Generalized Picard singular integrals

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\textbf{A B S T R A C T}

In this article, we introduce and study a new type of Picard singular integral operators on \(\mathbb{R}^n\) constructed by means of the concept of the nonisotropic \(\beta\)-distance and the \(q\)-exponential functions. The central role here is played by the concept of nonisotropic \(\beta\)-distance, which allows one to improve and generalize the results given for classical Picard and \(q\)-Picard singular integral operators. In order to obtain the rate of convergence we introduce a new type of modulus of continuity depending on the nonisotropic \(\beta\)-distance with respect to the uniform norm. Then we give the definition of \(\beta\)-Lebesgue points depending on nonisotropic \(\beta\)-distance and a pointwise approximation result shown at these points. Furthermore, we study the global smoothness preservation property of these new type Picard singular integral operators and prove a sharp inequality.

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\textbf{1. Introduction}

The \(q\)-analysis is extensively used in approximation theory, especially in the study of various sequences of linear positive operators such as Bernstein [1], Szász Mirakyan [2], Meyer, König and Zeller operators [3], Bleimann, Butzer and Hahn operators [4] and singular integral operators such as the Picard and Gauss–Weierstrass operators (see [5–7]). Recently in [6] we introduced a generalization of the well known Picard singular integral operators (see [8]) by using the \(q\)-analogue of the Euler Gamma integral, and termed the operators as \(q\)-the Picard singular integral operators. We have shown that these generalized operators have a more flexible rate of convergence than the classical Picard singular integral operators. Also these operators retain some approximation properties regarding, direct and pointwise approximation results in \(L_p(\mathbb{R})\) and weighted \(-L_p(\mathbb{R})\) spaces, global smoothness preservation properties and a Voronovskaya type theorem (see [4–7,9,10]).

In this paper, we introduce the multivariate variant of the \(q\)-Picard singular integral defined by (1.1) depending on the nonisotropic \(\beta\)-distance. Then we will show that from the rate of convergence point of view these operators with this construction are more flexible than both of the classical Picard and \(q\)-Picard singular integral operators. That is, depending on our selection of the parameter \(q\) and the parameter \(\beta\) (which is defined below) the rate of convergence can be refined. Also we define a new modulus of continuity which is harmonious with these operators. Finally for these operators a pointwise approximation result is shown and the global smoothness preservation property is presented.

Recall that, the generalization of the Picard singular integral in the multivariate case given [8, Chapter 12] and some approximation properties of them have been studied initially (see [8,11–14]). Also the generalization of the classical Picard and Gauss–Weierstrass operators depending on \(\beta\)-distance and some pointwise approximation results have been presented in [9,10].

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Now we give the concept of the nonisotropic $\beta$-distance. Let $n \in \mathbb{N}$ and $\beta_1, \beta_2, \ldots, \beta_n$ be positive numbers with $|\beta| = \beta_1 + \beta_2 + \cdots + \beta_n$ and
\[
\|x\|_\beta = \left( |x_1|^{\beta_1} + \cdots + |x_n|^{\beta_n} \right)^{\frac{1}{\beta}} , \quad x \in \mathbb{R}^n .
\]
The expression $\|x\|_\beta$ is called the nonisotropic $\beta$-distance between $x$ and $0$. Note that this distance has the following properties of homogeneity for positive $t$:
\[
\left( |t^{\beta_1}x_1|^{\beta_1} + \cdots + |t^{\beta_n}x_n|^{\beta_n} \right)^{\frac{1}{\beta}} = t |x|_\beta .
\]
Also, nonisotropic $\beta$-distance has following properties.
\begin{enumerate}
\item $\|x\|_\beta = 0 \iff x = 0$,
\item $\|t^\beta x\|_\beta = t \|x\|_\beta$,
\item $\|x + y\|_\beta \leq M_\beta \left( \|x\|_\beta + \|y\|_\beta \right)$,
\end{enumerate}
where $\beta_{\text{min}} = \min \{ \beta_1, \beta_2, \ldots, \beta_n \}$ and $M_\beta = 2 \left( 1 + \frac{1}{\beta_{\text{min}}} \right)^{\frac{1}{\beta}}$ (see [15]).

It can be seen that nonisotropic $\beta$-distance becomes the ordinary Euclidean distance $|x|$ for $\beta_i = \frac{1}{2}$, $i = 1, 2, \ldots, n$. Also, this distance does not satisfy the triangle inequality.

Now we recall that the $q$-generalizations of Picard singular integrals given in [6]. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. For $\lambda > 0$ and $0 < q < 1$, the $q$-generalizations of Picard singular integrals of $f$ are
\[
P_\lambda (f; \lambda x) \equiv P_\lambda (f; x) := \frac{(1 - q)}{2|\lambda|_q \ln q^{-1}} \int_{-\infty}^\infty f(x + t) E_q \left( \frac{(1 - q)|t|}{|\lambda|_q} \right) dt ,
\]
where the $q$-extension of exponential function $e^x$ is
\[
E_q (x) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q; q)_n} x^n = (-x; q)_\infty ,
\]
with $(a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k)$ and $(-x; q)_\infty = \prod_{k=0}^{\infty} (1 + xq^k)$.

For $q > 0$, $q$-number is
\[
[\lambda]_q = \begin{cases} 1 - q^\lambda, & q \neq 1 \\ \lambda, & q = 1 \end{cases}
\]
for all nonnegative $\lambda$. If $\lambda$ is an integer, i.e. $\lambda = n$ for some $n$, we write $[n]_q$ and call it $q$-integer. Also, we define a $q$-factorial as
\[
[n]_q ! = \begin{cases} 1, & n = 0, 1, 2, \ldots \\ [n]_q [n-1]_q \cdots [1]_q, & n = 1, 2, \ldots 
\end{cases}
\]
For integers $0 \leq k \leq n$, the $q$-binomial coefficients are given by
\[
\binom{n}{k}_q = \frac{[n]_q !}{[k]_q ! [n-k]_q !}.
\]
For details see [16].

Another needed formula is $q$-extension of Euler integral representation for the gamma function given in [17,18] for $0 < q < 1$
\[
c_q (x) \Gamma_q (x) = \frac{1 - q}{\ln q^{-1}} \int_0^{\infty} t^{x-1} E_q \left( \left( 1 - q \right) t \right) dt , \quad \text{Re} \ x > 0
\]
where $\Gamma_q (x)$ is the $q$-gamma function defined by
\[
\Gamma_q (x) = \frac{(q; q)_\infty (1 - q)^{1-x}}{(q^x; q)_\infty} , \quad 0 < q < 1
\]
and $c_q (x)$ satisfies the following conditions:
\begin{enumerate}
\item $c_q (x+1) = c_q (x)$
\item $c_q (n) = 1, n = 0, 1, 2, \ldots$
\item $\lim_{q \to 1^-} c_q (x) = 1$.
\end{enumerate}

When $x = n + 1$ with $n$ a nonnegative integer, we obtain
\[
c_q (n + 1) = [n]_q ! .
\]
2. Construction of a family of singular integral operators

In order to introduce the new singular integral operators, we start with the following elementary lemma.

**Lemma 1.** For all \( \lambda > 0, n \in \mathbb{N} \) and \( \beta_i \in (0, \infty) \) (\( i = 1, 2, \ldots, n \)) with \( |\beta| = \beta_1 + \beta_2 + \cdots + \beta_n \) we have

\[
c(n, \beta, q) \int_{\mathbb{R}^n} \mathcal{P}_\lambda (\beta, t) \, dt = 1,
\]

where

\[
\mathcal{P}_\lambda (\beta, t) = 1/E_\beta \left( \frac{(1-q) \|t\|_\beta}{|\lambda|^\beta} \right),
\]

and

\[
c(n, \beta, q)^{-1} = \frac{n}{2|\beta|} \omega_{\beta, n-1} \Gamma_q (n) \frac{\ln q^{-1}}{(1-q) q^{\frac{n-1}{2}}}.
\]

**Proof.** The \( t = [\lambda]^\beta_\nu \cdot x \) change of variable gives that

\[
c(n, \beta, q) \int_{\mathbb{R}^n} \mathcal{P}_\lambda (\beta, t) \, dt = c(n, \beta, q) \int_{\mathbb{R}^n} \frac{dx}{E_\beta \left( (1-q) \|x\|_\beta \right)}.
\]

We use generalized \( \beta \)-spherical coordinates [15] and consider the transformation

\[
x_1 = (u \cos \theta_1)^{2\beta_1} \\
x_2 = (u \sin \theta_1 \cos \theta_2)^{2\beta_2} \\
\vdots \\
x_{n-1} = (u \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-2} \cos \theta_{n-1})^{2\beta_{n-1}} \\
x_n = (u \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-1})^{2\beta_n},
\]

where \( 0 \leq \theta_1, \theta_2, \ldots, \theta_{n-2} \leq \pi, 0 \leq \theta_{n-1} \leq 2\pi, u \geq 0 \). Denoting the Jacobian of this transformation by \( J_\beta (u, \theta_1, \ldots, \theta_{n-1}) \) we obtain

\[
J_\beta (u, \theta_1, \ldots, \theta_{n-1}) = u^{2|\beta|-1} \Omega_\beta (\theta),
\]

where \( \Omega_\beta (\theta) = 2^n \beta_1 \cdots \beta_n \prod_{j=1}^{n-1} (\cos \theta_j)^{2\beta_j-1} (\sin \theta_j)^{\sum_{k=j+1}^{n} 2\beta_k-1} \). We can easily see that the integral

\[
\omega_{\beta, n-1} = \int_{S^{n-1}} \Omega_\beta (\theta) \, d\theta
\]

is finite, where \( S^{n-1} \) is the unit sphere in \( \mathbb{R}^n \).

Thus we have

\[
c(n, \beta, q) \int_{\mathbb{R}^n} \frac{dx}{E_\beta \left( (1-q) \|x\|_\beta \right)} = c(n, \beta, q) \int_{0}^{\infty} \int_{S^{n-1}} \frac{u^{2|\beta|-1} \Omega_\beta (\theta) \, d\theta \, du}{E_\beta \left( (1-q) u^{\frac{2|\beta|}{n}} \right)}.
\]

Using (2.3), we have

\[
\int_{\mathbb{R}^n} \frac{dx}{E_\beta \left( (1-q) \|x\|_\beta \right)} = c(n, \beta, q) \frac{n}{2|\beta|} \omega_{\beta, n-1} \int_{0}^{\infty} \frac{u^{n-1} \, du}{E_\beta \left( (1-q) u^{\frac{2|\beta|}{n}} \right)}.
\]

If we use (1.3) and choose

\[
c(n, \beta, q)^{-1} = \frac{n}{2|\beta|} \omega_{\beta, n-1} \Gamma_q (n) \frac{\ln q^{-1}}{(1-q) q^{\frac{n-1}{2}}},
\]

then we have the desired result. \( \square \)
Definition 1. Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a function. For \( 0 < q < 1, \lambda > 0, n \in \mathbb{N} \) and \( \beta_i \in (0, \infty) \) \((i = 1, 2, \ldots, n)\) with \(|\beta| = \beta_1 + \beta_2 + \cdots + \beta_n\), the \( q \)-Picard integral depending on \( \beta \)-distance of \( f \) is
\[
P_{\lambda, \beta} (f; q, x) \equiv P_{\lambda, \beta} (f; x) := \frac{c(n, \beta, q)}{|x|_\lambda^\beta} \int_{\mathbb{R}^n} f (x + t) P_\lambda (\beta, t) \, dt, \tag{2.4}
\]
where \( P_\lambda (\beta, t) \) and \( c(n, \beta, q) \) defined as in (2.1) and (2.2), respectively.

Note that, if we take \( \beta_i = \frac{1}{2}, i = 1, 2, \ldots, n \), it appears \( P_{\lambda, \frac{1}{2}} (f; q, x) \) operators introduced in [6]. If we take \( q \to 1 \), then \( P_{\lambda, \frac{1}{2}} (f; 1, x) \) operators are classical Picard singular integral [see [8]].

3. Approximation properties of the operator \( P_{\lambda, \beta} (f; \cdot) \)

In this section, we first introduce a nonisotropic modulus of continuity reflecting the nonisotropic \( \beta \)-distance and the operator \( P_{\lambda, \beta} (f; \cdot) \). Then we estimate the rate of convergence. Secondly, we introduce \( \beta \)-Lebesgue points of \( f \) and give a pointwise approximation theorem on these points.

Definition 2. Let \( f \in C (\mathbb{R}^n), n \in \mathbb{N} \) and \( \beta_i \in (0, \infty) \) \((i = 1, 2, \ldots, n)\) with \(|\beta| = \beta_1 + \beta_2 + \cdots + \beta_n\). For every \( \delta > 0 \), nonisotropic moduli of continuity of \( f \) is
\[
\omega_{\beta} (f; \delta) = \sup_{|h|_\beta \leq \delta} |f (x + h) - f (x)|.
\]

Lemma 2. Let \( f \in C (\mathbb{R}^n) \) and \( \beta_i \in (0, 1] \) \((i = 1, 2, \ldots, n)\) with \(|\beta| = \beta_1 + \cdots + \beta_n\). For \( \delta > 0 \) and \( C > 0 \), then
\[
\omega_{\beta} (f; C |\beta| \delta) \leq (1 + C) \omega_{\beta} (f; \delta).
\]

Proof. For positive integer \( k \), we can write
\[
\omega_{\beta} (f; k |\beta| \delta) = \sup_{|h|_\beta \leq \delta} |f (x + k^{\beta} h) - f (x)|
\]
\[
= \sup_{x \in \mathbb{R}^n, |h|_\beta \leq \delta} \left| \sum_{i=1}^{k} f (x + s^\beta h) - f (x + (s - 1)^\beta h) \right|
\]
\[
\leq \sup_{x \in \mathbb{R}^n, |h|_\beta \leq \delta} \left| \sum_{i=1}^{k} f (x + s^\beta h) - f (x + (s - 1)^\beta h) \right|
\]
\[
\leq k \omega_{\beta} (f; \delta),
\]
where \( \|s^\beta h - (s - 1)^\beta h\|_\beta \leq \|h\|_\beta \), by \( s^\beta - (s - 1)^\beta \leq 1 \) for \( i = 1, 2, \ldots, n \). Since \( \omega_{\beta} (f; \delta) \) is a nondecreasing function of \( \delta \), we have
\[
\omega_{\beta} (f; C |\beta| \delta) \leq (1 + C) \omega_{\beta} (f; \delta). \quad \square
\]

Theorem 1. Let \( 0 < q < 1, \lambda > 0, n \in \mathbb{N} \) and \( \beta_i \in (0, 1] \) \((i = 1, 2, \ldots, n)\) with \(|\beta| = \beta_1 + \beta_2 + \cdots + \beta_n\). If \( f \in C (\mathbb{R}^n) \), \( \omega_{\beta} (f; \delta) < \infty \) for \( \delta > 0 \), then we have for every \( x \in \mathbb{R}^n \)
\[
|P_{\lambda, \beta} (f; q, x) - f (x)| \leq K (q, \beta) \omega_{\beta} (f; \lambda |\beta| q^\frac{n \beta - 1}{2}),
\]
where
\[
K (q, \beta) = 1 + \frac{q^{n(\beta - 1)} \Gamma_q \left(n + \frac{n}{|\beta|}\right) \zeta_q \left(n + \frac{n}{|\beta|}\right)}{\Gamma_q \left(n \frac{(n + \frac{n}{|\beta|})(n + \frac{n}{|\beta|} - 1)}{2}\right)}.
\]
Proof. From Lemma 1 and definition of nonisotropic modulus of continuity, we can write

\[ P_{\lambda, \beta} (f; q, x) - f (x) = \frac{c(n, \beta, q)}{\|x\|_q^{|\beta|}} \int_{\mathbb{R}^n} (f (x + t) - f (x)) \mathcal{P}_{\lambda, \beta} (\beta, t) \, dt \]

\[ \leq \frac{c(n, \beta, q)}{\|x\|_q^{|\beta|}} \int_{\mathbb{R}^n} \omega_\beta (f; \|t\|_\beta) \mathcal{P}_{\lambda, \beta} (\beta, t) \, dt. \]

Since

\[ \omega_\beta (f; \|t\|_\beta) = \omega_\beta \left( f; \left( \left\| \frac{t}{\|t\|_\beta} \right\|_\beta \right) \frac{|\beta|}{\|x\|_q^{|\beta|}} \right), \]

using Lemma 2 with \( C = \frac{c(n, \beta, q)}{\|x\|_q^{|\beta|}} \) for \( \mathbf{t} \in \mathbb{R}^n \), we have

\[ |P_{\lambda, \beta} (f; q, x) - f (x)| \leq \omega_\beta \left( f; \left( \frac{\|t\|_\beta}{\|x\|_q^{|\beta|}} \right) \right) \left( 1 + \frac{c(n, \beta, q)}{\|x\|_q^{|\beta|}} \int_{\mathbb{R}^n} \|t\|_\beta \mathcal{P}_{\lambda, \beta} (\beta, t) \, dt \right). \]

We apply change of variable with

\[ \mathbf{t} = \|x\|_q^{|\beta|} \mathbf{y} \]
\[ d\mathbf{t} = \|x\|_q^{|\beta|} d\mathbf{y}, \]

where \( \mathbf{y} \in \mathbb{R}^n \) such that \( \|x\|_q^{|\beta|} \mathbf{y} = (\|x\|_q^{|\beta|} y_1, \ldots, \|x\|_q^{|\beta|} y_n) \) and then by using the generalized \( \beta \)-spherical coordinates as in Lemma 1, for \( x \in \mathbb{R}^n \) given we have

\[ |P_{\lambda, \beta} (f; q, x) - f (x)| \leq \omega_\beta \left( f; \left( \frac{\|\mathbf{y}\|_\beta}{\|x\|_q^{|\beta|}} \right) \right) \left( 1 + c(n, \beta, q) \int_{\mathbb{R}^n} \|\mathbf{y}\|_\beta \mathcal{P}_{\lambda, \beta} (\beta, t) \, dt \right). \]

By (2.3) we have

\[ |P_{\lambda, \beta} (f; q, x) - f (x)| \leq \omega_\beta \left( f; \left( \frac{\|\mathbf{y}\|_\beta}{\|x\|_q^{|\beta|}} \right) \right) \left( 1 + c(n, \beta, q) \frac{\|\mathbf{y}\|_\beta}{\|x\|_q^{|\beta|}} \omega_{\beta, n-1} \int_{0}^{\infty} \frac{u^{n+\frac{n-1}{|\beta|}}}{E_q (\|x\|_q^{|\beta|} u)} \, du \right). \]

Also, using (1.3), we have

\[ \int_{0}^{\infty} \frac{u^{n+\frac{n-1}{|\beta|}}}{E_q ((1-q) u)} \, du = \frac{\Gamma_q \left( \frac{n}{|\beta|} + \frac{n}{|\beta|} \right) c_q \left( \frac{n}{|\beta|} + \frac{n}{|\beta|} \right) \ln q^{-1}}{(1-q) q^{\left( \frac{n}{|\beta|} + \frac{n}{|\beta|} \right) \ln q^{-1}}}. \]

Substituting this equality into (3.1) and using (2.2), we have the desired result. \( \square \)

Remark 1. Let \( X := C_U (\mathbb{R}^n), n \geq 1 \), be the space of uniformly continuous functions from \( \mathbb{R}^n \) into \( \mathbb{R} \). For \( f \in X \), we consider the first order modulus of continuity of \( f \) by

\[ \omega (f; \delta) := \sup_{x \in \mathbb{R}^n} |f (x) - f (y)|, \quad \delta > 0. \]

Here \( \|\cdot\| \) is an arbitrary norm in \( \mathbb{R}^n \). We know that \( \omega (f; \delta) \) is finite for all \( \delta > 0 \) (see [8, p. 297, 298]) and trivially we see that

\[ \lim_{\delta \to 0} \omega (f; \delta) = 0, \quad \text{iff} \quad f \in X. \]

Also the above properties are true for the Euclidean norm and its equivalent, the maximum norm.
If \( f \in X \), where \( \mathbb{R}^n \) is equipped with maximum norm, we observe the following:

Let \( \delta > 0 \) small enough, \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \), and \( \| x \|_{\max} \) the maximum norm. Let \( A = \{ x, y \in \mathbb{R}^n : \| x - y \|_{\beta} \leq \delta \} \). For \( x, y \in \mathbb{R}^n \) and \( i = 1, \ldots, n \), we have

\[
|x_i - y_i| \leq \| x - y \|_{\beta} \frac{\beta_i}{\lambda},
\]

and for \( x, y \in A \) we get

\[
|x_i - y_i| \leq \delta \frac{\beta_i}{\lambda} \leq \delta \frac{\beta^*}{\lambda^*}, \quad i = 1, \ldots, n,
\]

where \( \beta^* = \min \{\beta_1, \ldots, \beta_n\} \). Thus we have

\[
\| x - y \|_{\max} \leq \delta \frac{\beta^*}{\lambda^*}.
\]

That is \( A \subset B \), where \( B := \{ x, y \in \mathbb{R}^n : \| x - y \|_{\max} \leq \delta \frac{\beta^*}{\lambda^*} \} \). Thus

\[
\omega_{\beta}(f; \delta) = \sup_{x, y \in \mathbb{R}^n} |f(x) - f(y)|_{\max} \leq \delta \frac{\beta^*}{\lambda^*}.
\]

Using above inequality and (3.2), for \( f \in X \),

\[
\lim_{\delta \to 0} \omega_{\beta}(f; \delta) = 0. \tag{3.3}
\]

Using Theorem 1 and (3.3), we can give following result:

**Corollary 1.** Let \( P_{\lambda,\beta}(f, \cdot) \) be a positive linear operators, defined by (2.4). If \( f \in X \), \( n \in \mathbb{N} \), \( \beta_i \in (0, 1) \ (i = 1, 2, \ldots, n) \) with \( |\beta| = \beta_1 + \beta_2 + \cdots + \beta_n \), \( \lambda > 0 \), and \( 0 < q < 1 \), then

\[
\lim_{\lambda \to 0} \| P_{\lambda,\beta}(f; q, x) - f(x) \|_{\infty} = 0.
\]

Now we introduce an analogy of the classical Lipschitz space \( \text{Lip}_M(\alpha) \).

**Definition 3.** For a given \( M > 0 \), \( n \in \mathbb{N} \), \( \beta_i \in (0, \infty) \ (i = 1, 2, \ldots, n) \) with \( |\beta| = \beta_1 + \beta_2 + \cdots + \beta_n \) and \( 0 \leq \alpha < 1 \), we denote by \( \text{Lip}_{M,\beta}(\alpha) \) the subset of all functions \( f \in C(\mathbb{R}^n) \) such that

\[
|f(t) - f(x)| \leq M \| t - x \|_{\beta}^\alpha, \quad \text{for every } x, t \in \mathbb{R}^n.
\]

**Remark 2.** Call \( |t_\alpha - x_\alpha| = \max \{|t_1 - x_1|, \ldots, |t_n - x_n|\} \). We have

\[
\| t - x \|_{\beta}^\alpha \leq \| t_\alpha - x_\alpha \|_{\beta}^\alpha \leq n \| t_\alpha - x_\alpha \|_{\beta}^\alpha \leq n \| t - x \|_{\beta}^\alpha.
\]

where \( \frac{1}{\beta_\alpha} = \min \left\{ \frac{1}{\beta_1}, \ldots, \frac{1}{\beta_n} \right\} \) same as \( \beta_\alpha = \max \{\beta_1, \ldots, \beta_n\} \) if \( |t_\alpha - x_\alpha| \leq 1 \) and \( \frac{1}{\beta_\alpha} = \max \left\{ \frac{1}{\beta_1}, \ldots, \frac{1}{\beta_n} \right\} \) same as \( \beta_\alpha = \min \{\beta_1, \ldots, \beta_n\} \) if \( |t_\alpha - x_\alpha| > 1 \). Therefore, we have

\[
\| t - x \|_{\beta}^\alpha \leq n^{\beta_\alpha} |t_\alpha - x_\alpha| \leq n^{\beta_\alpha} |t - x|
\]

and

\[
\| t - x \|_{\beta}^\alpha \leq n^{\beta_\alpha} |t - x|^{\beta_\alpha}. \tag{3.4}
\]

If \( f \in \text{Lip}_{M,\beta}(\alpha) \) then we have

\[
|f(t) - f(x)| \leq M n^{\beta_\alpha} |t - x|^{\beta_\alpha}. 
\]
For small $\delta > 0$ the last implies

$$\omega^\text{Euclidean}_\beta(f; \delta) \leq M n^{\frac{\beta}{|\alpha|}} \delta^{\frac{\beta}{|\alpha|}} \alpha,$$

where $\beta_\alpha = \max \{\beta_1, \ldots, \beta_n\}$, that is $f$ is uniformly continuous.

Using Definitions 2 and 3, we have

$$\omega_\beta(f; \delta) \leq M \delta^\alpha$$

for any function $f \in \text{Lip}_{M, \beta}(\alpha)$.

Using Theorem 1 and (3.4), we can give following result:

**Corollary 2.** Let $P_{\lambda, \beta}(f, \cdot)$ be a positive linear operators, defined by (2.4). If $f \in \text{Lip}_{M, \beta}(\alpha)$ for some $0 \leq \alpha < 1$, $n \in \mathbb{N}$, $\beta_i \in (0, 1)$ $(i = 1, 2, \ldots, n)$ with $|\beta| = \beta_1 + \beta_2 + \cdots + \beta_n$, $\lambda > 0$ and $0 < q < 1$, then we have for every $x \in \mathbb{R}^n$

$$|P_{\lambda, \beta}(f; q, x) - f(x)| \leq MK (q, \beta) [\lambda]_n^{q \beta} \alpha,$$

where $M$ is a positive constant independent of $\lambda$ and $K (q, \beta)$ is defined as in Theorem 1.

**Remark 3.** As a consequence of Corollary 2 we can say that the convergence rate of the operators (2.4) to $f$ is $O \left( \frac{[\lambda]_n^{q \beta}}{n^{\alpha}} \right)$.

which can be made better depending on not only the chosen $q$ but also the choice of $\beta$. Also, for suitable $q$ and $\beta$ this rate coincides with the rates of convergence of the $q$-Picard and classical Picard singular integral operators, respectively, to the identity.

Now we give a result which is a pointwise version of the theorem of approximation to the identity (see [19]). For this purpose we first give the following definition.

**Definition 4.** Let $f \in L_p(\mathbb{R}^n), p > 1$ and $\beta_i \in (0, \infty)$ $(i = 1, 2, \ldots, n)$ with $|\beta| = \beta_1 + \beta_2 + \cdots + \beta_n$. We say that $x$ is a $\beta$-Lebesgue point of $f$, if the condition

$$\lim_{h \to 0} \left( \frac{1}{n^{2|\beta|}} \int_{|y|_\beta < h} |f(x + y) - f(x)|^p \, dy \right)^{\frac{1}{p}} = 0$$

holds.

**Theorem 2.** Let $n \in \mathbb{N}$ and $\beta_i \in (0, \infty)$ $(i = 1, 2, \ldots, n)$ with $|\beta| = \beta_1 + \beta_2 + \cdots + \beta_n$, $\lambda > 0$ and $0 < q < 1$. If $f \in L_p(\mathbb{R}^n), 1 \leq p < \infty$, then

$$\lim_{\lambda \to 0} P_{\lambda, \beta}(f; q, x) = f(x)$$

whenever $x$ is a Lebesgue point of $f$.

**Proof.** Let $x$ be a Lebesgue point of $f$. This means that for any $\varepsilon > 0$ one can find $\eta > 0$ such that $\eta > h$ implies that

$$\left( \frac{1}{n^{2|\beta|}} \int_{|y|_\beta < h} |f(x + y) - f(x)|^p \, dy \right)^{\frac{1}{p}} < \varepsilon.$$

Changing to generalized $\beta$-polar coordinates we can reinterpret the former condition as: if $\eta > h$ then

$$G_\beta(h) = \int_0^h s^{2|\beta| - 1} g(s) \, ds < h^{2|\beta|} \varepsilon^p$$

where

$$g(s) = \int_{S^{n-1}} |f(x + (g\theta)^\beta) - f(x)|^p \Omega_\beta(\theta) \, d\theta.$$

On the other hand, for all $\eta > 0$ we have

$$|P_{\lambda, \beta}(f; q, x) - f(x)| \leq \frac{c(n, \beta, q)}{[\lambda]_n^{q \beta}} \int_{|y|_\beta < \eta} |f(x + y) - f(x)| \, d\beta + \frac{c(n, \beta, q)}{[\lambda]_n^{q \beta}} \int_{|y|_\beta \geq \eta} |f(x + y) - f(x)| \, d\beta \leq I_1 + I_2.$$
To estimate $I_1$ first we use Hölder’s inequality and later the generalized $\beta$-spherical coordinates, so we have

$$I_1 \leq \left( \frac{c(n, \beta, q)}{[\lambda_q^{|\beta|}]} \int_{|y|_{\beta}}^{\infty} \frac{f(x + y) - f(x)^p \mathcal{P}_\lambda(\beta, y)}{[\lambda_q^{|\beta|}]} \, dy \right)^{\frac{1}{p}}$$

$$= \left( \int_0^\eta \left\{ \int_{s=1}^\eta \left| f(x + (s\theta)^\beta) - f(x)^\beta \Omega_\beta(\theta) \, d\theta \right| s^{2|\beta|-1} \mathcal{P}_\lambda^0(\beta, s) \, ds \right\} \right)^{\frac{1}{p}}$$

where

$$\mathcal{P}_\lambda^0(\beta, s) = \frac{c(n, \beta, q)}{[\lambda_q^{|\beta|}]} E_q \left( \frac{(1 - q^2) |\beta|}{|\lambda_q^{|\beta|}} \right).$$

Using integration by parts twice and the above observations we have

$$I_1 \leq \left( G_\beta(s) \mathcal{P}_\lambda^0(\beta, s) \right)_{\mathcal{P}_\lambda^0(\beta, s)}^\eta - \int_0^\eta G_\beta(s) \, d \left( \mathcal{P}_\lambda^0(\beta, s) \right) \right)^{\frac{1}{p}}$$

$$\leq \left( s^{2|\beta|} \mathcal{P}_\lambda^0(\beta, s) \right)_{\mathcal{P}_\lambda^0(\beta, s)}^\eta - \int_0^\eta s^{2|\beta|} \, d \left( \mathcal{P}_\lambda^0(\beta, s) \right) \right)^{\frac{1}{p}}$$

$$\leq \left( \eta^{2|\beta|} \mathcal{P}_\lambda^0(\beta, \eta) - \int_0^\infty s^{2|\beta|} \, d \left( \mathcal{P}_\lambda^0(\beta, s) \right) \right)^{\frac{1}{p}}$$

Since

$$2 |\beta| \int_0^\infty s^{2|\beta| - 1} \mathcal{P}_\lambda^0(\beta, s) \, ds = \frac{2 |\beta| c(n, \beta, q)}{\omega_{\beta, n-1} [\lambda_q^{|\beta|}]} \int_{\mathbb{R}^n} \mathcal{P}_\lambda(\beta, y) \, dy$$

there exists a constant $A$ such that $I_1 \leq \varepsilon A$.

To estimate $I_2$, using Hölder’s inequality for $\frac{1}{p} + \frac{1}{p^'} = 1$ we have

$$I_2 \leq \frac{c(n, \beta, q)}{[\lambda_q^{|\beta|}]} \left\| X_\eta \mathcal{P}_\lambda(\beta, \cdot) \right\|_{p^'} + \frac{c(n, \beta, q)}{[\lambda_q^{|\beta|}]} \left\| f(x) \right\| \left\| X_\eta \mathcal{P}_\lambda(\beta, \cdot) \right\|_1,$n

where $X_\eta$ is the characteristic function of the set of $y$ such that $\left\| y \right\|_{\beta} \geq \eta$. We see that

$$\frac{c(n, \beta, q)}{[\lambda_q^{|\beta|}]} \left\| X_\eta \mathcal{P}_\lambda(\beta, \cdot) \right\|_1 = \frac{c(n, \beta, q)}{[\lambda_q^{|\beta|}]} \int_{\left\| y \right\|_{\beta} \geq \eta} \mathcal{P}_\lambda(\beta, y) \, dy$$

$$= \frac{c(n, \beta, q)}{[\lambda_q^{|\beta|}]} \int_{\left\| y \right\|_{\beta} \geq \eta} \frac{1}{E_q \left( 1 - q \left\| y \right\|_{\beta} \right)} \, dy.$$
But by (1.2) we have

\[
\frac{c(n, \beta, q)}{[\lambda]_q^{[|\beta|]}} \| \chi_q \mathcal{P}_\lambda (\beta, \cdot) \|_\infty = \frac{c(n, \beta, q)}{[\lambda]_q^{[|\beta|]}} \sup_{t, |t|_q^{[|\beta|]} \geq \eta} 1 / E_q \left( \frac{(1 - q) \| t \|_q}{[\lambda]_q^{[|\beta|]}} \right) \\
\leq \frac{c(n, \beta, q)}{[\lambda]_q^{[|\beta|]}} \prod_{k=0}^{n} \left( [\lambda]_q^{[|\beta|]} + (1 - q) q^k \eta \frac{2(|\beta|)}{n} \right) \\
\leq \frac{c(n, \beta, q)}{[\lambda]_q^{[|\beta|]}} \prod_{k=0}^{n} \left( [\lambda]_q^{[|\beta|]} + (1 - q) q^k \eta \frac{2(|\beta|)}{n} \right) \\
\leq c(n, \beta, q) [\lambda]_q^{[|\beta|]} \rightarrow 0 \quad \text{as} \quad \lambda \rightarrow 0.
\]

Thus the proof is completed. \( \square \)

4. Global smoothness preservation property

In this chapter, we show that the \(q\)-Picard integral operators depending on the \(\beta\)-distance given by (2.4) satisfy the global smoothness preservation property. The global smoothness inequalities involve a different modulus of continuity given in [8, 11].

**Theorem 3.** Let the function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) with \( \omega_\beta (f; \delta) < \infty \), for any \( \delta > 0 \) and \( \beta_i \in (0, \infty) \) \((i = 1, 2, \ldots, n)\) with \(|\beta| = \beta_1 + \beta_2 + \cdots + \beta_n\), such that \( P_{\lambda, \beta} (f; q; x) \in \mathbb{R} \) for \( 0 < q < 1 \). Then we have

\[
\omega_\beta (P_{\lambda, \beta} (f; q; \cdot); \delta) \leq \omega_\beta (f; \delta).
\]

**Proof.** Notice that

\[
P_{\lambda, \beta} (f; q; x) - P_{\lambda, \beta} (f; q; y) = \frac{c(n, \beta, q)}{[\lambda]_q^{[|\beta|]}} \int_{\mathbb{R}^n} \frac{1}{E_q} \left( \frac{(1 - q) \| t \|_q}{[\lambda]_q^{[|\beta|]}} \right) f (\mathbf{x} + t) - f (\mathbf{y} + t) \mathcal{P}_\lambda (\beta, t) \, dt.
\]

By Lemma 1, we have

\[
|P_{\lambda, \beta} (f; q; x) - P_{\lambda, \beta} (f; q; y)| = \frac{c(n, \beta, q)}{[\lambda]_q^{[|\beta|]}} \int_{\mathbb{R}^n} |f (\mathbf{x} + t) - f (\mathbf{y} + t)| \mathcal{P}_\lambda (\beta, t) \, dt \\
\leq \omega_\beta (f; \delta). \quad \square
\]

We finish with

**Theorem 4.** Inequality (4.1) is sharp, namely it is attained by the projection \( f_s (\mathbf{x}) = x_j \), where \( \mathbf{x} = (x_1, \ldots, x_j, \ldots, x_n) \in \mathbb{R}^n \) and \( j \in \{1, \ldots, n\} \) is fixed.

**Proof.** We observe that

\[
P_{\lambda, \beta} (f_s; q; x) - P_{\lambda, \beta} (f_s; q; y) = \frac{c(n, \beta, q)}{[\lambda]_q^{[|\beta|]}} \int_{\mathbb{R}^n} \left[ (x_j + t_j) - (y_j + t_j) \right] \mathcal{P}_\lambda (\beta, t) \, dt \\
= x_j - y_j \\
= f_s (\mathbf{x}) - f_s (\mathbf{y}).
\]

Hence for any \( \mathbf{x}, \mathbf{y} \in \mathbb{R}^n \) with \( \| \mathbf{x} - \mathbf{y} \|_p \leq \delta, \delta > 0 \) we have

\[
|P_{\lambda, \beta} (f_s; q; \mathbf{x}) - P_{\lambda, \beta} (f_s; q; \mathbf{y})| = |f_s (\mathbf{x}) - f_s (\mathbf{y})|
\]

and

\[
\omega_\beta (P_{\lambda, \beta} (f_s; q; \cdot); \delta) = \omega_\beta (f_s; \delta), \quad \text{for any} \quad \delta > 0.
\]
Further notice that
\[ |x_i - y_i| = \left( |x_i - y_i|^{\frac{1}{n}} \right)^{\frac{n}{\beta}} \leq \|x - y\|_\beta^{\frac{n}{\beta}} \]
and
\[ \omega_\beta (f_\ast; \delta) < \infty. \]

At the end we see that
\[ P_{\lambda, \beta} (f_\ast; q, x) = x_j + c(n, \beta, q) \left[ \lambda \right] \int_{n}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{t_j}{E_q \left( \frac{1 - q}{\|t\|_\beta} \right)} dt_1 \cdots dt_j \cdots dt_n \]

That is \( P_{\lambda, \beta} (f_\ast; q, x) = x_j \in \mathbb{R} \). So \( f_\ast \) fulfills all the assumptions of Theorem 3. □

References