative Poisson ratio (see [20]). For metals the value σ lies around 0.3 (see [21, p. 105]).

of $\Delta u = 0$ on $\partial\Omega$ one obtains $\Delta u + (1-\sigma)(2u_{xy}\nu_1\nu_2 - u_{xx}\nu_2^2 - u_{yy}\nu_1^2) = 0$ on $\partial\Omega$. Let us recall that for $u = 0$ on $\partial\Omega$ it holds that

$$\begin{aligned}
&\Delta u + (1-\sigma)\big(2u_{xy}\nu_1\nu_2 - u_{xx}\nu_2^2 - u_{yy}\nu_1^2\big)\\
&\quad = \sigma\Delta u + (1-\sigma)\big(2u_{xy}\nu_1\nu_2 + u_{xx}\nu_1^2 + u_{yy}\nu_2^2\big)\\
&\quad = \sigma(u_{nn} + \kappa u_n) + (1-\sigma)u_{nn} = u_{nn} + \sigma\kappa u_n = \Delta u + (1-\sigma)\kappa u_n.
\end{aligned}$$

Here κ is the curvature of the boundary. This implies that the physically relevant boundary value problem reads as

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega,\\ u = \Delta u + (1-\sigma)\kappa u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

On polygonal domains (1.4) leads to (1.1) with some singularity in the corners. Note that through an approximation of the boundary the corresponding approximating solutions not necessarily converge to a solution for the original domain. The difference between the solution of (1.1) on a disk and the approximation by the solutions on regular m-polygons was noticed in [2]. This so-called Babuška paradox was studied by Maz'ya et al. in [23]. Finally we would like to refer to [12] for the positivity question under Dirichlet boundary conditions.

1.2. The two types of solutions

Throughout this paper we will assume that Ω is a bounded uniformly Lipschitz domain in $\mathbb{R}^n$. If we assume more regularity such will be stated in the theorem. We will recall some of the known results for the Dirichlet Laplace and the consequences for (1.1). For convex domains both approaches will lead to the same solution.

1.2.1. The $(H^1)^2$-solution

Let us recall that no matter what regularity the boundary satisfies the following result holds true for

$$\begin{cases} -\Delta v = f & \text{in } \Omega,\\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

Theorem 1.1. (See [19].) *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then for every $f \in L^2(\Omega)$ there exists a unique weak solution $v \in \mathring{H}^1(\Omega)$ of* (1.5). *Moreover $f \geqslant 0$ implies $v \geqslant 0$.*

As usual by a weak solution of (1.5) we mean a function $v \in \mathring{H}^1(\Omega)$ such that

$$\int_\Omega (\nabla v . \nabla\varphi - f\varphi)\,dx = 0 \quad \text{for all } \varphi \in \mathring{H}^1(\Omega).$$

For this solution operator we will use $\mathcal{G}_\Omega$; one has $\mathcal{G}_\Omega \in L(L^2(\Omega); \mathring{H}^1(\Omega))$. The solution for (1.1) obtained by $\mathcal{G}_\Omega^2$ we will call a $(H^1)^2$-solution.

Theorem 1.2. (See [13].) *Let Ω be convex. Then for every $f \in L^2(\Omega)$ the weak solution $v \in \mathring{H}^1(\Omega)$ of* (1.5) *lies in $H^2(\Omega)$. Moreover, there exists $c > 0$ such that for every $f \in L^2(\Omega)$ and corresponding v it holds that*

$$\|v\|_{H^2(\Omega)} \leqslant c\|f\|_{L^2(\Omega)}.$$

Remark 1.2.1. Instead of Ω being convex it is sufficient that there exist C^2-diffeomorphisms that map $\overline{\Omega}$ 'locally' onto a convex domain.

Remark 1.2.2. Convexity is not sufficient to conclude for arbitrary p from $f \in L^p(\Omega)$ that $v \in W^{2,p}(\Omega)$. In a two-dimensional domain with a smooth boundary except for finitely many corners of opening angle α_i one needs that $\alpha_i < \frac{p}{2(p-1)}\pi$ for all i.

Corollary 1.3. *Let Ω be convex. Then for every $f \in L^2(\Omega)$ there exists a unique $w \in H^2(\Omega) \cap \mathring{H}^1(\Omega)$ with $\Delta w \in H^2(\Omega) \cap \mathring{H}^1(\Omega)$ that solves problem* (1.1).

Proof. Existence follows form the previous theorem; uniqueness through the maximum principle. □

1.2.2. The H^2-solution

Let us fix the Hilbert space

$$\mathcal{H}(\Omega) = H^2(\Omega) \cap \mathring{H}^1(\Omega) \tag{1.6}$$

supplied with the usual $\|\cdot\|_{H^2(\Omega)}$-norm.

Definition 1.4. We will call u a H^2-solution of (1.1) if

1. $u \in \mathcal{H}(\Omega)$;
2. $\int_\Omega (\Delta u \Delta v - f v)\,dx = 0$ for all $v \in \mathcal{H}(\Omega)$.

Remark 1.4.1. Note that when $u \in H^4(\Omega)$ an integration by parts of the integral identity in Definition 1.4 shows

$$\int_\Omega \left(\Delta^2 u - f\right) v\,dx - \int_{\partial\Omega} \Delta u \frac{\partial v}{\partial n}\,ds = 0 \quad \text{for all } v \in \mathcal{H}(\Omega), \tag{1.7}$$

and hence $\Delta u|_{\partial\Omega} = 0$. If $\Gamma \subset \partial\Omega$ is C^4 then one may use local arguments to find $\Delta u|_\Gamma = 0$.

Existence of such an H^2-solution is not guaranteed on general non-smooth domains. However, using the result of Theorem 1.2 we may conclude that:

Proposition 1.5. *If Ω is a convex domain, then problem* (1.1) *with the right-hand side $f \in L^2(\Omega)$ has a unique H^2-solution u. Moreover, there exists $c > 0$ such that for every $f \in L^2(\Omega)$ and corresponding u it holds that*

$$\|u\|_{H^2(\Omega)} \leqslant c\|f\|_{L_2(\Omega)}. \tag{1.8}$$

Proof. Indeed by Theorem 1.2 we find that since Ω is convex the "second fundamental inequality" ([13], [18, p. 65]) is satisfied for $u \in \mathcal{H}(\Omega)$:

$$\|u\|_{H^2(\Omega)} \leqslant c_\Omega \|\Delta u\|_{L_2(\Omega)} \tag{1.9}$$

and therefore $((u,v)) := \int_\Omega \Delta u \Delta v \, dx$ is a inner product in $\mathcal{H}(\Omega)$. Since $\Psi(v) = \int_\Omega f v \, dx$ is a continuous functional on $\mathcal{H}(\Omega)$ the Riesz representation theorem gives a unique H^2-solution. Moreover

$$\|u\|^2_{H^2(\Omega)} \leqslant c_\Omega \int\limits_\Omega (\Delta u)^2 \, dx = c_\Omega \int\limits_\Omega f u \, dx \leqslant c_\Omega \|f\|_{L_2(\Omega)} \|u\|_{L_2(\Omega)}$$

and one finds that (1.8) holds with $c = c_\Omega^{1/2}$. □

Corollary 1.6. *If Ω is a convex domain, then for $f \in L^2(\Omega)$ the $(H^1)^2$- and H^2-solutions to problem* (1.1) *coincide.*

Remark 1.6.1. Although these solutions are identical for a convex domain and hence this solution u satisfies $u, \Delta u \in H^2(\Omega)$, additional conditions are necessary in order to conclude that $u \in H^4(\Omega)$.

2. Higher-dimensional domains with a conic point

We will restrict ourselves to domains with only one 'concave' boundary point and assume that the domain near this one point is like a cone.

Definition 2.1. Let $\omega \subset \mathbb{S}^{n-1}$ (the unit sphere in $\mathbb{R}^n$) and set

$$K_\omega^R := \{r\theta;\ 0 < r < R \text{ and } \theta \in \omega\}.$$

Open sets in $\mathbb{R}^n$ of the form $K_\omega^\infty := \{r\theta;\ 0 < r \text{ and } \theta \in \omega\}$ will be called cones.

Some special domains we define for $\alpha \in (0, 2\pi)$:

$$\Omega_\alpha := K^1_{\omega_\alpha} \quad \text{with } \omega_\alpha = \left\{\theta \in \mathbb{S}^{n-1};\ \arccos(\theta_1) < \frac{1}{2}\alpha\right\}. \tag{2.1}$$

Notice that $\arccos(\theta_1)$ is the angle between θ and $\hat{e}_1 = (1, 0, \ldots, 0)$. See Fig. 1.
To avoid other non-smooth boundary parts of Ω_α the general setting will be as follows.

Condition 2.2. We will assume that $\Omega \subset \mathbb{R}^n$ is bounded and is such that

- $\partial\Omega \backslash \mathcal{O}$ is C^∞;
- there exist $R > 0$ and a proper subdomain ω of $\mathbb{S}^{n-1}$ with $\Omega \cap B_R(\mathcal{O}) = K_\omega^R$.

Fig. 1. Ω_α for $\alpha = 2\pi - 2$ in dimension 2 and 3.

Here $\mathcal{O} = (0, \dots, 0)$ and $B_R(\mathcal{O}) = \{x \in \mathbb{R}^n \colon |x| < R\}$ is the ball of radius $R > 0$ centered at $\mathcal{O}$. In other words, Ω has a smooth boundary except at $\mathcal{O}$ where it locally coincides with a cone.

Due to the singularity at $\mathcal{O}$ the equalities in (1.1) are not necessarily pointwise and the appropriate formulation of (1.1) becomes

$$\begin{cases} \Delta^2 u(x) = f(x) & \text{for } x \in \Omega, \\ u(x) = \Delta u(x) = 0 & \text{for } x \in \partial\Omega \backslash \mathcal{O}, \\ \text{specified behaviour near } \mathcal{O}. \end{cases} \tag{2.2}$$

For a convex domain Proposition 1.5 gives the existence and uniqueness of a H^2-solution. This result is based on the estimate in (1.9). In general (1.9) does not hold true for concave domains and, thus, even the solvability of problem (2.2) in the Sobolev space $H^2(\Omega)$ cannot be concluded directly from the variational formulation. We are forced to proceed by considering the iterated Dirichlet Laplacian as in (1.2).

3. The Dirichlet problem for the Poisson equation

Let Ω be as in Condition 2.2. In order to study solutions in a domain with a conical boundary point such, like Ω satisfying Condition 2.2, we know from [15] that a possible approach starts with considering non-trivial power-law solutions

$$U(x) = r^{\Lambda}\Phi(\theta) \tag{3.1}$$

of the following model problem in the infinite cone

$$\begin{cases} -\Delta U(x) = 0 & \text{for } x \in K_\omega^\infty, \\ U(x) = 0 & \text{for } x \in \partial K_\omega^\infty \backslash \mathcal{O}. \end{cases} \tag{3.2}$$

Since the Laplace operator in the spherical coordinates takes the form

$$\Delta = r^{1-n}\frac{\partial}{\partial r} r^{n-1}\frac{\partial}{\partial r} + r^{-2}\widetilde{\Delta},$$

where $\widetilde{\Delta}$ denotes the Laplace–Beltrami operator on the unit sphere, the functions in (3.1) get the exponents

$$\Lambda_k^{\pm} = -\frac{n-2}{2} \pm \sqrt{\frac{(n-2)^2}{4} + \mu_k}. \tag{3.3}$$

Here μ_k is the kth eigenvalue of the problem

$$\begin{cases} -\widetilde{\Delta}\Phi(\theta) = \mu\Phi(\theta) & \text{for } \theta \in \omega, \\ \Phi(\theta) = 0 & \text{for } \theta \in \partial\omega, \end{cases} \tag{3.4}$$

and the angular part Φ in (3.1) is a corresponding eigenfunction. Eigenvalues of problem (3.4) form the sequence

$$0 < \mu_1 < \mu_2 \leqslant \mu_3 \leqslant \cdots \quad \text{and} \quad \mu_k \to \infty \text{ for } k \to \infty.$$

The first eigenvalue μ_1 is simple and the corresponding eigenfunction Φ_1 can be chosen positive in ω. The positive exponents of power-law solutions (3.1) also form the sequence

$$0 < \Lambda_1^+ < \Lambda_2^+ \leqslant \Lambda_3^+ \leqslant \cdots \quad \text{and} \quad \Lambda_k^+ \to \infty \text{ for } k \to \infty. \tag{3.5}$$

The negative exponents in (3.3) are related to (3.5) by the formula $\Lambda_k^- = 2 - n - \Lambda_k^+$.

Let us recall the function spaces which fit to problem (1.5) on Ω satisfying Condition 2.2. First we set

$$C_c^\infty(\overline{\Omega}\backslash\mathcal{O}) := \left\{u \in C^\infty(\overline{\Omega});\ \text{support}(u) \subset \overline{\Omega}\backslash B_\varepsilon(\mathcal{O}) \text{ for some } \varepsilon > 0\right\}.$$

Definition 3.1. Let $V_\beta^{l,p}(\Omega)$ be defined as the completion of $C_c^\infty(\overline{\Omega}\backslash\mathcal{O})$ with respect to the weighted norm below. That is:

$$V_\beta^{l,p}(\Omega) = \overline{C_c^\infty(\overline{\Omega}\backslash\mathcal{O})}^{\|\cdot\|}, \tag{3.6}$$

$$\|z\| := \|z\|_{V_\beta^{l,p}(\Omega)} = \left(\sum_{j=0}^{l} \left\| |x|^{\beta-l+j}\nabla_x^j z\right\|_{L_p(\Omega)}^p\right)^{1/p}, \tag{3.7}$$

where $l \in \{0, 1, \ldots\}$, $p \in (1, +\infty)$ and $\beta \in \mathbb{R}$.

Remark 3.1.1. The coefficients l, p and β are respectively the indices of smoothness, summability, and weight. If $l \leqslant s$ and $\beta \leqslant \gamma$ then the inclusion $V_{\gamma+s-l}^{s,p}(\Omega) \subset V_\beta^{l,p}(\Omega)$ holds. This inclusion becomes compact only under the restrictions $l > s$ and $\beta < \gamma$.

In order to define the appropriate space for zero Dirichlet boundary conditions we set

$$C_0^\infty(\Omega) := \left\{u \in C^\infty(\overline{\Omega});\ \text{support}(u) \subset \Omega\right\}.$$

Definition 3.2. Set $\mathring{V}^{l,p}_{\beta}(\Omega) = \overline{C_c^{\infty}(\Omega)}^{\|\cdot\|}$ with the norm as in (3.7) for the same parameters $l \in \{0, 1, \ldots\}$, $p \in (1, +\infty)$ and $\beta \in \mathbb{R}$.

Remark 3.2.1. The space $\mathring{V}^{l,p}_{\beta}(\Omega)$ is the subspace of functions $v \in V^{l,p}_{\beta}(\Omega)$ satisfying $(\frac{\partial}{\partial n})^k v = 0$ for $k \in \{0, \ldots, l-1\}$ on $\partial\Omega \backslash \mathcal{O}$.

The following assertion is a direct consequence of general results in the theory of elliptic problems in domains with piecewise smooth boundaries. See the key works by Kondratiev [15], by Maz'ya and collaborators [17,27,28] or by Grisvard [11]. For precise statements we may also refer to [30], namely Chapter 6 Theorem 1.4 (p. 226), Chapter 4 Theorem 1.7 (p. 105) and Chapter 3 Theorem 6.10 (p. 82).

Proposition 3.3. *Suppose Ω satisfies Condition* 2.2 *and let Λ_1^+ be defined by* (3.3), (3.4). *The operator of problem* (1.5) *regarded as the mapping*

$$A^{l,p}_{\beta} : V^{l+1,p}_{\beta}(\Omega) \cap \mathring{V}^{1,p}_{\beta-l}(\Omega) \to V^{l-1,p}_{\beta}(\Omega) \tag{3.8}$$

is an isomorphism if and only if

$$1 - \Lambda_1^+ < \beta - l + \frac{n}{p} < n - 1 + \Lambda_1^+. \tag{3.9}$$

More precisely,

1. *if $\beta - l + \frac{n}{p} < 1 - \Lambda_1^+$, then there exist $f \in V^{l-1,p}_{\beta}(\Omega)$ for which problem* (1.5) *has no solution in $V^{l+1,p}_{\beta}(\Omega) \cap \mathring{V}^{1,p}_{\beta-l}(\Omega)$;*
2. *if $\beta - l + \frac{n}{p} > n - 1 + \Lambda_1^+$, problem* (1.5) *with $f = 0$ has a non-trivial solution $v \in V^{l+1,p}_{\beta}(\Omega)$;*
3. *in the cases $\beta - l + \frac{n}{p} = 1 - \Lambda_1^+$ and $\beta - l + \frac{n}{p} = n - 1 + \Lambda_1^+$, mapping* (3.8) *is not Fredholm, namely, the range* $\operatorname{Im} A^{l,p}_{\beta}$ *is not closed in $V^{l-1,p}_{\beta}(\Omega)$.*

For further consideration we need the particular indices

$$l = 1, \qquad p = 2, \qquad \beta = 0 \tag{3.10}$$

that provide $V^{l-1,p}_{\beta}(\Omega) = V^{0,2}_{0}(\Omega) = L_2(\Omega)$.

Lemma 3.4. *It holds that*

$$V^{2,2}_{0}(\Omega) \cap \mathring{V}^{1,2}_{-1}(\Omega) = H^2(\Omega) \cap \mathring{H}^1(\Omega). \tag{3.11}$$

Proof. Let us write $\mathcal{V}(\Omega) = V^{2,2}_{0}(\Omega) \cap \mathring{V}^{1,2}_{-1}(\Omega)$. Then we observe that the exponents of r in the weighted norm (3.7) with $\beta = 0$ and $l = 2$ are non-positive for $j = 0, 1, 2$. Thus, the inclusion

$\mathcal{V}(\Omega) \subset \mathcal{H}(\Omega)$ is evident. To verify the inverse inclusion $\mathcal{H}(\Omega) \subset \mathcal{V}(\Omega)$, we first need the one-dimensional Hardy inequality

$$\int_0^\infty r^{n-3} w(r)^2\,dr \leqslant \frac{4}{(n-2)^2} \int_0^\infty r^{n-1} \left|\frac{dw}{dr}(r)\right|^2 dr \quad \text{for all } w \in C_0^1[0,\infty)$$

which assures that

$$\left\| |x|^{-1} \nabla v \right\|_{L_2(\Omega)} \leqslant c \|\nabla v\|_{H^1(\Omega)} \leqslant c' \|v\|_{H^2(\Omega)}.$$

Secondly, by using the Dirichlet condition $v = 0$ on $\partial\Omega$ and the Poincaré–Friedrichs inequality on the domain $\omega \subset \mathbb{S}^{n-1}$ we find

$$\int_\omega \left|v(\theta)\right|^2 ds_\theta \leqslant c \int_\omega \left|\widetilde{\nabla} v(\theta)\right|^2 ds_\theta \quad \text{for all } v \in \mathring{H}^1(\omega),$$

where $\widetilde{\nabla}$ stands for the angular part of the gradient $\nabla = (\partial/\partial r, r^{-1}\widetilde{\nabla})$. As a result we obtain

$$\begin{aligned}\left\| |x|^{-2} v \right\|^2_{L_2(\Omega \cap B_R(\mathcal{O}))} &= \int_0^R r^{n-5} \int_\omega \left|v(r\theta)\right|^2 ds_\theta\,dr \leqslant c \int_0^R r^{n-5} \int_\omega \left|\widetilde{\nabla} v(r\theta)\right|^2 ds_\theta\,dr \\ &\leqslant c' \int_0^R r^{n-3} \int_\omega \left|\nabla v(r\theta)\right|^2 ds_\theta\,dr = c \left\| |x|^{-1} \nabla_x v \right\|^2_{L_2(\Omega \cap B_R(\mathcal{O}))}.\end{aligned}$$

Since $r > R$ on $\Omega \backslash B_R(\mathcal{O})$, we conclude that $\||x|^{-2} v\|_{L_2(\Omega)} \leqslant c \||x|^{-1} \nabla_x v\|_{L_2(\Omega)}$ and finish the proof. □

4. Domains in dimension $n \geqslant 4$

As a consequence of Proposition 3.3 we may conclude that typical four and higher-dimensional concave boundary points do not destroy the positivity preserving property.

Proposition 4.1. *Let $n \geqslant 4$ and suppose that Ω is as in Condition* 2.2. *If $f \in L_2(\Omega)$ and $f \geqslant 0$, then there is a unique weak solution $u \in \mathcal{H}(\Omega)$ of* (1.1) *and $u \geqslant 0$.*

Proof. Taking the indices as in (3.10) Proposition 3.3 states that

$$A_0^{1,2} : V_0^{2,2}(\Omega) \cap \mathring{V}_{-1}^{1,2}(\Omega) \to V_0^{0,2}(\Omega)$$

is an isomorphism whenever, see (3.9),

$$1 - \Lambda_1^+ < 0 - 1 + \frac{n}{2} < n - 1 + \Lambda_1^+, \tag{4.1}$$

or, equivalently,

$$n > 4 - 2\Lambda_1^+. \tag{4.2}$$

For $n \geqslant 4$ the condition in (4.2) is satisfied. We have $V_0^{0,2}(\Omega) = L_2(\Omega)$ and from Lemma 3.4 we know that $\mathcal{H}(\Omega) = V_0^{2,2}(\Omega) \cap \mathring{V}_{-1}^{1,2}(\Omega)$. So $A_0^{1,2} : \mathcal{H}(\Omega) \to L_2(\Omega)$ is an isomorphism and both problems (1.1) and (1.2) have unique solutions in $H^2(\Omega)$. In particular, this means that problem (2.2) has a unique generalized solution $u \in \mathcal{H}(\Omega)$ and estimate (1.8) is valid. □

Remark 4.1.1.

1. Since $v \in V_0^{2,2}(\Omega)$ Proposition 4.1 gives a necessary and sufficient condition for the existence of the solution $u = w \in V_0^{4,2}(\Omega) \cap \mathring{V}_{-1}^{1,2}(\Omega) \subset H^4(\Omega)$ to problem (2.2) whenever

$$1 - \Lambda_1^+ < 0 - 3 + \frac{n}{2} < n - 1 + \Lambda_1^+ \tag{4.3}$$

   or, equivalently, $n > 8 - 2\Lambda_1^+$. So inequality (4.3) is satisfied for the case $n \geqslant 8$. Hence for $n \geqslant 8$ we always have that $u \in H^4(\Omega)$.
2. If the cone K_ω^∞ is convex, i.e. $\omega \subset \mathbb{S}_+^{n-1}$, then by Rayleigh's principle $\mu_1(\omega) > \mu_1(\mathbb{S}^{n-1} \cap \mathbb{R}_+^n) = 2$ and hence

$$\Lambda_1^+ > 1. \tag{4.4}$$

   The last equality follows from the fact that $U(x) = x_n$ is a positive power-law solution to the model problem (3.2) in the cone $K_\omega^\infty = \mathbb{R}_+^n = \{x;\ x_n > 0\}$. The estimate in (4.4) shows that (4.3) is valid for $n \geqslant 6$ and $u \in H^4(\Omega)$.
3. Finally, if $K_\omega^\infty \subset \mathbb{R}_{++}^n = \{x\colon x_n > 0,\ x_{n-1} > 0\}$, then $\mu_1(\omega) > \mu_1(\mathbb{S}^{n-1} \cap \mathbb{R}_{++}^n) = 6$ and

$$\Lambda_1^+ > 2 \tag{4.5}$$

   because the model problem (3.2) in $\mathbb{R}_{++}^n$ has the positive power-law solution $U(x) = x_n x_{n-1}$ and $\Lambda_1^+(\mathbb{S}^{n-1} \cap \mathbb{R}_{++}^n) = 2$. In that case (4.3) is satisfied for any dimension $n \geqslant 4$ and $u \in H^4(\Omega)$.

Remark 4.1.2. Since $v \in V_0^{2,2}(\Omega) \subset V_1^{2,2}(\Omega)$ it may be of interest to get a solution $u = w \in V_1^{4,2}(\Omega) \subset H^3(\Omega)$. The condition

$$1 - \Lambda_1^+ < 1 - 3 + \frac{n}{2} < n - 1 + \Lambda_1^+ \tag{4.6}$$

is verified for any $n \geqslant 6$. By virtue of (4.4), inequalities (4.6) hold true for any convex cone $K \subset \mathbb{R}^n$ whenever $n \geqslant 4$.

5. Domains in dimension $n = 3$

Now condition (3.9) with indices (3.10) reads as $1 - \Lambda_1^+ < \frac{1}{2} < 2 + \Lambda_1^+$. So in order to use the above argument to conclude the existence of the generalized solution $u \in H^2(\Omega)$ to problem (2.2) we need

$$\Lambda_1^+ > \frac{1}{2}. \tag{5.1}$$

Lemma 5.1. *There exist cones K_ω^∞ with $\omega \subset \mathbb{S}^2$ as in Definition* 2.1 *such that $\mu_1 = \frac{3}{4}$.*

Remark 5.1.1. If $\Omega \subset \mathbb{R}^3$ satisfies Condition 2.2 with ω such that $\mu_1 = \frac{3}{4}$ then $\Lambda_1^+ = \frac{1}{2}$. With the lemma such domains Ω can be constructed.

Proof. Let K be a slender cone, $K \subset \mathcal{K}_\varepsilon = \{x \in \mathbb{R}_+^n\colon |x|^2 - x_n^2 < \varepsilon x_n^2\}$ where $\varepsilon > 0$ is a small parameter. As shown in [24] (see also [26, Chapter 10]), there exist positive constants ε_K and c_K such that, for $\varepsilon \in (0, \varepsilon_K)$ and the exterior cone $K^* = \mathbb{R}^3 \backslash \overline{K}$ with the cross-section $\omega^* = \mathbb{S}^{n-1} \backslash \overline{\omega}_K$, we have

$$\Lambda_1^+(\omega^*) < c_{K^*} |\log \varepsilon|^{-1}. \tag{5.2}$$

Estimate (5.2) with a sufficiently small ε provides the inequality

$$\Lambda_1^+ < 1/2, \tag{5.3}$$

and hence $\mu_1(\omega) < 1$. Furthermore, since the first eigenvalue $\mu_1(\omega)$ of problem (3.4) depends continuously on the domain ω (see [14] for details) and according to (4.4) $\mu_1(\mathbb{S}^2 \cap \mathbb{R}_+^3) = 2$, there exist (infinitely many, non-convex) cones K_ω^∞, for which $\mu_1(\omega) = \frac{3}{4}$, that is with formula (3.3),

$$\Lambda_1^+ = 1/2, \tag{5.4}$$

which completes the proof. □

Lemma 5.2. *Suppose that $\Omega \subset \mathbb{R}^3$ satisfies Condition* 2.2. *Then $\Lambda_2^+ > 1$.*

Proof. With formula (3.3) it is sufficient to prove that $\mu_2(\omega) > 2$. By [10, Satz 3, Chapter VI, §2] one knows that $\Omega_a \subset \Omega_b$ implies that $\mu_m(\Omega_a) \geqslant \mu_m(\Omega_b)$. Indeed, setting for $v_i \in L_2(\Omega)$, $i = 1, \ldots, m$,

$$d_\Omega(v_1, \ldots, v_m) := \inf\left\{ \frac{\int_\Omega |\nabla\varphi|^2\,dx}{\int_\Omega \varphi^2\,dx};\ \varphi \in \mathring{H}^1(\Omega) \backslash \{0\} \text{ with } \int_\Omega \varphi v_i\,dx = 0 \right\},$$

it holds that

$$\mu_m(\Omega) = \sup\{ d(v_1, \ldots, v_m);\ v_i \in L_2(\Omega),\ i = 1, \ldots, m \}.$$

Since every $v_i \in L_2(\Omega_b)$ can be restricted to $Rv_i \in L_2(\Omega_a)$ and $\varphi \in \mathring{H}^1(\Omega_a)$ can be extended to $E\varphi \in \mathring{H}^1(\Omega_b)$ with $\int_{\Omega_a} \varphi Rv_i\,dx = \int_{\Omega_b} E\varphi v_i\,dx$ one finds

$$d_{\Omega_a}(Rv_1, \dots, Rv_m) \geqslant d_{\Omega_b}(v_1, \dots, v_m).$$

Since $R: L_2(\Omega_b) \to L_2(\Omega_a)$ is onto we find, by taking the suprema on both sides, $\mu_n(\Omega_a) \geqslant \mu_n(\Omega_b)$. If Ω_a lies strictly within Ω_b then the inequality is strict. Indeed, suppose that $\mu_n(\Omega_a) = \mu_n(\Omega_b)$ and let φ_n be the corresponding eigenfunction on Ω_a. Then $E\varphi_n$ is an eigenfunction on Ω_b which is zero on an open set. Since we assumed that Ω_b is connected we find by the unique continuation that $E\varphi_n \equiv 0$, a contradiction. So

$$\mu_n(\Omega_a) > \mu_n(\Omega_b).$$

For the present case we may conclude that $\mu_2(\omega) > \mu_2(\mathbb{S}^2) = 2$. □

For every $f \in L_2(\Omega)$ the iterated Dirichlet Laplacian (1.2) gives a solution $w \in H^1(\Omega)$ of problem (2.2). To demonstrate that if $\Lambda_1^+ \in (0, 1/2)$ this solution w does not belong to $H^2(\Omega)$ for at least some positive right-hand side $f \in L_2(\Omega)$, we will introduce a weight function (see [27]). Let Φ_1 and μ_1 be the first eigenfunction and eigenvalue on ω and let Φ_1 be normalized by $\|\Phi_1\|_{L_2(\omega)} = 1$ and $\Phi_1 > 0$. Set

$$U_1^-(r\theta) = r^{-1-\Lambda_1^+}\Phi_1(\theta) \tag{5.5}$$

and define

$$\zeta_1(x) = \frac{1}{1+2\Lambda_1^+}\chi(r)U_1^-(x) + \hat{\zeta}_1(x), \tag{5.6}$$

where $\chi \in C_0^\infty[0, R)$ is a cut-off function such that $\chi(r) = 1$ as $r \in (0, R/2)$, and where $\hat{\zeta}_1 \in \mathring{H}^1(\Omega)$ is the solution to (1.5) with

$$f = -\frac{2\nabla\chi \cdot \nabla U_1^- + U_1^-\Delta\chi}{1+2\Lambda_1^+} \in C_c^\infty(\overline{\Omega}\setminus\mathcal{O}).$$

Clearly, $\chi U_1^- \notin H^1(\Omega)$ and, therefore, ζ_1 is non-trivial irregular solution to the homogeneous problem (1.1). Since $c_1 > 0$ and $\Phi_1(\theta) > 0$ for $\theta \in \omega$, the harmonics ζ_1 is positive inside Ω.

Lemma 5.3. *Suppose that $\Lambda_1^+ < \frac{1}{2}$. Then the solution $v \in \mathring{H}^1(\Omega)$ of problem* (1.5) *with the right-hand side $f \in L_2(\Omega)$ admits the asymptotic representation*

$$v(x) = \chi(r)c_1U_1^+(x) + \tilde{v}(x) \tag{5.7}$$

where $\tilde{v} \in H^2(\Omega)$, $U_1^+(x) = r^{\Lambda_1^+}\Phi_1(\theta)$, and

$$c_1 = \int_\Omega \zeta_1(x)f(x)\,dx. \tag{5.8}$$

Furthermore, it holds true that

$$\|\tilde{v}\|_{H^2(\Omega)} + |c_1| \leqslant c\|f\|_{L_2(\Omega)}. \tag{5.9}$$

Proof. In view of Lemma 5.2 and under assumption (5.3), the representation (5.7) together with the estimate in (5.9) follow from the general results in [15] which are also presented in Proposition 3.3. See [30, Theorem 3.5.6 (p. 68) and Theorem 4.2.1 (p. 106)]. We also like to refer to [4] where a formula of type (5.7) is used to calculate the defect index of the Laplacian with the domain $H^2(\Omega) \cap \mathring{H}^1(\Omega)$ in the two-dimensional case. Let us verify the integral formulae (5.8) by applying method in [27] (see also [30, Theorem 3.5.10 (p. 72) and Theorem 3.3.9 (p. 117)]). In view of (5.7) and (5.6) the Green formula in the domain $\Omega_\delta = \Omega \backslash B_\delta(\mathcal{O})$ yields

$$\begin{aligned}
\int_\Omega \zeta_1 f\,dx &= \lim_{\delta\to 0}\int_{\Omega_\delta} \zeta_1 f\,dx = \lim_{\delta\to 0}\int_{\Omega_\delta}(v\Delta\zeta - \zeta\,\Delta v)\,dx \\
&= \lim_{\delta\to 0}\int_{\partial\mathbb{B}_\delta\cap K_\omega^\infty}\left(\zeta_m\frac{\partial v}{\partial r} - v\frac{\partial\zeta_1}{\partial r}\right)ds_x \\
&= \lim_{\delta\to 0}\frac{\delta^2}{1+2\Lambda_1^+}c_j\left(\Lambda_1^+\delta^{-1-\Lambda_1^+}\delta^{\Lambda_1^+-1} + \left(1+\Lambda_1^+\right)\delta^{\Lambda_1^+}\delta^{-2-\Lambda_1^+}\right) \\
&\quad\times\int_\omega \Phi_1(\theta)^2\,ds_\theta = c_1. \qquad \square
\end{aligned}$$

If $f \geqslant 0$ and $f \not\equiv 0$, then the coefficient c_1 in (5.7) is positive and $v \notin H^2(\Omega)$ because $U_1^+ \notin H^2(\Omega \cap K_\omega^R)$ according to (5.3). A way to construct the generalized solution $u \in H^2(\Omega)$ of problem (1.1) was proposed in [25]. Observing that by (5.3), $\zeta_1 \in L_2(\Omega)$, we take

$$v = v_0 + a_1\zeta_1 \in L_2(\Omega) \tag{5.10}$$

as a solution to problem (1.1) while a_1 is a constant to be fixed and $v_0 \in \mathring{H}^1(\Omega)$ is the energy solution of problem (1.1). According to Lemma 5.3, the solution $u = w \in \mathring{H}^1(\Omega)$ to problem (1.5) with the right-hand side (5.10) belongs to $H^2(\Omega)$ provided

$$c_1^0 = \int_\Omega \zeta_1(x)v_0(x)\,dx + a_1\int_\Omega \zeta_1(x)^2\,dx = 0. \tag{5.11}$$

The factor at a_1 in (5.11) is positive. Thus, we may compute a_1 and fix the function v as in (5.10). The inequalities

$$|a_1| \leqslant c\|v_0\|_{L_2(\Omega)} \leqslant c\|f\|_{L_2(\Omega)},$$

are valid so that, due to (5.9), estimate (1.9) holds true for the solution u of problem (2.2).

Remark 5.3.1. Let us compare the asymptotics of the constructed solution $u \in H^2(\Omega)$ to problem (1.1) with the asymptotics

$$w(x) = \chi(r) C_1^w r^{\Lambda_1^+} \Phi_1(\theta) + O(r), \quad r \to 0, \tag{5.12}$$

of the solution $w \in H^1(\Omega)$ of the iterated Dirichlet Laplacian (1.2). We wrote $r\theta = x$. Since the main singularity of the right-hand side (5.10) in problem (1.5) for u is of the form

$$\chi(r) \frac{a_1}{1 + 2\Lambda_1^+} r^{-1-\Lambda_1^+} \Phi_1(\theta) + o(1), \quad r \to 0,$$

and coefficient (5.11) is annulled, we find the solution

$$V^1(x) = -\frac{1/2}{1 - 2\Lambda_1^+} r^{1-\Lambda_1^+} \Phi_1(\theta) \tag{5.13}$$

of the inhomogeneous model problem

$$-\Delta V^1(x) = r^{-1-\Lambda_1^+} \Phi_1(\theta), \quad x \in K_\omega^\infty; \qquad V^1(x) = 0, \quad x \in \partial K_\omega^\infty \backslash \mathcal{O}, \tag{5.14}$$

and derive the asymptotic formula

$$u(x) = -\chi(r) \frac{a_1/2}{1 - (2\Lambda_1^+)^2} r^{1-\Lambda_1^+} \Phi_1(\theta) + O(r) \quad \text{for } r \to 0. \tag{5.15}$$

Comparing (5.12) and (5.15), we observe that the above modification of solving the iterated Dirichlet Laplacian changes the singularity $\chi(r) r^{\Lambda_1^+} \Phi_1(\theta) \in H^1(\Omega) \backslash H^2(\Omega)$ into the singularity $\chi(r) r^{1-\Lambda_1^+} \Phi_1(\theta) \in H^2(\Omega)$ (the inclusions are ensured since $\Lambda_1^+ \in (0, 1/2)$). Note that the singular term $r^{\Lambda_1^+} \Phi_1(\theta)$ in (5.12) has the positive angular part Φ_1 while in (5.15) the same angular part is attributed to $r^{1-\Lambda_1^+} \Phi_1(\theta)$ which, owing to Lemma 5.2, remains to be the main asymptotic term.

Let us now consider the case (5.4), $\Lambda_1^+ = \frac{1}{2}$. Since the left inequality in (3.9) with $\beta = 0$, $l = 1$, $n = 3$ and $\Lambda_1^+ = \frac{1}{2}$ is invalid, Proposition 3.3 does not supply us with the solutions v and w of problem (1.2) in $H^2(\Omega) \cap \mathring{H}^1(\Omega)$. This fact can be explained by observing that the asymptotics of both, v and w, must contain term $r^{1/2} \Phi_1(\theta)$ which is not in $H^2(\Omega)$. At the same time, there is no generalized solution $u \in H^2(\Omega)$ of problem (1.1). Indeed, an attempt to improve the regularity property of w by considering the solution (5.10) to problem (1.1) provides an even worse singularity of u. Indeed, since now $1 - 2\Lambda_1^+ = 0$, formula (5.13) cannot be used and a solution to the model problem (5.14) takes the form

$$V^1(x) = -\frac{1}{2} r^{1/2} (\log r + C) \Phi_1(\theta)$$

where the constant C is arbitrary.

The following example explains why properties of solutions to the problems in question change crucially in the case $\Lambda_1^+ = 1/2$.

Example 5.3.2. Let $r^{1/2}\Phi(\theta)$ be a non-trivial solution to the model problem (3.2). We introduce the cut-off function

$$\chi_N(t) = \bigl(1-\chi(t)\bigr)\chi(t-N-1)$$

where $N \in \mathbb{N}$ and $\chi \in C^\infty(\mathbb{R})$, $0 \leqslant \chi(t) \leqslant 1$, $\chi(t) = 1$ for $t < 0$ and $\chi(t) = 0$ for $t \geqslant 1$. The function

$$v_N(x) = \chi_N\left(\log\frac{R}{r}\right) r^{1/2}\Phi(\theta)$$

belongs to $C_0^\infty(\overline{\Omega}\backslash\mathcal{O})$ and

$$\begin{aligned}\bigl\|v_N; H^2(\Omega)\bigr\|^2 &\geqslant \bigl\|r^{-2}\widetilde{\nabla}^2 v_N; L_2(\Omega)\bigr\|^2 \geqslant \int\limits_{Re^{-N-1}}^{Re^{-1}} r^{-4}\bigl(r^{1/2}\bigr)^2 r^2\,dr\,\bigl\|\widetilde{\nabla}^2\Phi_1; L_2(\omega)\bigr\|^2 \\ &= c_\Phi\left(\log\frac{R}{e} - \log\frac{R}{e^{N+1}}\right) = Nc_\Phi \qquad (5.16)\end{aligned}$$

where $c_\Phi > 0$. On the other hand,

$$f_N(x) := -\Delta v_N(x) = \Phi(\theta) r^{-2}\frac{\partial}{\partial r} r^{3/2}\chi'\left(\log\frac{R}{r}\right)$$

where $\chi'(t) = \frac{d\chi}{dt}(t)$. The support of f_N is located in the union of the sets $K_\omega^R \backslash K_\omega^{e^{-1}R}$ and $K_\omega^{e^{-N-1}R}\backslash K_\omega^{e^{-N-2}R}$. Hence,

$$\|f_N\|^2_{L^2(\Omega)} \leqslant c\left(1 + \int\limits_{Re^{-N-2}}^{Re^{-N-1}} r^{2(-2+1/2)} r^2\,dr\right) \leqslant C. \qquad (5.17)$$

Comparing (5.16) and (5.17) shows that the range of the operator of problem (1.1),

$$A : H^2(\Omega) \cap \mathring{H}^1(\Omega) \to L_2(\Omega)$$

is not closed because the estimate $\|v\|_{H^2(\Omega)} \leqslant C\|f\|_{L^2(\Omega)}$ cannot hold with a constant C independent of the right-hand side f.

6. Domains in dimension $n = 2$

We start by recalling the following result from [25].

Theorem 6.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain that has a boundary which is C^2 except for finitely many corners. Then for every $f \in L_2(\Omega)$ there exists a unique solution $u \in H^2(\Omega)$. Moreover, there is a constant c_Ω such that for all f*

$$\|u\|_{H^2(\Omega)} \leqslant c\|f\|_{L_2(\Omega)}.$$

For simplicity we will not only assume that Ω satisfies Condition 2.2 but even the following.

Condition 6.2. The domain $\Omega \subset \mathbb{R}^2$ is bounded and simply connected with

- $\partial\Omega \backslash \mathcal{O} \in C^\infty$;
- there exist $\alpha \in (\pi, 2\pi]$ such that $K^1_{\omega_\alpha} \subset \Omega \subset K^\infty_{\omega_\alpha}$.

An important application appears for $\alpha = 2\pi$; then the domain contains a so-called crack. Such problems have been studied for example in [22].

We are interested in domains with a non-convex angular point so $\alpha > \pi$. For the definition of ω_α see (2.1). Note that the only extra restriction is that Ω itself lies inside a multiple of its interior cone. This restriction is not fundamental but will be convenient for positivity statements. Since most of our arguments use the asymptotic behavior near $\mathcal{O}$ we may replace the assumption $K^1_{\omega_\alpha} \subset \Omega \subset K^\infty_{\omega_\alpha}$ by $\{x \in \Omega;\ |x| < 1\} = K^1_{\omega_\alpha}$ if we multiply by an appropriate cut-off function.

6.1. Comparing the $(H^1)^2$ and the H^2-solutions

Let us recall the weight function from [30, Chapter 2 (p. 32)]. Notice that weight functions for general boundary value problems were introduced in [27].

Definition 6.3. Set

$$\zeta(x) = \frac{1}{\pi} r^{-\pi/\alpha} \cos\left(\frac{\pi}{\alpha}\varphi\right) + \hat{\zeta}(x)$$

with $x = (r\cos\varphi, r\sin\varphi)$ and where $\hat{\zeta} \in H^1(\Omega)$ is the unique solution of

$$\begin{cases} -\Delta\hat{\zeta} = 0 & \text{in } \Omega, \\ \hat{\zeta} = -\frac{1}{\pi} r^{-\pi/\alpha} \cos\left(\frac{\pi}{\alpha}\varphi\right) & \text{on } \partial\Omega. \end{cases}$$

Remark 6.3.1. If $\Omega = \Omega_\alpha \subset \mathbb{R}^2$ then $\zeta(x) = \frac{1}{\pi}(r^{-\pi/\alpha} - r^{\pi/\alpha}) \cos(\frac{\pi}{\alpha}\varphi)$.

If Ω satisfies Condition 6.2 we find that $\zeta \in C^2(\overline{\Omega} \backslash \{0\})$, and that ζ satisfies

$$\begin{cases} -\Delta\zeta = 0 & \text{in } \Omega, \\ \zeta = 0 & \text{on } \partial\Omega \backslash \{0\}. \end{cases}$$

Notice that due to the assumption above the boundary values of $\hat{\zeta}$ are zero on $\partial\Omega \cap B_1(0)$ and since the boundary values are hence regular we find hence $\hat{\zeta} \in H^1(\Omega)$. Also notice that a direct computation shows $\zeta \in L^2(\Omega)$ and $\zeta \notin H^1(\Omega)$ when $\alpha > \pi$.

Lemma 6.4. *Let $\Omega \subset \mathbb{R}^2$ satisfy Condition* 6.2. *One finds that $\zeta > 0$ in Ω.*

Proof. Since $\hat{\zeta}_{|\partial\Omega}$ is bounded the maximum principle shows that $\hat{\zeta}$ is bounded in $\overline{\Omega}$. Since the singularity of ζ goes to $+\infty$ we find $\min(0, \zeta) \in H^1(\Omega)$ and even $\min(0, \zeta) \in \mathring{H}^1(\Omega)$.

Moreover, $\min(0,\zeta)$ is superharmonic. So the strong maximum principle implies $\min(0,\zeta) > 0$ in Ω or $\min(0,\zeta) = 0$ in Ω. Since $\zeta \not\equiv 0$ we get $\zeta > 0$ in Ω. □

Using this auxiliary function $\zeta \in L^2(\Omega)$ we may solve the second part of (1.2) except in 0, that is

$$\begin{cases} -\Delta v = f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega\backslash\{0\}, \end{cases} \tag{6.1}$$

by

$$v_a(x) = v(x) + a\zeta(x), \tag{6.2}$$

where $v = \mathcal{G}_\Omega f$ is the unique solution in $\mathring{H}^1(\Omega)$ of (6.1) and a some arbitrary constant. Next, solving the remaining part of the system with the right-hand side v_a:

$$\begin{cases} -\Delta w_a = v_a & \text{in } \Omega, \\ w_a = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.3}$$

Notice that $v \in L^2(\Omega)$ and hence we find for each v_a a unique solution $w_a \in \mathring{H}^1(\Omega)$. So all functions $w_a = \mathcal{G}_\Omega^2 f + a\mathcal{G}_\Omega \zeta$, with $a \in \mathbb{R}$, satisfy (2.2). If we demand that $w_a, \Delta w_a \in \mathring{H}^1(\Omega)$, then $a = 0$. We will show that the solution u of (1.1) in $H^2(\Omega) \cap \mathring{H}^1(\Omega)$ has $a \neq 0$. This procedure to find a solution in $H^2(\Omega)$ can also be found in [24]. It is similar as in the 3d-case discussed in the previous section.

Let us introduce a second auxiliary function.

Definition 6.5. Let Ω satisfy Condition 6.2. Set

$$\eta(x) = r^{\pi/\alpha} \cos\left(\frac{\pi}{\alpha}\varphi\right)\chi(r)$$

with $x = (r\cos\varphi, r\sin\varphi)$ and where $\chi \in C^\infty(\mathbb{R})$ satisfies $0 \leqslant \chi \leqslant 1$ with

$$\chi(r) = 1 \quad \text{for } r \leqslant \frac{1}{2}R \quad \text{and} \quad \chi(r) = 0 \quad \text{for } r \geqslant R.$$

Remark 6.5.1. One may show that $\eta \in \mathring{H}^1(\Omega)$ and $\eta \notin H^2(\Omega)$.

The function w that solves (6.3) has the unique representation

$$w_a(x) = c\eta(x) + \tilde{w}(x) \tag{6.4}$$

with $\tilde{w} \in H^2(\Omega) \cap \mathring{H}^1(\Omega)$. Indeed, see [4] or [30, Chapter 2 (pp. 33, 34 and 36–38)]. So if we want $w_a \in H^2(\Omega)$ we need $c = 0$. This c is the so-called intensity factor for which the following integral representation holds (see [30, Chapter 2, formula 3.7 (p. 33)])

$$c = \int_\Omega v_a(x)\zeta(x)\,dx \tag{6.5}$$

with v_a the right-hand side in (6.3). For v_a as in (6.2) we find that

$$a = -\|\zeta\|_{L^2(\Omega)}^{-2} \int_\Omega v(x)\zeta(x)\,dx.$$

So the solution $u \in H^2(\Omega) \cap \mathring{H}^1(\Omega)$ of (1.1) is as follows

$$u = \mathcal{G}_\Omega^2 f - \frac{\int_\Omega v(x)\zeta(x)\,dx}{\|\zeta\|_{L^2(\Omega)}^2}\mathcal{G}_\Omega\zeta = \mathcal{G}_\Omega^2 f - \frac{\int_\Omega (\mathcal{G}_\Omega f)(x)\zeta(x)\,dx}{\|\zeta\|_{L^2(\Omega)}^2}\mathcal{G}_\Omega\zeta,$$

or, if we set $\mathcal{P}_\zeta$ the $L_2(\Omega)$-projection on ζ

$$u = \mathcal{H}_\Omega f := \left(\mathcal{G}_\Omega^2 - \mathcal{G}_\Omega\mathcal{P}_\zeta\mathcal{G}_\Omega\right) f. \tag{6.6}$$

Since $\zeta > 0$ we may immediately state.

Corollary 6.6. *Let $\Omega \subset \mathbb{R}^2$ satisfy Condition* 6.2. *If $f \in L^2(\Omega)$ satisfies $f > 0$ then the solutions $u \in H^2(\Omega) \cap \mathring{H}^1(\Omega)$ of* (1.1) *and $w \in \mathring{H}^1(\Omega)$ with $\Delta w \in \mathring{H}^1(\Omega)$ of* (1.2) *are ordered: $u \leqslant w$ in Ω.*

Notice that the solution operator for the plate problem (1.1) is well defined for $f \in H^{-1}(\Omega)$. So $\mathcal{G}_\Omega$ may be extended to $H^{-1}(\Omega)$. One finds $\mathcal{G}_\Omega f \in \mathring{H}^1(\Omega) \subset L^2(\Omega)$ and $\mathcal{H}_\Omega : H^{-1}(\Omega) \to L^2(\Omega)$ is well defined. Hence the operator $\mathcal{H}_\Omega$ is also defined for $(C(\overline{\Omega}))'$ and it will be sufficient (and also necessary) for the positivity of $\mathcal{H}_\Omega$ to study if $\mathcal{H}_\Omega\delta_x \geqslant 0$ for all $x \in \Omega$. Here δ_x is the Dirac measure at $x \in \Omega$. We have that $\mathcal{H}_\Omega$ is positive if and only if

$$\int_\Omega G_\Omega(x,z)G_\Omega(z,y)\,dz \int_\Omega \zeta(z)\zeta(z)\,dz \geqslant \int_\Omega G_\Omega(z,y)\zeta(z)\,dz \int_\Omega G_\Omega(x,z)\zeta(z)\,dz \tag{6.7}$$

where $G_\Omega(x,y)$ is the kernel of $\mathcal{G}_\Omega$.

6.2. Asymptotic expansion

6.2.1. Formal asymptotics

By the work of Kondratiev [15] one knows that a solution of a boundary value problem such as (1.1) for Ω a cone and $f \equiv 0$ near the vertex has formally the following asymptotic expansion near that vertex

$$u(x) \sim \sum_i \sum_{k=0}^{\kappa_i - 1} c_{i,k} r^{\lambda_i} (\log r)^k \psi_{i,k}(\varphi). \tag{6.8}$$

Here we used $x = (r\cos\varphi, r\sin\varphi)$. In the present planar case the values λ_i (with κ_i the multiplicity of λ_i) and functions $\psi_{i,k}$ are determined by the spectral problem on the arc:

$$\begin{cases} \left(\lambda_i^2 + \frac{\partial^2}{\partial\varphi^2}\right)\left((\lambda_i - 2)^2 + \frac{\partial^2}{\partial\varphi^2}\right)\psi_{\lambda_i,k}(\varphi) = \psi_{\lambda_i,k-1}(\varphi) \quad \text{for } |\varphi| < \frac{1}{2}\alpha, \\ \psi_{\lambda_i,k}\left(\frac{1}{2}\alpha\right) = \psi''_{\lambda_i,k}\left(\frac{1}{2}\alpha\right) = \psi_{\lambda_i,k}\left(-\frac{1}{2}\alpha\right) = \psi''_{\lambda_i,k}\left(-\frac{1}{2}\alpha\right) = 0, \end{cases}$$

where we take $\psi_{\lambda_i,-1}(\varphi) \equiv 0$. One finds, allowing $j \in \{0,1\}$ and $n \in \mathbb{N}$, that

$$\lambda_i = 2j \pm n\frac{\pi}{\alpha} \quad \text{and} \quad \psi_{\lambda_i,0}(\varphi) = \sin\left(n\frac{\pi}{\alpha}\left(\varphi + \frac{1}{2}\alpha\right)\right).$$

Let us define for $n \in \mathbb{Z}\backslash\{0\}$

$$U^{n,j}(r,\varphi) = r^{2j+n\frac{\pi}{\alpha}} \sin\left(n\frac{\pi}{\alpha}\left(\varphi + \frac{1}{2}\alpha\right)\right). \tag{6.9}$$

Notice that the multiplicity of λ_i is 1 except when $2 \pm n\frac{\pi}{\alpha} = n\frac{\pi}{\alpha}$ for some $n \in \mathbb{Z}$, that is, for $\alpha = n\pi$. Since we are interested in $\alpha \in (\pi, 2\pi]$ this does not occur except when $\alpha = 2\pi$. So we find that $\kappa_i = 0$ for all eigenvalues whenever $\alpha \in (\pi, 2\pi)$ and (6.8) becomes

$$u \sim \sum_{n\in\mathbb{Z}} \sum_{j\in\{0,1\}} c_{n,j} r^{2j+n\frac{\pi}{\alpha}} \sin\left(n\frac{\pi}{\alpha}\left(\varphi + \frac{1}{2}\alpha\right)\right). \tag{6.10}$$

For $\alpha = 2\pi$ the situation degenerates since $U^{-2,1}(r,\varphi) = U^{2,0}(r,\varphi) = -r\sin(\varphi)$ and we are forced to add $U^2_{\ln}(r,\varphi) = r\ln(r)\sin(\varphi)$ to the functions in (6.9). However, since $U^2_{\ln}$ does not belong to H^2 locally near 0 it does not influence the asymptotic behaviour of an H^2-solution.

6.2.2. Solutions

For $f \in L_2(\Omega)$ only some asymptotic terms in (6.10) play a role. Since $V^{0,2}_\delta(\Omega) \supset V^{0,2}_0(\Omega) = L_2(\Omega)$ for $\delta > 0$ we have $f \in V^{0,2}_\delta(\Omega)$ with $\delta \geqslant 0$.

If $2j + n\frac{\pi}{\alpha} \neq 3 - \delta$ for $n \in \mathbb{Z}\backslash\{0\}$, then a solution u of (1.1) in $H^2(\Omega)$ can be written as

$$u(x) = \sum_{1<2j+n\frac{\pi}{\alpha}<3-\delta} c_{n,j} U^{n,j}(r,\varphi) + \tilde{u}(x) \quad \text{for } \alpha \in (\pi, 2\pi) \tag{6.11}$$

with $\tilde{u} \in V^{4,2}_\delta(\Omega)$. Indeed $r^{2j+n\frac{\pi}{\alpha}} \sin(n\frac{\pi}{\alpha}(\varphi + \frac{1}{2}\alpha)) \in H^2(\Omega)$ with $\alpha \in (\pi, 2\pi)$ if and only if $2j + n\frac{\pi}{\alpha} > 1$. The restriction $1 < 2j + n\frac{\pi}{\alpha} < 3 - \delta$ brings at most six asymptotic terms.

In the case $\alpha \in (\pi, 2\pi)$ these are

$$\begin{aligned} U^{-1,1}(r,\varphi) &= r^{2-\frac{\pi}{\alpha}} \cos\left(\frac{\pi}{\alpha}\varphi\right), & U^{1,1}(r,\varphi) &= r^{2+\frac{\pi}{\alpha}} \cos\left(\frac{\pi}{\alpha}\varphi\right), \\ U^{2,0}(r,\varphi) &= r^{2\frac{\pi}{\alpha}} \sin\left(2\frac{\pi}{\alpha}\varphi\right), & U^{3,0}(r,\varphi) &= r^{3\frac{\pi}{\alpha}} \cos\left(3\frac{\pi}{\alpha}\varphi\right), \\ U^{4,0}(r,\varphi) &= r^{4\frac{\pi}{\alpha}} \sin\left(4\frac{\pi}{\alpha}\varphi\right), & U^{5,0}(r,\varphi) &= r^{5\frac{\pi}{\alpha}} \cos\left(5\frac{\pi}{\alpha}\varphi\right). \end{aligned} \tag{6.12}$$

For $\alpha = 2\pi$ we obtain

$$u(x) = \sum_{1\leqslant 2j+\frac{1}{2}n<3-\delta} c_{n,j} U^{n,j}(r,\varphi) + \tilde{u}(x). \tag{6.13}$$

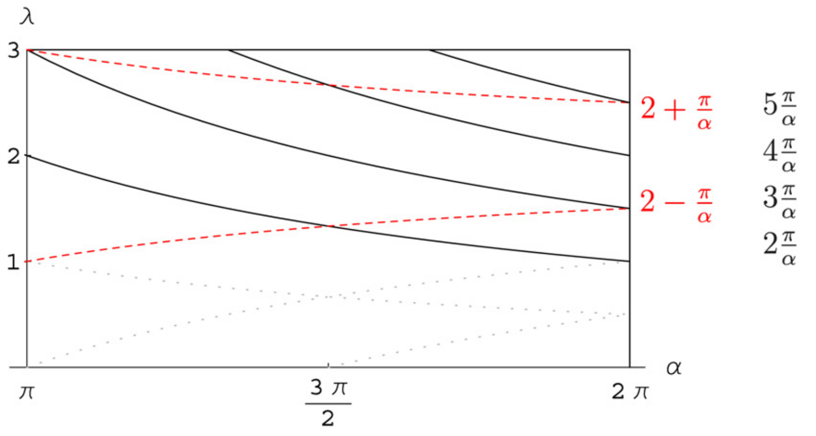


Fig. 2. The relation between λ_i and α.

Now the functions $U^{-1,1}$ and $U^{2,0}$ coincide and lie in $C^\infty(\overline{\Omega})^{\bullet}$, indeed

$$U^{2,0}(r,\varphi) = U^{-1,1}(r,\varphi) = r\sin(\varphi) = y.$$

Note that the function $U^2_{\ln}$ does not lie in $H^2(\Omega)$ and hence does not appear in the expansion.

Remark 6.6.1. We will show that the sign near $(0,0)$ is determined by the leading term in the expansion (6.10), that is, the one with the smallest exponent. Figure 2 reminds us that the function $U^{n,j}$ with the smallest exponent, that is, the lowest order coefficient $2j + n\frac{\pi}{\alpha}$, is as follows:

1. for $\alpha > \frac{3}{2}\pi$ the sign-changing function $U^{2,0}$ leads;
2. for $\pi < \alpha < \frac{3}{2}\pi$ the positive function $U^{-1,1}$ leads.

Remark 6.6.2. One may also notice that $U^{4,0}, U^{5,0} \in V^{4,2}_1(\Omega)$ for respectively $\alpha < \frac{4}{3}\pi$ and $\alpha < \frac{5}{3}\pi$.

6.2.3. Embedding

Lemma 6.7. *There exist $c_M > 0$ such that for all $w \in V^{2,2}_\gamma(K^1_{\omega_\alpha})$ with $|\gamma| \leqslant M$ and $|\alpha| \leqslant \pi$ the following holds*

$$\left|v(r,\varphi)\right| \leqslant c_M\, r^{1-\gamma} \|v\|_{V^{2,2}_\gamma(K^1_{\omega_\alpha})}. \tag{6.14}$$

Proof. Setting $\hat{v}(t,\varphi) = e^{(1-\gamma)t} v(e^{-t},\varphi)$ we find that there exists $c > 0$ such that

$$\|\hat{v}\|_{H^2(\mathbb{R}^+\times(-\alpha/2,\alpha/2))} = \sum_{k=0}^{2}\sum_{p=0}^{k} \int_{-\alpha/2}^{\alpha/2}\int_0^\infty \left|\partial_t^{k-p}\partial_\varphi^p \tilde{v}(t,\varphi)\right|^2 dt\, d\varphi$$

$$
\begin{aligned}
&= \sum_{k=0}^{2} \sum_{p=0}^{k} \int_{-\alpha/2}^{\alpha/2} \int_{0}^{1} \left|(r\partial_r)^{k-p}\partial_\varphi^p\left(r^{\gamma-1}v(r,\varphi)\right)\right|^2 \frac{1}{r}\,dr\,d\varphi \\
&\leqslant c \sum_{k=0}^{2} \sum_{p=0}^{k} \int_{-\alpha/2}^{\alpha/2} \int_{0}^{1} r^{2(\gamma-2+k)} \left|\partial_r^{k-p}\left(\frac{1}{r}\partial_\varphi\right)^p v(r,\varphi)\right|^2 r\,dr\,d\varphi \\
&\leqslant c' \|v\|_{V_\gamma^{2,2}(K^1_{\omega_\alpha})}.
\end{aligned}
\tag{6.15}
$$

By the Sobolev embedding $H^2(D) \subset L^\infty(D)$ for any $D \subset \mathbb{R}^2$ and one finds

$$
\left|\hat{v}(t,\varphi)\right| \leqslant c\|\hat{v}\|_{H^2(\mathbb{R}^+\times(-\alpha/2,\alpha/2))}. \tag{6.16}
$$

Combining (6.15) and (6.16) implies (6.14). $\square$

Corollary 6.8. *There exist $c_\delta > 0$ such that for all $w \in V_\delta^{4,2}(K^1_{\omega_\alpha})$:*

$$
r^{\delta-3}\left|w(x)\right| + r^{\delta-2}\left|\nabla w(x)\right| \leqslant c_\delta \|w\|_{V_\delta^{4,2}(K^1_{\omega_\alpha})}. \tag{6.17}
$$

Proof. Since $V_\delta^{4,2}(K^1_{\omega_\alpha}) \subset V_{\delta-2}^{2,2}(K^1_{\omega_\alpha})$ we find, by taking $\gamma = \delta - 2$ in (6.14), that for some $\tilde{c}_1 > 0$

$$
r^{\delta-3}\left|w(x)\right| \leqslant \tilde{c}_1 \|w\|_{V_\delta^{4,2}(K^1_{\omega_\alpha})}.
$$

Similarly, since $\nabla w \in V_\delta^{3,2}(K^1_{\omega_\alpha})$, we have from (6.14) that

$$
r^{\delta-2}\left|\nabla w(x)\right| \leqslant \tilde{c}_2 \|w\|_{V_\delta^{4,2}(K^1_{\omega_\alpha})}. \qquad \square
$$

Corollary 6.9. *There exist $c_\delta > 0$ such that for all $w \in V_\delta^{4,2}(K^1_{\omega_\alpha}) \cap \mathring{V}_{-1}^{1,2}(\Omega)$:*

$$
\left|w(x)\right| \leqslant c_\delta\, r^{3-\delta} \cos\left(\frac{\pi}{\alpha}\varphi\right) \|w\|_{V_\delta^{4,2}(K^1_{\omega_\alpha})}.
$$

Proof. If $w \in V_\delta^{4,2}(K^1_{\omega_\alpha})$ then $r^{-2}\partial_\varphi^2 w \in V_\delta^{2,2}(K^1_{\omega_\alpha})$ and hence by Lemma 6.7

$$
\left|r^{-2}\partial_\varphi^2 w(r,\varphi)\right| \leqslant c_M\, r^{1-\delta} \|w\|_{V_\delta^{4,2}(K^1_{\omega_\alpha})}.
$$

Since $w(r,-\alpha/2) = w(r,\alpha/2) = 0$ it follows from (6.14) that

$$
\begin{aligned}
\left|w(r,\varphi)\right| &\leqslant \frac{1}{2}\left(\frac{1}{4}\alpha^2 - \varphi^2\right) \left\|\partial_\varphi^2 w(\cdot,\varphi)\right\|_{L^\infty(-\alpha/2,\alpha/2)} \\
&\leqslant 2\pi c_M r^{3-\delta} \cos\left(\frac{\pi}{\alpha}\varphi\right) \|w\|_{V_\delta^{4,2}(K^1_{\omega_\alpha})}. \qquad \square
\end{aligned}
$$

6.2.4. Coefficients

The weight functions that define the coefficients for general boundary value problems were introduced in [27]. Let us recall the particular weight functions from [30, Chapter 2 (p. 32)] that correspond by duality to the power type solutions in (6.12):

$$\begin{aligned}
Y^{-1,1}(r,\varphi) &= \frac{1}{2}\frac{1}{2-2\frac{\pi}{\alpha}} r^{\frac{\pi}{\alpha}} \cos\left(\frac{\pi}{\alpha}\varphi\right), &\quad Y^{1,1}(r,\varphi) &= \frac{1}{2}\frac{1}{2+2\frac{\pi}{\alpha}} r^{-\frac{\pi}{\alpha}} \cos\left(\frac{\pi}{\alpha}\varphi\right),\\
Y^{2,0}(r,\varphi) &= \frac{1}{4}\frac{1}{4\frac{\pi}{\alpha}-2} r^{2-2\frac{\pi}{\alpha}} \sin\left(2\frac{\pi}{\alpha}\varphi\right), &\quad Y^{3,0}(r,\varphi) &= \frac{1}{6}\frac{1}{6\frac{\pi}{\alpha}-2} r^{2-3\frac{\pi}{\alpha}} \cos\left(3\frac{\pi}{\alpha}\varphi\right),\\
Y^{4,0}(r,\varphi) &= \frac{1}{8}\frac{1}{8\frac{\pi}{\alpha}-2} r^{2-4\frac{\pi}{\alpha}} \sin\left(4\frac{\pi}{\alpha}\varphi\right), &\quad Y^{5,0}(r,\varphi) &= \frac{1}{10}\frac{1}{10\frac{\pi}{\alpha}-2} r^{2-5\frac{\pi}{\alpha}} \cos\left(5\frac{\pi}{\alpha}\varphi\right).
\end{aligned} \tag{6.18}$$

Definition 6.10. Set

$$\zeta_{n,j}(r,\varphi) = Y^{n,j}(r,\varphi) + \hat{\zeta}_{n,j}(r,\varphi) \tag{6.19}$$

where $\hat{\zeta}_{n,j} \in H^1(\Omega)$ is the unique solution of

$$\begin{cases} -\Delta\hat{\zeta}_{n,j} = 0 & \text{in } \Omega,\\ \hat{\zeta}_{n,j} = -Y^{n,j} & \text{on } \partial\Omega. \end{cases}$$

Note that the restriction of $Y^{n,j}$ to $\partial\Omega$ are smooth.

Similar as in Lemma 5.3 the coefficients in (6.11) are defined by

$$c_{n,j} = c_{n,j}(f) := \int_\Omega f(x)\zeta_{n,j}(x)\,dx.$$

6.2.5. Conclusion

Proposition 6.11. *Suppose that Ω satisfies Condition* 6.2. *Let $f \in L_2(\Omega)$ and let u be the H^2-solution of* (1.1).

1. *If $\frac{3}{2}\pi < \alpha \leqslant 2\pi$ and $c_{2,0}(f) \neq 0$ then u changes sign near* 0.
2. *If $\alpha = \frac{3}{2}\pi$ and $|c_{2,0}(f)| > \frac{1}{2}|c_{-1,1}(f)|$ then u changes sign near* 0.
   If $\alpha = \frac{3}{2}\pi$ and $|c_{2,0}(f)| < \frac{1}{2}|c_{-1,1}(f)|$ then u has a fixed sign near 0.
3. *If $\pi < \alpha < \frac{3}{2}\pi$ and $c_{-1,1}(f) \neq 0$ then u has a fixed sign near* 0.

Proof. From the expansion (6.11) and the embedding of Lemma 6.7 of we find that

$$u(x) = \sum_{1\leqslant 2j+n\frac{\pi}{\alpha}<3-\delta} c_{n,j}(f)U^{n,j}(r,\varphi) + o\left(r^{3/2}\cos\left(\frac{\pi}{\alpha}\varphi\right)\right).$$

If $\alpha \neq \frac{3}{2}\pi$ and the appropriate coefficient is non-zero we may conclude by Remark 6.6.1.

For $\alpha = \frac{3}{2}\pi$ we have $U^{-1,1}(r,\varphi) = r^{\frac{4}{3}}\cos(\frac{2}{3}\varphi)$ and $U^{2,0}(r,\varphi) = r^{\frac{4}{3}}\sin(\frac{4}{3}\varphi)$. Since $\sin(\frac{4}{3}\varphi) \leqslant 2\cos(\frac{2}{3}\varphi)$ the solution is of fixed sign whenever $|c_{2,0}(f)| < \frac{1}{2}|c_{-1,1}(f)|$. On the other hand, for any $\varepsilon > 0$ one has that $\cos(\frac{2}{3}\varphi) - (\frac{1}{2}+\varepsilon)\sin(\frac{4}{3}\varphi)$ is of opposite sign near $\pm\frac{3}{4}\pi$. So the solution changes sign whenever $|c_{2,0}(f)| > \frac{1}{2}|c_{-1,1}(f)|$. □

Lemma 6.12. *If $\alpha \in (\frac{3}{2}\pi, 2\pi]$ there exist $f \in L_2(\Omega)$ with $f \geqslant 0$ and $c_{2,0}(f) \neq 0$.*

Proof. The coefficient is fixed by $c_{2,0}(f) = \int_\Omega f(x)\zeta_{2,0}(x)\,dx$. The function $\zeta_{2,0}$ is sign-changing so support$(\zeta_{2,0}^+)$ is non-empty and we may take any non-trivial non-negative $f \in L_2(\Omega)$ with support$(f) \subset$ support$(\zeta_{2,0}^+)$ in order to find that $c_{2,0}(f) > 0$. □

Corollary 6.13. *Let Ω satisfy Condition* 6.2. *For $\alpha \in (\frac{3}{2}\pi, 2\pi]$ the H^2-solution operator is not positivity preserving.*

6.3. Alternative approach

Let $\mathcal{G}_\Omega$, $\mathcal{H}_\Omega$ and ζ be as in Section 6.1 so if we set $\mathcal{P}_\zeta$ the $L_2(\Omega)$-projection on ζ the H^2-solution u satisfies

$$u = \mathcal{H}_\Omega f := \left(\mathcal{G}_\Omega^2 - \mathcal{G}_\Omega \mathcal{P}_\zeta \mathcal{G}_\Omega\right) f. \tag{6.20}$$

A powerful tool for the Laplace equation in 2 dimensions are conformal mappings. We may use a conformal mapping that straightens the corner. This will allow us to transfer system (1.1) to a system with a smooth domain but with singularities in the right-hand sides. On the smooth image domain we may use standard techniques for smooth domains.

For the two-dimensional domains that satisfy Condition 2.2 such a conformal mapping is $h(z) = z^{\pi/\alpha}$. Remember that for a conformal bijection

$$G_\Omega(x,z) = G_{h(\Omega)}\big(h(x), h(z)\big).$$

A first observation is that $\tilde{u} \in C^1(\overline{h(\Omega)}) \cap C_0(\overline{h(\Omega)})$ with $u = \tilde{u} \circ h \in H^{2,2}(\Omega)$ may only hold if

$$\frac{\partial}{\partial x_1}\tilde{u}(0,0) = 0. \tag{6.21}$$

The second observation is concerned with regularity. Since $f \in L^2(\Omega)$ implies $\tilde{f} := (f \circ h^{-1}).J_{h^{-1}} \in L^2(h(\Omega))$ one finds $\tilde{v} := \mathcal{G}_{h(\Omega)}(\tilde{f}.J_{h^{-1}}) \in H^{2,2}(h(\Omega)) \subset C^\gamma(\overline{h(\Omega)})$ for all $\gamma \in (0,1)$. A direct computation shows that $J_{h^{-1}} \in C^{\min(1,2(\frac{\alpha}{\pi}-1))}(\overline{h(\Omega)})$ and hence for $\alpha > \pi$ that

$$\left(\mathcal{G}_\Omega^2 f\right) \circ h^{-1} = \mathcal{G}_{h(\Omega)}(\tilde{v}.J_{h^{-1}}) \in C^{2,\gamma}\left(\overline{h(\Omega)}\right) \quad \text{for some } \gamma > 0. \tag{6.22}$$

Next we address the second term in (6.6), namely $\mathcal{G}_\Omega \mathcal{P}_\zeta \mathcal{G}_\Omega f = c_f(\mathcal{G}_\Omega \zeta)$. The Dirichlet Laplace with ζ for a right-hand side on the infinite cone K_ω^∞ has $\frac{\alpha}{4(\alpha-\pi)} x_1 |x|^{2(\frac{\alpha}{\pi}-1)}$ as a solution. Taking into account the boundary contribution by $\partial\Omega\backslash\partial K_\omega^\infty$ for the solution on Ω one finds that for some $w \in C^2(\overline{h(\Omega)})$

$$\mathcal{G}_\Omega \zeta \circ h^{-1} = w - \frac{\alpha}{4(\alpha-\pi)} x_1 |x|^{2(\frac{\alpha}{\pi}-1)}. \tag{6.23}$$

So we have

$$\begin{aligned} \tilde{u} &= \left(\mathcal{G}_\Omega^2 f\right) \circ h^{-1} - c_f\left(w - \frac{\alpha}{4(\alpha-\pi)} x_1 |x|^{2(\frac{\alpha}{\pi}-1)}\right) \\ &= \left(\left(\mathcal{G}_\Omega^2 f\right) \circ h^{-1} - c_f w\right) + c_f \frac{\alpha}{4(\alpha-\pi)} x_1 |x|^{2(\frac{\alpha}{\pi}-1)} \end{aligned}$$

with

$$\frac{\partial}{\partial x_1}\left(\left(\mathcal{G}_\Omega^2 f\right) \circ h^{-1} - c_f w\right)(0,0) = 0. \tag{6.24}$$

Finally notice that for $2(\frac{\alpha}{\pi}-1) > 1$ we have $x_1|x|^{2(\frac{\alpha}{\pi}-1)} = o(|x|^2)$ and $\tilde{u} \in C^2(\overline{h(\Omega)})$. By (6.24) the sign near $(0,0)$ is determined by the second order derivatives, or even more specifically, since $\frac{\partial^2}{\partial x_2^2}\tilde{u}(0,0) = 0$ and hence by the differential equation also $\frac{\partial^2}{\partial x_1^2}\tilde{u}(0,0) = 0$, by $\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}\tilde{u}(0,0)$. So we may conclude that if $\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}\tilde{u}(0,0) \neq 0$ the solution u changes sign near $(0,0)$.

That such a sign change can occur follows from the next observation. If for example Ω is symmetric in $x_2 = 0$ so is w and hence $\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}\tilde{u}(0,0) = 0$. In that case the solution changes sign if and only if $\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}\mathcal{G}_{h(\Omega)}(\tilde{v}.J_{h^{-1}}) \neq 0$. For generic f this will indeed be the case. If one chooses f which has a support in the upper part Ω such a non-zero mixed derivative will follow from Serrin's Maximum Principle at a corner [31].

We conclude by remarking that we find a similar sign changing result as before since $2(\frac{\alpha}{\pi} - 1) > 1$ is equivalent to $\alpha > \frac{3}{2}\pi$.

6.4. *Open problems*

In the case $\alpha \in (\pi, \frac{3}{2}\pi)$ one knows that whenever $f \in L^2(\Omega)$ with $0 \neq f \geqslant 0$ the H^2-solution u will not display a sign change near 0. However, numerical evidence shows that there are still sign changing solutions with positive right-hand side at least for α near $\frac{3}{2}\pi$. We expect such a sign change for all values between π and $\frac{3}{2}\pi$. So let us fix that claim.

Conjecture 6.14. *For each planar domain Ω which has a concave corner with angle in $(\pi, \frac{3}{2}\pi]$ there is a non-negative right-hand side $f \in L^2(\Omega)$ such that the H^2-solution u of* (1.1) *changes sign.*

Suppose the domain with a concave corner in the origin is symmetric with respect to y and if we take $f \geqslant 0$ such that $f(x,y) = f(x,-y)$ then the coefficient for $U^{2,0}$ equals 0. Hence at least locally near the origin positivity is preserved for all angles in $(\pi, 2\pi)$. Are such solutions positive on the whole domain? We expect so but not been able to prove such a result. Let us state a precise claim.

Conjecture 6.15. *Suppose that Ω is a planar domain satisfying Condition* 6.2 *which is symmetric with respect to $y = 0$. Then for every non-trivial non-negative right-hand side $f \in L_2(\Omega)$ that satisfies $f(x, y) = f(x, -y)$, the H^2-solution u is positive.*

Remark 6.15.1. Note that the difference between the H^2-solution u and the $(H^1)^2$ solution w is, see (6.6), a multiple of the ζ from Definition 6.3. In the notation of (6.12) we have that $\zeta = U^{-1,0}$. A right-hand side with the above symmetry implies that the $U^{2,0}$-component both of u and w is 0. Near 0 both for w and for u the contribution of the leading term, respectively $U^{-1,0}$ and $U^{-1,1}$, is positive when f is positive. But only for w the maximum principle yields global positivity.

One might pose a related conjecture for non-symmetric domains.

Conjecture 6.16. *Suppose that Ω is a planar domain satisfying Condition* 6.2*. Then for every non-trivial non-negative right-hand side $f \in L_2(\Omega)$ that satisfies $\int_\Omega f(x)\zeta_{2,0}(x)\,dx = 0$, the H^2-solution u is positive.*

These conjectures are quite blunt. It could very well be that some number $\alpha_1 \in (\pi, \frac{3}{2}\pi)$ exists with positivity preserving for angles $\alpha \in (\pi, \alpha_1]$. Under the additional condition that $\int_\Omega f(x)\zeta_{2,0}(x)\,dx = 0$ the problem could be positivity preserving only for $\alpha \in (\pi, \alpha_2]$ where α_2 is strictly less than 2π. In order to show that indeed $\alpha < 2\pi$ one might try to use for f a combination of $U^{-1,1}$, $U^{3,0}$ and $U^{1,1}$ similar as in the last remark in [32].

7. Numerical evidence

Numerical approximations for higher order problems in the presence of non-convex boundary singularities are by no means easy. In this section we do not want to present rigorous numerical results. Our aim is twofold. First we want to illustrate that on a standard L-shaped domain both solution are crucially different. See Figs. 3, 4 and 5. Secondly, we use sectorial domains to give numerical evidence for our claim that also for angles just below $\frac{3}{2}\pi$ sign-change occurs. See Fig. 6.

7.1. Finite differences

Using finite differences for $(\mathcal{G}_\Omega)^2$ and for the H^2-solution operator on an L-shaped domain we obtained numerical results that confirmed the above features. See Fig. 3. The right-hand side f is defined by a vector with a single non-zero entry. The number of grid points (degrees of freedom) is ± 1800.

7.2. Finite elements

Using finite elements the available software often restricts one to second order equations. If one does so here to find a solution by the iteration through (1.2) one would find an approximation of the $(H^1)^2$-solution w. In order to find the physically more relevant H^2-solution one either needs a direct approach using at least second order elementary functions or one proceeds through the system and subtracts a numerical approximation of the $\mathcal{G}_\Omega \mathcal{P}_\zeta \mathcal{G}_\Omega f$-term in (6.6). The limitations of the available software forced us to proceed through this second approach. Illustrations of the obtained results can be found in Figs. 4 and 5.

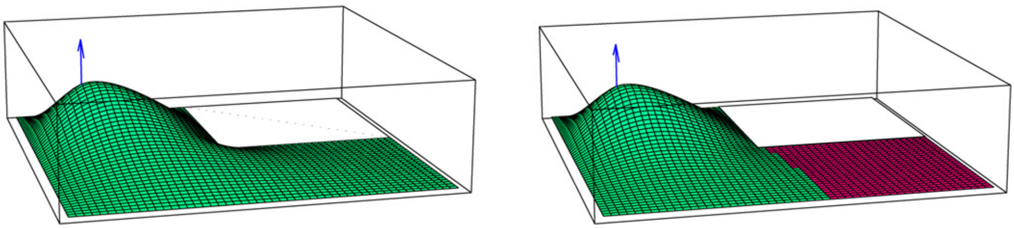


Fig. 3. The approximation for w (left) and u (right), the solution of (1.1), respectively in $(H^1)^2$- and H^2-sense, with a point source on the right-hand side. The arrow shows the location of the point source. Both solutions have been obtained by finite differences with uniformly distributed nodes. Only near the concave corner the discretization-matrices are different. The red part shows the area where the approximation of u is negative. These two graphs have been obtained using Mathematica [A]. (For interpretation of the references to colour in this figure legend, the reader is reffered to the web version of this article.)

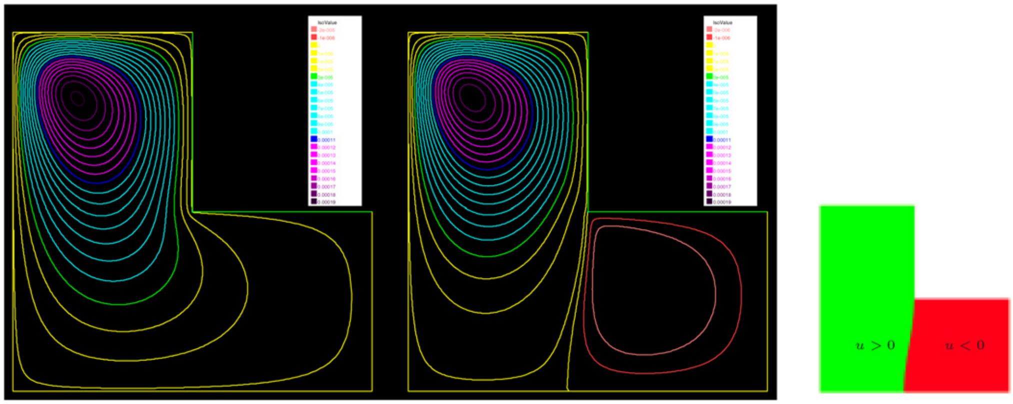


Fig. 4. On the left the level lines for respectively the approximation of the w-solution in $(H^1)^2$ and the u-solution in H^2 by finite elements for a point source. In small on the right the approximation level sets of the u-solution with the subdomains for the positive (green = light) and negative (red = dark) part. The approximations were obtained by FreeFem++ software [B] using adaptive mesh generation.

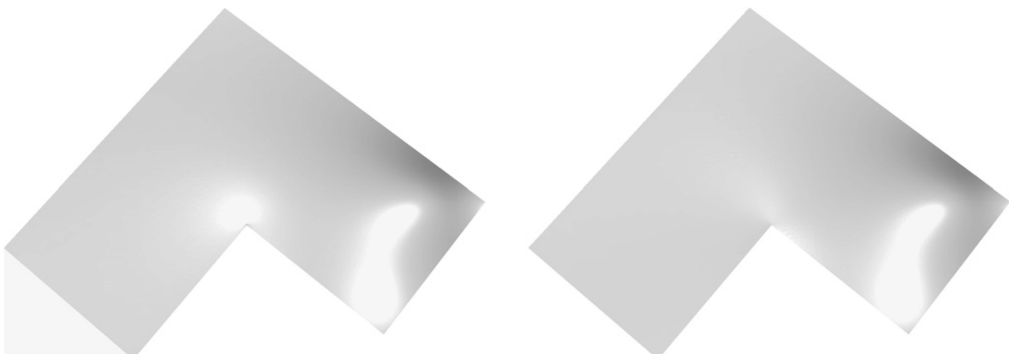


Fig. 5. On the left the $(H^1)^2$-solution w and on the right the H^2-solution u. The virtual reflection was produced by the Medit-software [C] of the 'Laboratoire Jacques Louis Lions.' It enhances the fact that the w is much more 'curved' near the concave corner: in fact there the energy integral is divergent.

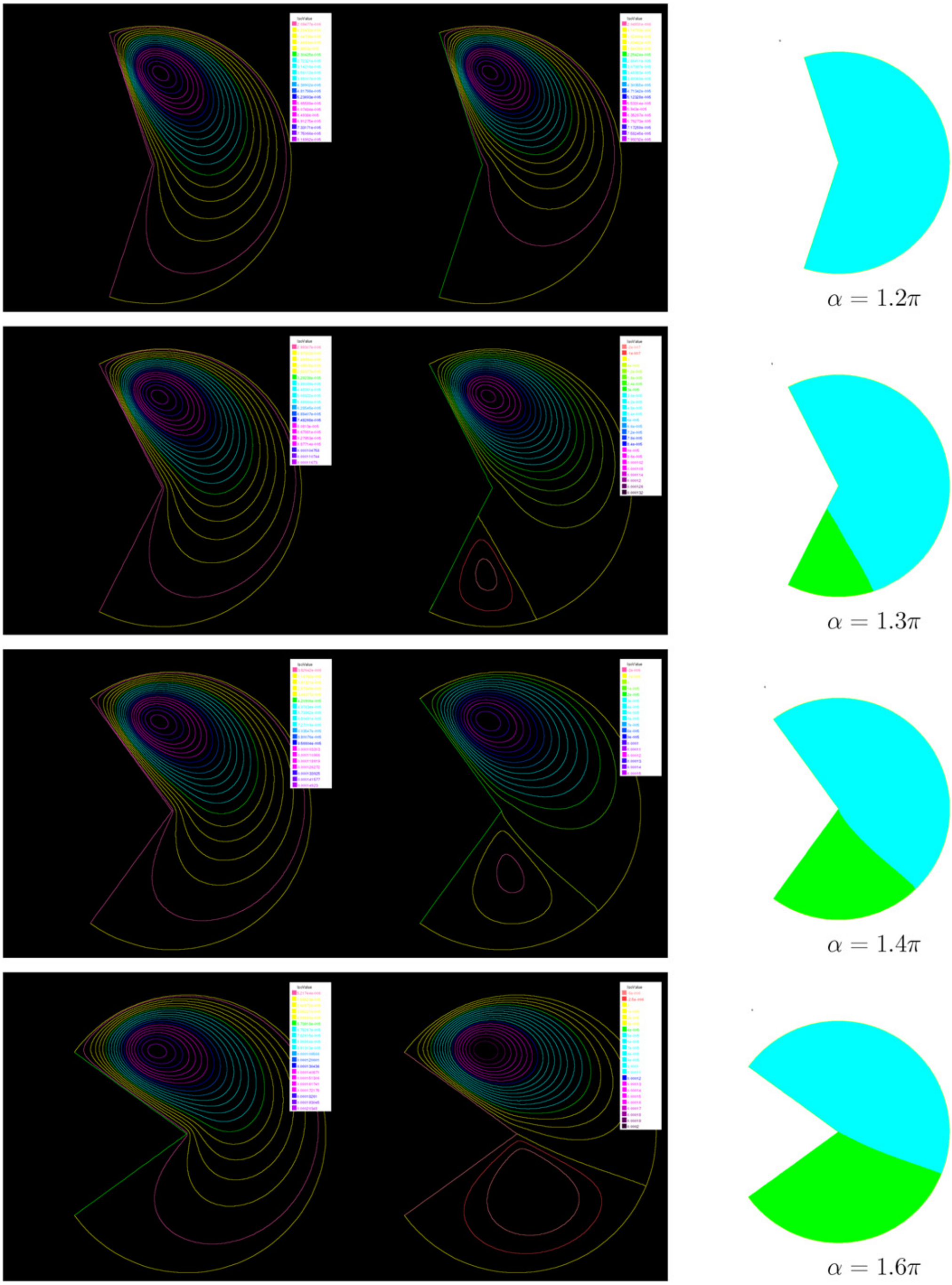


Fig. 6. On the left the level lines for the approximation by finite elements of the w-solution in $(H^1)^2$ respectively the u-solution in H^2 for a point source. In small on the right the approximation for the u-solution in H^2 with the subdomains for the positive (blue or light) and negative (green or dark) part. For each graph the number of triangles used lies around 13 000. Obtained by FreeFem++ software.

In our knowledge one of the early papers that addresses numerical treatment of a elliptic system with a reentrant corner, is [3] by Bernardi and Raugel.

7.2.1. Technical details

For both approximations we used the Free++ software [B] using an adaptive mesh generation with $P2$-elements. For the approximation of the singular ζ the domain was slightly altered by removing a small disk around the concave corner. Although ζ itself is not bounded it lies in L_2 and hence also the integrals $\int_\Omega \zeta v\, dx$ converges. The finite element problem is numerically solved as follows:

- $f := e^{-100((x-x_0)^2+(y-y_0)^2)}$ where the position of (x_0, y_0) has been picked by hand for optimal visual effect;
- $v := \mathcal{G} f$ is solved by a conjugate gradient method using piecewise linear elements;
- $w := \mathcal{G} v$ is solved by the same method;
- a mesh adaptation according to a numerical estimate of the error in w is applied and v, w are recomputed. As a result of the preset error bound the process terminates with the number of triangles around 15 000 and the number of interior nodes (= degrees of freedom) around 30 000;
- for the computation of u the stress intensity factor c_i is approximated and $u := w - c_i \mathcal{G}\zeta$. Also $\mathcal{G}\zeta$ is computed by the same conjugate gradient method.

Remark 7.0.1. Notice that the results for u by finite differences and by finite elements only roughly coincide. One should remark that the grid for the finite differences is rather course. Nevertheless, in both cases the nodal line seems to go straight down from the reentrant corner although the asymptotic formula tells us that it should bisect the angle.

Remark 7.0.2. We have shown that sign-changing occurs in the H^2-solution u whenever $\alpha \in (\frac{3}{2}\pi, \pi)$. From the numerical evidence shown in Fig. 6 one may guess that $\frac{3}{2}\pi$ is not optimal.

Software

[A] Mathematica 4, http://www.wolfram.com/.
[B] FreeFem++, http://www.ann.jussieu.fr/~hecht/freefem++.htm.
[C] Medit, http://www.ann.jussieu.fr/~frey/logiciels/medit.html.

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