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Robust synchronization of a class of uncertain complex networks via discontinuous control

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ABSTRACT

We propose robust controller designs to synchronize networks with uncertainties in their node dynamics and their connections. We consider two situations: in the first, we assume that the effect of uncertainties vanishes as synchronization is achieved. In the second, disturbances are assume nonvanishing but bounded. To achieve robust synchronization on these situations, we design local feedback controllers, which are smooth in the first case, and discontinuous in the latter. These designs allow us to establish synchronization criteria for this class of uncertain dynamical networks. We use numerical experiments to illustrate our results.

1. Introduction

In recent years synchronization of dynamical networks has become a very active area of research [1–3]. In particular, studies on the synchronization of networks with small-world [4] and scale-free [5] topologies have significantly advance our understanding of the synchronization phenomenon in real-world complex networks, highlighting their potential applications to the Internet, electric power distribution, social and economical groups, and even biological systems [6–9]. In contrast to their wide potential applicability, the bulk of research on network synchronization has concentrated on networks with identical nodes, linearly and diffusively coupled, where full knowledge of the dynamical description of its nodes and the structure of their interconnections is available. Under these conditions, approaches like the Master Stability Function (MSF) [10,11], and other methods based on linearized analysis of the network's transverse dynamics [6,12,13], can be used to determine the stability of the overall synchronized behavior of the network. Unfortunately, when considering more realistic situations, where complete knowledge is not available linearized approaches are not directly applicable.

Although dynamical networks may synchronize spontaneously, in most cases it is necessary to take actions to force the network into a synchronized state. This situation is referred to as *controlled synchronization*. In [14], an adaptive robust controller was proposed to achieve synchronization on uncertain networks that preserve their diffusive structure under perturbations. That is, networks where perturbations and control inputs vanish at the synchronized solution. For this type of uncertain networks in [15], synchronization was robustly achieved designing the coupling functions of the network. In [16], the problem of adaptive synchronization of uncertain networks was reconsidered describing local and global synchronization designs. Linear feedback controllers to achieve robust synchronization on uncertain networks with uniform and nonuniform inner coupling matrices were proposed in [17]. The effect of coupling delays on the synchronization of uncertain networks was considered in [18]. Following a linearized analysis under vanishing perturbations in [19,20] conditions for synchronization of a network with slightly different nodes were derived using the MSF approach. In the above results it is required that the uncertain network remain diffusive under the effects of perturbations and controls. When

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considering that the topology can be perturbed, the problem becomes more complex. In [21], adaptive synchronization was considered in the context of networks under the action of slow varying time dependent network topology. In another paper [22], a similar solution was found from an MSF approach.

In this paper we extend previous results by relaxing the requirements of identical nodes and vanishing coupling functions. In particular, we design controllers for two situations: vanishing and non-vanishing perturbations. In the first, we propose smooth synchronizing controllers. While for the latter, we propose discontinuous local feedback controllers to achieve robust synchronization. These controller designs allow us to derive several criteria for robust synchronization of uncertain networks, which relate emergence of synchronization to: the dynamics of an isolated nominal node, topological features of the nominal network, and bounds of the uncertainties affecting the network.

The remainder of the paper is as follows: In Section 2, we describe in detail the synchronization problem for uncertain networks. In Section 3, we provide controllers for two distinct situations; namely, vanishing and non-vanishing perturbations. While in Section 4, we use different numerical experiments to show the validity of our results. Finally, we conclude the paper with some closing comments and remarks.

2. Uncertain dynamical network model

The state space description of a network with uncertain couplings, where each node is a dynamical system with uncertain parameters and a local controller, is given by:

$$\dot{x}_i = \tilde{f}_i(x_i, \tilde{\rho}_i) + \tilde{g}_i(X) + u_i, \quad \text{for } i = 1, \ldots, N \tag{1}$$

where $x_i \in \mathbf{R}^n$ are the state variables of the ith node; $X = [x_1, \ldots, x_N] \in \mathbf{R}^{n\times N}$ is a row vector of the state variables of each node in the network; and $u_i \in \mathbf{R}^n$ is a local feedback controller to be designed.

The parameters of each dynamical node are assume to be $\tilde{\rho}_i = \rho + \Delta\hat{\rho}_i \in \mathbf{R}^p$, with $\Delta = \pm 1$, where ρ and $\hat{\rho}_i$ are the nominal and uncertain parts of the parameters of the ith node, respectively. The interactions of the ith node within the network are given by the uncertain coupling function $\tilde{g}_i(X) = g_i(X) + \Delta\hat{g}_i(X) : \mathbf{R}^{n\times N} \to \mathbf{R}^n$; where $g_i(X)$ describes the nominal coupling, and $\hat{g}_i(X)$ the uncertain part of the interactions between the ith node and the rest of the network. The uncertain function $\tilde{f}_i(x_i, \tilde{\rho}_i) = f(x_i, \rho) + \Delta\hat{f}_i(x_i, \hat{\rho}_i) : \mathbf{R}^{n\times p} \to \mathbf{R}^n$ describes the dynamics of the ith node in isolation ($\tilde{g}_i(X) = 0 \in \mathbf{R}^n$), *i.e.*, disconnected from the network; with the nonlinear Lipschitz function $f(x_i, \rho)$ describing the dynamics of the node with nominal parameters, and the unknown but bounded function $\hat{f}_i(x_i, \hat{\rho}_i)$ describing the effects of the parameter uncertainties on the ith node dynamics.

Remark 1. The uncertain dynamical network model in (1) is similar to the one used in [14,15,17,18]. However, our model differs in that it considers uncertainty in both, dynamical description and coupling structure. Further, we express the uncertainties as deviations from a nominal description which directly depend on unknown parameters and perturbations. In this way different situations can be considered, including the case of perturbations which render the coupling structure not vanishing.

We assume that the perturbations affecting the uncertain dynamical network (1) are small and independent of time, such that for appropriate choices of the feedback control inputs u_i the solutions of each node can be made to remain close to each other. Then, the trajectories of each individual perturbed node are well approximated by the average trajectory of the network, $\bar{s} = \frac{1}{N}\sum_{j=1}^{N} x_j$, which evolves according to:

$$\dot{\bar{s}} = \frac{1}{N}\sum_{j=1}^{N}\dot{x}_j = \frac{1}{N}\sum_{j=1}^{N}\left(\tilde{f}_j(x_j, \tilde{\rho}_j) + \tilde{g}_j(X)\right). \tag{2}$$

For the uncertain network (1) the dynamics of average trajectory becomes

$$\dot{\bar{s}} = \frac{1}{N}\sum_{j=1}^{N} f(x_j, \rho) + \frac{1}{N}\sum_{j=1}^{N} g_j(X) + \frac{1}{N}\sum_{j=1}^{N}\Delta\left(\hat{f}_j(x_j, \hat{\rho}_j) + \hat{g}_j(X)\right). \tag{3}$$

In particular, considering the nominal controlled dynamical network

$$\dot{x}_i = f(x_i, \rho) + g_i(X) + u_i, \quad \text{for } i = 1, \ldots, N \tag{4}$$

we have that synchronization is asymptotically achieve if the state solutions of every node in the network evolve at unison, in the sense that

$$\lim_{t\to\infty} \|x_i - s\| = 0, \quad \text{for } i = 1, \ldots, N \tag{5}$$

where $s \in \mathbf{R}^n$ is the *synchronized solution*.

The existence of a synchronized solution for (4) depends on the properties of the coupling function $g_i(X)$. In particular, in the case of diffusive coupling, that is, a nominal coupling function such that $g_i(X) = 0\ \forall i$ when $x_1 = x_2 = \cdots = x_N$, the nominal controlled dynamical network has a synchronized solution with a dynamical evolution given by

$$\dot{s} = f(s, \rho) \tag{6}$$

where $f(\cdot)$ describes the dynamics of an isolated nominal node.

From (4) and (6) the synchronization error ($\varepsilon_i = x_i - s$) for the nominal network evolves according to:

$$\dot{\varepsilon}_i = f(x_i, \rho) - f(s, \rho) + g_i(X) + u_i, \quad \text{for } i = 1, \ldots, N. \tag{7}$$

Then, the synchronization of the nominal dynamical network (4) becomes a control problem, where the objective is to design local controllers such that (7) becomes asymptotically stable about its zero equilibrium point.

Notice that (3) for a diffusive nominal coupling function and vanishing perturbations when $x_i = x_j\ \forall i, j$, becomes (6). Then, we can write:

$$\dot{\bar{s}} = f(s, \rho) + \frac{1}{N}\sum_{j=1}^{N} g_j(S) + \frac{1}{N}\sum_{j=1}^{N} \Delta\left(\hat{f}_j(x_j, \hat{\rho}_j) + \hat{g}_j(X)\right) \tag{8}$$

where $S = (x_1, \ldots, x_N) \in \mathbf{R}^{n\times N}$ when $x_i = x_j\ \forall i, j$.

Remark 2. Under potentially non vanishing perturbations the synchronized solutions (6) is no longer possible for network (1), however, it is reasonable to presume that under small perturbations and appropriate control action the nearly identical nodes evolve on a near-synchronous state [20] which is well approximated by their average trajectory. Furthermore, if the perturbations vanish at the synchronized solution, the average trajectory becomes the dynamics of an isolated node. Then, our average trajectory can be conceived as the synchronized solution of the nominal network plus the average effects of the perturbations.

From (1) and (2) the dynamical evolution of the synchronization error ($e_i = x_i - \bar{s}$) is found to be:

$$\dot{e}_i = \tilde{f}_i(x_i, \tilde{\rho}_i) + \tilde{g}_i(X) - \dot{\bar{s}} + u_i, \quad \text{for } i = 1, \ldots, N. \tag{9}$$

Then, the synchronization of the uncertain network (1) becomes a control problem, where the objective is to design local controllers u_i, such that the error dynamics (9) becomes robustly stable about the zero equilibrium point.

3. Robust synchronization design

We start our design assuming that the nominal coupling functions, $g_i(X)$, are diffusive linear combinations of the network state variables, such that:

$$g_i(X) = c\sum_{j=1}^{N} a_{ij}\Gamma x_j \tag{10}$$

where the inner coupling matrix, $\Gamma \in \mathbf{R}^{n\times n}$, is a 0–1 matrix describing the way the state variables are coupled between two connected nodes. The network topology is described by the connectivity matrix, $\mathcal{A} = \{a_{ij}\} \in \mathbf{R}^{N\times N}$, which is a matrix constructed as follows: if there is a connection between the ith and jth nodes ($j \neq i$), the entries $a_{ij} = a_{ji} = 1$; otherwise $a_{ij} = a_{ji} = 0$, with the diagonal entries given by $a_{ii} = -\sum_{j=1, j\neq i}^{N} a_{ij}$. The variable $c \in \mathbf{R}$ is the coupling strength between the nodes, which is taken to be uniform for the entire network.

If the network is connected such that there are no isolated clusters, the eigenvalues of $\mathcal{A}$ are real, nonpositive, and can be ordered as follows [2]:

$$0 = \lambda_1 > \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_N. \tag{11}$$

Further, the connectivity matrix can be rewritten as $\mathcal{A} = \Omega^{-1}\Lambda\Omega$, where $\Lambda = \mathrm{Diag}(\lambda_1, \ldots, \lambda_N)$ and $\Omega = [\omega_1, \ldots, \omega_N] \in \mathbf{R}^{N\times N}$, with $\omega_i = [\omega_{i1}, \ldots, \omega_{iN}]^\top \in \mathbf{R}^N$ $i = 1, \ldots, N$ the eigenvectors of $\mathcal{A}$.

Notice that with $\mathcal{A}$ constructed as described above, the nominal coupling functions vanish when synchronization is achieved, i.e., $g_i(X) = 0$, when $x_i = x_j, \forall i, j$. Then, $g_j(S) = 0, \forall j$.

It follows that the synchronization error dynamics in (9) become

$$\dot{e}_i = \bar{f}_i(x_i, \tilde{\rho}_i) + \bar{g}_i(X) + c\sum_{j=1}^{N} a_{ij}\Gamma e_j + u_i, \quad \text{for } i = 1, \ldots, N \tag{12}$$

where $\bar{f}_i(x_i, \tilde{\rho}_i) = f(x_i, \rho) - f(s, \rho) + \Delta\left(\hat{f}_i(x_i, \hat{\rho}_i) - \frac{1}{N}\sum_{j=1}^{N}\hat{f}_j(x_j, \hat{\rho}_j)\right) \in \mathbf{R}^n$ and $\bar{g}_i(X) = \Delta\left(\hat{g}_i(X) - \frac{1}{N}\sum_{j=1}^{N}\hat{g}_j(X)\right) \in \mathbf{R}^n$.

In what follows, the local controllers u_i in (12) are designed such that synchronization, in the sense of (4), is robustly achieved for two distinct situation: In the first, the perturbations due to uncertainties in the network are assume to be bounded in terms of the synchronization error, as such, they vanish as the network synchronizes. In the second, we consider these perturbations to be bounded but not vanishing.

3.1. Under vanishing perturbations

In this subsection we assume that the uncertain parts of the dynamical network (1) satisfy the following bounds:

$$\left\| \sum_{m=1}^{N} \bar{f}_m\left(x_m, \tilde{\rho}_m\right) \omega_{im} \right\| \leq \alpha_i \left\| \sum_{m=1}^{N} e_m \omega_{im} \right\|, \quad \text{and} \tag{13}$$

$$\left\| \sum_{m=1}^{N} \bar{g}_m\left(X\right) \omega_{im} \right\| \leq \sum_{p=1}^{N} \beta_{ip} \left\| \sum_{m=1}^{N} e_m \omega_{pm} \right\| \tag{14}$$

for $i = 1, \ldots, N$, where $\alpha_i \geq 0 \in \mathbf{R}$ and $\beta_{ip} \geq 0 \in \mathbf{R}$ are nonnegative constants.

In this case we have the following result:

Theorem 1. *Suppose that (13) and (14) hold. If the local controllers u_i are constructed as*

$$u_i = -c\; k\; \Gamma\; e_i, \quad \text{for } i = 1, \ldots, N \tag{15}$$

with the controller gain $k > 0$, satisfying the bound

$$k > \frac{N(\alpha + \beta)}{c} - \delta \tag{16}$$

where $\delta = \min\{|\lambda_i|\}_{\lambda_i \neq 0}$, $\alpha = \max\{\alpha_i\}$ and $\beta = N \max\{\beta_{ji}\}$. Then, the uncertain dynamical network (1) robustly synchronizes, in the sense that the zero fixed point of (12) is robustly stable.

Proof. Defining the vector variables $\mathbf{e} = [e_1, \ldots, e_N] \in \mathbf{R}^{n\times N}$, $\bar{\mathbf{f}} = \left[\bar{f}_1\left(\cdot\right), \ldots, \bar{f}_N\left(\cdot\right)\right] \in \mathbf{R}^{n\times N}$ and $\bar{\mathbf{g}} = \left[\bar{g}_1\left(\cdot\right), \ldots, \bar{g}_N\left(\cdot\right)\right] \in \mathbf{R}^{n\times N}$ the error dynamics (12) are rewritten as: $\dot{\mathbf{e}} = \bar{\mathbf{f}}\left(X, \rho\right) + \bar{\mathbf{g}}(X) + c\Gamma\mathbf{e}\left(\mathcal{A} - K\right)$; where $\rho = [\rho_1, \ldots, \rho_N]$ and $K = \mathrm{Diag}(k, \ldots, k) \in \mathbf{R}^{N\times N}$. In the transformed coordinates $\eta = \mathbf{e}\Omega$, the error dynamics become $\dot{\eta} = \bar{\mathbf{f}}\left(X, \rho\right)\Omega + \bar{\mathbf{g}}\left(X\right)\Omega + c\Gamma\eta\left(\Lambda - K\right)$; where $\eta = [\eta_1, \ldots, \eta_N] \in \mathbf{R}^{n\times N}$, with $\eta_i = \mathbf{e}\;\omega_i \in \mathbf{R}^n$; or equivalently,

$$\dot{\eta}_i = \bar{\mathbf{f}}\left(X, \rho\right)\omega_i + \bar{\mathbf{g}}\left(X\right)\omega_i + c\left(\lambda_i - k\right)\Gamma\eta_i, \quad \text{for } i = 1, \ldots, N. \tag{17}$$

The stability of the error dynamics (12) around its zero equilibrium point is determine using the Lyapunov candidate function: $V = \frac{1}{2}\sum_{j=1}^{N} \eta_j^\top \eta_j$, the time derivative of V along the trajectories of (17) is given by

$$\dot{V} = \sum_{j=1}^{N} \eta_j^\top c\left(\lambda_j - k\right)\Gamma\eta_j + \sum_{j=1}^{N} \eta_j^\top \left(\sum_{m=1}^{N} \bar{f}_m\left(x_m, \tilde{\rho}_m\right)\omega_{jm}\right) + \sum_{j=1}^{N} \eta_j^\top \left(\sum_{m=1}^{N} \bar{g}_m(X)\omega_{jm}\right). \tag{18}$$

The first term on the righthand side of (18) is bounded as:

$$\sum_{j=1}^{N} \eta_j^\top c\left(\lambda_j - k\right)\Gamma\eta_j \leq \sum_{j=1}^{N} \eta_j^\top(-\gamma)\eta_j \leq -\gamma\sum_{j=1}^{N} \|\eta_j\|^2$$

where $-\gamma I_n \geq c\left(\lambda_j - k\right)\Gamma$ for every j, with $\gamma > 0$. Since the eigenvalues of $\mathcal{A}$ are all nonpositive the bounds on γ are $-c(|\lambda_j| + k)\Gamma$ with $\lambda_j \neq 0$. Letting $\delta = \min\{|\lambda_j|\}_{\lambda_j \neq 0}$ the bound becomes $\gamma \leq c\;(\delta + k)$. From (14) we have the second term in the righthand side of (18) bounded as:

$$\sum_{j=1}^{N} \eta_j^\top \sum_{m=1}^{N} \bar{f}_m\left(x_m, \tilde{\rho}_m\right)\omega_{jm} \leq \sum_{j=1}^{N} \eta_j^\top \alpha_j \left\| \sum_{m=1}^{N} e_m\omega_{jm} \right\| \leq \alpha\sum_{j=1}^{N} \|\eta_j\|^2$$

with $\alpha = \max\{\alpha_i\}$. From (15), it follows that the third term in the righthand side of (18) is bounded as:

$$\sum_{j=1}^{N} \eta_j^\top \sum_{m=1}^{N} \bar{g}_m\left(X\right)\omega_{jm} \leq \sum_{j=1}^{N} \eta_j^\top \sum_{i=1}^{N} \beta_{ji} \left\| \sum_{m=1}^{N} e_m\omega_{im} \right\| \leq \beta\sum_{j=1}^{N}\sum_{i=1}^{N} \|\eta_j\|\;\|\eta_i\|$$

with $\beta = N\max\{\beta_{ji}\}$. Then, $\dot{V}$ can be rewritten as a quadratic function of $\|\eta\| = [\|\eta_1\|, \ldots, \|\eta_N\|]^\top \in \mathbf{R}^N$ as:

$$\dot{V} \leq -|\eta|^\top Q|\eta|$$

where Q is a $N \times N$ matrix whose elements are given by

$$q_{ij} = \begin{cases} -\beta & \text{for } i \neq j \\ -(\alpha + \beta) + \gamma & \text{for } i = j. \end{cases}$$

By choosing $\gamma > N(\alpha+\beta)$, the matrix Q is positive definite ($Q > 0$) which means that $\dot{V} < 0$. Then, the error dynamics in the transform coordinates (18) are globally uniformly asymptotically stable about the zero fixed point ($\eta = 0$), which implies that the uncertain dynamical network (1) under assumptions (14) and (15) with the controller (15), achieves robust synchronization. From the above conditions on γ, the relation in (16) is readily obtained from:

$$c(\delta + k) \geq \gamma > N(\alpha+\beta). \quad \square \tag{19}$$

In a similar way, the following result is obtained directly from the previous theorem.

Corollary 2. *For the uncertain dynamical network (1) with no controllers ($u_i = 0$, for all i), assuming that the conditions on (14) and (15) hold. If the coupling strength satisfies the following criterion*

$$c > \frac{N(\alpha+\beta)}{\delta} \tag{20}$$

where $\delta = \min\{|\lambda_i|\}_{\lambda_i \neq 0}$, $\alpha = \max\{\alpha_i\}$ and $\beta = N\max\{\beta_{ji}\}$. Then, the uncertain dynamical network robustly synchronizes to the solution $\bar{s}(t)$ described in (2), in the sense of (2).

Proof. From (20), when the controller is removed ($k = 0$), the criterion for robust synchronization in (19) is obtained following a similar procedure as in the proof of Theorem 1. $\square$

3.2. Under non-vanishing perturbations

In the case where the perturbations in the network do not vanish at the synchronized state, the bounds on the uncertain parts of the network cannot be expressed in terms of the synchronization error as in (14) and (15). Instead, we assume that the perturbations are bounded as follows:

$$\left\| \sum_{m=1}^{N} \bar{f}_m\left(x_m, \tilde{\rho}_m\right) \omega_{im} \right\| \leq a_i, \quad \text{and} \tag{21}$$

$$\left\| \sum_{m=1}^{N} \bar{g}_m\left(X\right) \omega_{im} \right\| \leq b_i \tag{22}$$

for $i = 1, \ldots, N$, where $a_i > 0 \in \mathbf{R}$ and $b_i > 0 \in \mathbf{R}$ are small positive constants.

To robustly achieve synchronization under these non-vanishing perturbations, we propose the use of discontinuous local controllers as described in the following result.

Theorem 3. *Suppose that (21) and (22) hold. If the local controllers are constructed as:*

$$u_i = -c\ k\ \Gamma\ e_i - \mu\ \mathrm{sgn}(e\Omega)\omega_i^*, \quad \text{for } i = 1, \ldots, N \tag{23}$$

where $e = [e_1, \ldots, e_N] \in \mathbf{R}^{n\times N}$, the matrix $\Omega \in \mathbf{R}^{N\times N}$ is such that the connectivity matrix can be rewritten as $\mathcal{A} = \Omega^{-1}\Lambda\Omega$, with $\Lambda = \mathrm{Diag}(\lambda_1, \ldots, \lambda_N)$; and $\omega_i^* \in \mathbf{R}^N$ is the ith column of the matrix Ω^{-1}. $\mathrm{sgn}(\cdot)$ represents the conventional sign function, $\mathrm{sgn}(\epsilon) = \{1, \text{for } \epsilon > 0; 0, \text{for } \epsilon = 0; -1, \text{for } \epsilon < 0\}$, with $\mathrm{sgn}(e\Omega) = \mathrm{sgn}(\eta) = [\mathrm{sgn}(\eta_1), \ldots, \mathrm{sgn}(\eta_N)] \in \mathbf{R}^{n\times N}$, with $\mathrm{sgn}(\eta_i) = [\mathrm{sgn}(\eta_{i1}), \ldots, \mathrm{sgn}(\eta_{in})]^\top \in \mathbf{R}^n$. Furthermore, if the smooth $k > 0$ and discontinuous $\mu > 0$ controller gains are designed such that

$$k > \frac{\gamma}{c} - \delta, \quad \text{and} \tag{24}$$

$$\mu > a + b \tag{25}$$

where $\gamma > 0$, $a = \max\{a_i\}$, and $b = \max\{b_i\}$. Then, the uncertain dynamical network (1), robustly synchronizes to the solution $\bar{s}$ described in (2).

Proof. In terms of the vector variables described above, the error dynamics in (12) under control (23) can be rewritten as: $\dot{\mathbf{e}} = \bar{\mathbf{f}}(X,\rho) + \bar{\mathbf{g}}(X) + c\Gamma\mathbf{e}(\mathcal{A} - K) - \mu\ \mathrm{sgn}(\eta)\Omega^{-1}$. In the transform coordinates $\eta = \mathbf{e}\Omega$, the error dynamics become $\dot{\eta} = (\bar{\mathbf{f}}(X,\rho) + \bar{\mathbf{g}}(X))\Omega - \mu\ \mathrm{sgn}(\eta) + c\Gamma\eta(\Lambda - K)$, or equivalently:

$$\dot{\eta}_i = (\bar{\mathbf{f}}(X,\rho) + \bar{\mathbf{g}}(X))\omega_i - \mu\ \mathrm{sgn}(\eta_i) + c(\lambda_i - k)\Gamma\eta_i, \quad \text{for } i = 1, \ldots, N. \tag{26}$$

The time derivative of the Lyapunov candidate function, $V = \frac{1}{2}\sum_{j=1}^{N}\eta_j^\top\eta_j$, along the trajectories of (26) is found to be

$$\begin{aligned}\dot V &= \sum_{j=1}^{N}\eta_j^\top c\left(\lambda_j - k\right)\Gamma\eta_j - \mu\sum_{j=1}^{N}\eta_j^\top \mathrm{sgn}(\eta_i) + \sum_{j=1}^{N}\eta_j^\top\left(\sum_{m=1}^{N}\bar f_m\left(x_m,\tilde\rho_m\right) + \bar g_m(X)\right)\omega_{jm}\\ &\le -\gamma\sum_{j=1}^{N}\|\eta_j\|^2 + (a+b-\mu)\sum_{j=1}^{N}\|\eta_j\|\end{aligned}$$

where $a = \max\{a_i\}$, $b = \max\{b_i\}$, and as before $-\gamma \le c(\delta + k)$ with $\delta = \min\{|\lambda_j|\}_{\lambda_j=0}$. Then, $\dot V < 0$ if $\mu > a+b$ and $\gamma > 0$, from which conditions (24) and (25) follow directly. □

4. Numerical simulations

Case 1. *Vanishing perturbations*:

Consider a dynamical network where each node is a chaotic Chen system given by Chen and Ueta [23]:

$$\begin{aligned}\dot x_1 &= p_1(x_2 - x_1)\\ \dot x_2 &= (p_3 - p_1)x_1 - x_1x_3 + p_3x_2\\ \dot x_3 &= x_1x_2 - p_2x_3\end{aligned} \tag{27}$$

with the nominal parameters $p_1 = 35$, $p_2 = 3$, and $p_3 = 28$. Using (27) as nominal nodes, the uncertain dynamical network (1) becomes:

$$\begin{bmatrix}\dot x_{i1}\\ \dot x_{i2}\\ \dot x_{i3}\end{bmatrix} = \begin{bmatrix}\tilde p_1(x_{i2} - x_{i1})\\ (\tilde p_3 - \tilde p_1)x_{i1} - x_{i1}x_{i3} + \tilde p_3 x_{i2}\\ x_{i1}x_{i2} - \tilde p_2 x_{i3}\end{bmatrix} + \tilde g_i(X) + u_i, \quad \text{for } i = 1,\ldots,N \tag{28}$$

where the uncertain parameters are $\tilde p_i = (1 + \Delta 0.1)p_i$ $(i = 1, 2, 3)$, and the uncertain component of each node is given by

$$\hat f_i(x_i,\hat\rho_i) = \begin{bmatrix}\hat p_1(x_{i2} - x_{i1})\\ (\hat p_3 - \hat p_1)x_{i1} - x_{i1}x_{i3} + \hat p_3 x_{i2}\\ x_{i1}x_{i2} - \hat p_2 x_{i3}\end{bmatrix}. \tag{29}$$

The uncertain coupling functions are $\tilde g_i(X) = (1 + \Delta 0.1)c\sum_{j=1}^{N} a_{ij}\Gamma x_j$, that is, the nominal coupling function is $g_i(X) = c\sum_{j=1}^{N} a_{ij}\Gamma x_j$ with $c = 1$ a nominal uniform coupling strength, and $\mathcal{A} = \{a_{ij}\}$ a 0–1 matrix satisfying the diffusive conditions such that its eigenvalue spectrum can be ordered as in (11). While the uncertain component of the coupling function for each node is:

$$\hat g_i(X) = \hat c\sum_{j=1}^{N} a_{ij}\Gamma x_j. \tag{30}$$

For simplicity, the internal coupling matrix is taken to be the identity ($\Gamma = I_3$), and the connectivity matrix $\mathcal{A}$ is constructed following the scale-free network model using the algorithm proposed by Wang and Chen [5] for $N = 50$. That is, we start with a 50×50 matrix of zeros, initially there are three nodes ($m_0 = 3$), then the entries $a_{ij} = a_{ji} = 1$, and the diagonal entries are obtain from $a_{ii} = -\sum_{j=1}^{N} a_{ij}$. At each time step a new node is added and is connected to $m = 3$ of the existing nodes chosen with a preferential attachment probability $\Pi_{ij} = \frac{d_i}{\sum_j d_j}$ with d_i the node degree of the i-th node. Once the nodes are chosen, the corresponding a_{ij} are change to one and the diagonal entries are recalculated. This process is repeated until the fifty nodes are connected.

For a network constructed as described above the uncertain components of the network (28), satisfies the bounds in (14) and (15), as both vanish at the synchronized state. Then, in order to satisfy the conditions of Theorem 1, the feedback controller gain is set to $k = 50$.

For our numerical simulation each node is initiated with different initial conditions, and their uncertain parameters are achieved as the multiplication of their nominal values times a randomly selected number of appropriate magnitude. After an uncontrolled transitory period, the synchronizing controller is activated at $t = 4$. As shown in Fig. 1, robust synchronization is achieved shortly after the controller is activated.

Case 2. *Non-vanishing perturbations*:

Next, we consider that each node is a chaotic Chua circuit:

$$\begin{aligned}\dot x_1 &= q_1(x_2 - x_1 - h(x_1))\\ \dot x_2 &= x_1 - x_2 + x_3\\ \dot x_3 &= -q_2x_2\end{aligned} \tag{31}$$

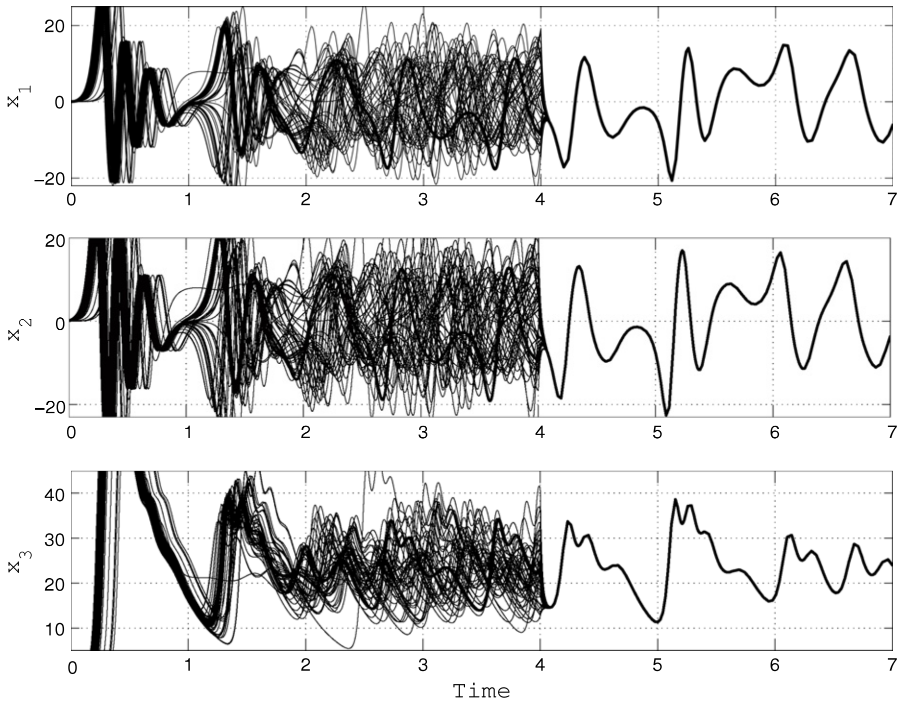

Fig. 1. Synchronization on a scale-free network of fifty Chen's systems under vanishing perturbations using the controllers described in Theorem 1.

where $h(x_1) = m_o x_1 + (\frac{1}{2})(m_1 - m_o)(|x_1 + 1| - |x_1 - 1|)$ and the nominal parameters are $q_1 = 9$, $q_2 = 100/7$, $m_o = -5/7$, $m_1 = -8/7$. As such, the uncertain dynamical network (1) becomes:

$$\begin{bmatrix} \dot{x}_{i1} \\ \dot{x}_{i2} \\ \dot{x}_{i3} \end{bmatrix} = \begin{bmatrix} \tilde{q}_1(x_{i2} - x_{i1} - h(x_{i1})) + \Delta\hat{d}_1 \\ x_{i1} - x_{i2} + x_{i3} + \Delta\hat{d}_2 \\ -\tilde{q}_2 x_{i2} + \Delta\hat{d}_3 \end{bmatrix} + \tilde{g}_i(X) + u_i, \tag{32}$$

for $i = 1, \dots, N$. The uncertain component of each node are given by

$$\hat{f}_i(x_i, \hat{\rho}_i) = \begin{bmatrix} \hat{q}_1(x_{i2} - x_{i1} - h(x_{i1})) + \hat{d}_1 \\ \hat{d}_2 \\ -\hat{q}_2 x_{i2} + \hat{d}_3 \end{bmatrix} \tag{33}$$

where the uncertain parameters are $\tilde{q}_i = (1 + \Delta 0.1)q_i$ $(i = 1, 2)$ and $\hat{d}_j = 0.2$ $(j = 1, 2, 3)$ are constant value perturbations affecting the uncertain nodes. The uncertain coupling functions are given by

$$\tilde{g}_i(X) = (1 + \Delta 0.1)c \sum_{j=1}^{N} a_{ij} \Gamma x_j + \Delta \hat{a}_{ii} \Gamma x_i \tag{34}$$

where $c = 1$, $\Gamma = I_3$, and $\mathcal{A}$ is a 0–1 diffusive 50×50 matrix constructed with the scale-free network model algorithm described above. Notice that here the uncertain component of the coupling functions has the additional perturbation $\hat{a}_{ii} = 0.05$ $(\forall i)$. Under these perturbations, the uncertain parts of the uncertain network (33) do not vanish at the synchronized solution. However, since the perturbations remain bounded, the bounds in (21) and (22) are satisfied. Then, a discontinuous local controller (23) can be designed to robustly synchronize the uncertain network to the average trajectory of the network. The synchronizing controller is designed with smooth and discontinuous gains set to $k = 10$ and $\mu = 5$, such that the conditions in (24) and (25) of Theorem 3 are satisfied.

The numerical simulations for this case were carried out in a similar way to the previous case, with different initial conditions with the uncertain parameters achieved through random number of appropriate magnitude. After an uncontrolled transitory period the discontinuous synchronizing controller was applied at $t = 40$. As shown in Fig. 2, robust synchronization is achieved even in the present of these non-vanishing perturbation.

5. Conclusion

We propose a solution to the synchronization problem for a class of uncertain dynamical networks, in which uncertainties are present in both their node descriptions and their interconnections. In particular, we design robust controllers for two

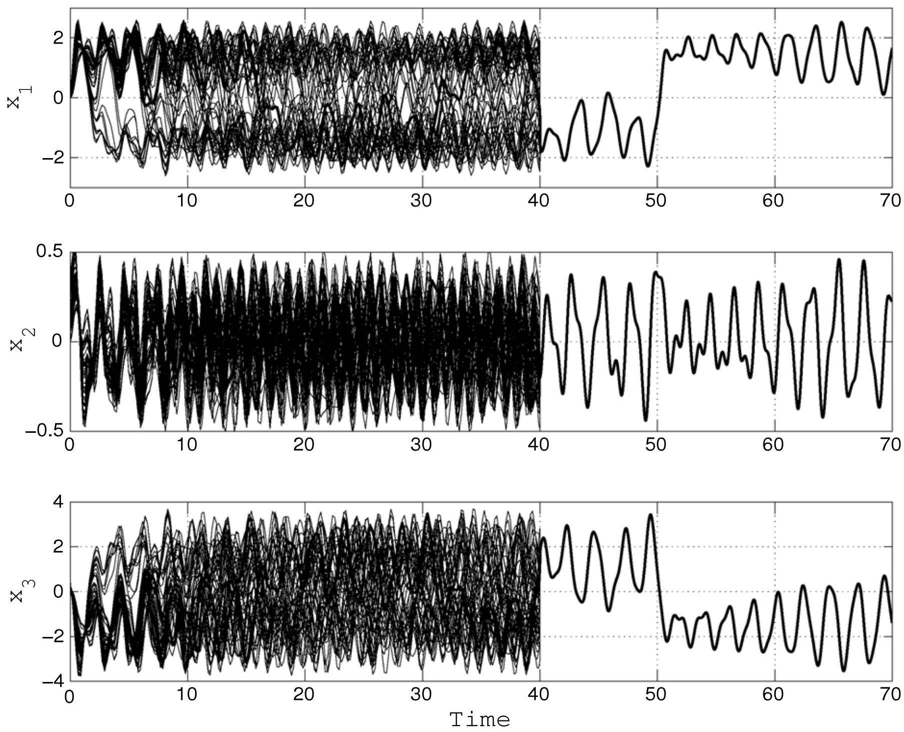

Fig. 2. Synchronization on a scale-free network of fifty Chua's circuits under non-vanishing perturbations using the controllers described in Theorem 3.

scenarios: In the first, uncertainties on both node dynamics and coupling structure vanish at the synchronized solution. In this case we have that the average trajectory of the network coincides with the dynamical evolution of an isolated nominal node, then local smooth feedback controllers can robustly synchronize the entire network. We illustrate this result using a scale-free network of Chen systems with multiplicative uncertainties in some of the system's parameters and in the uniform coupling strength. For these particular choices, we have vanishing perturbations at the synchronized solution, then our first Theorem is directly applicable. Complementary, in our second scenario, perturbations are nonvanishing. As such, the dynamical evolution of a nominal node is not at synchronized solution. However, we consider that all nodes are nearly identical, e.g. under the effects these perturbations, even potentially nonvanishing ones, the trajectories of the dynamical nodes remain close to each other. Under these conditions, an average trajectory is a reasonable objective for synchronization. For this particular situation, we propose a discontinuous feedback controller to robustly synchronize the network to its average trajectory. To illustrate the effectiveness of our design we use a network of Chua's circuit with multiplicative and additive uncertainties in some of the system's parameters and in the diagonal elements of the coupling matrix. With these type of perturbations the nodes in the network remain nearly identical and the results of our second Theorem are applicable.

Although the results presented above are restricted to dynamical networks with nearly identical uncertain nodes with possibly nonvanishing perturbations, there are many potential real-world applications, for example, these results can be applied to the consensus problem for automated vehicles [24]. Another potential application is to describe the coordinated operation of simplified cell models [25]. Research efforts are currently underway to expand the applicability of our designs to a wider type of dynamical networks, however those results will be reported elsewhere.

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