Multiple periodic oscillations in a nonlinear suspension bridge system

Zhonghai Ding

Department of Mathematical Sciences, University of Nevada, Las Vegas, NV 89154-4020, USA

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Abstract

In this paper, we study periodic oscillations in a suspension bridge system governed by the coupled nonlinear wave and beam equations describing oscillations in the supporting cable and roadbed under periodic external forces. By applying a variational reduction method, it is proved that the suspension bridge system has at least three periodic oscillations. © 2002 Elsevier Science (USA). All rights reserved.

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1. Introduction

It is well known that suspension bridges may display certain oscillations under external aerodynamic forces. Under the action of a strong wind, for example, a narrow and very flexible suspension bridge can undergo dangerous oscillations. The collapse of the Tacoma Narrows suspension bridge caused by a wind blowing at a speed of 42 miles per hour in the State of Washington on November 7, 1940, is one of the most striking examples. The Federal Works Agency Report [1] on the collapse has created a widespread demand for a comprehensive investigation of dynamic oscillation problems in suspension bridges in order to understand the causes of such destructive oscillations, and to develop design techniques to
prevent their recurrence in future. Based upon the observation of the fundamental nonlinearity in suspension bridges that the stays connecting the supporting cables and the roadbed resist expansion, but do not resist compression, new models describing oscillations in suspension bridges have been developed recently by Lazer and McKenna in [2,3]. The new models are described by systems of coupled nonlinear partial differential equations. The new study of suspension bridges initiated by Lazer and McKenna has obtained many important and interesting results. Multiple large amplitude periodic oscillations have been found theoretically and numerically in the single Lazer–McKenna suspension bridge equation (see [2–8,10] and references therein). However, except two recent papers [11,12] where the near-equilibrium periodic oscillation in a suspension bridge system with linear dampings was studied, there has been very little discussion in the existing literature on multiple periodic oscillations in suspension bridge systems governed by coupled nonlinear partial differential equations.

In this paper, we consider the following suspension bridge model proposed by Lazer and McKenna in [2]:

$$
\begin{align*}
\frac{d^2 m_c u_{tt}}{dx^2} - Q u_{xx} - K (w - u)^+ &= m_c g + \varepsilon h_1(x,t), \\
0 < x < L, \quad t > 0, \\
\frac{d^4 m_b w_{tt}}{dx^4} + EI w_{xxxx} + K (w - u)^+ &= m_b g + \varepsilon h_2(x,t), \\
0 < x < L, \quad t > 0, \\
u(0, t) = u(L, t) = 0, \\
w(0, t) = w(L, t) = 0, \\
w_{xx}(0, t) = w_{xx}(L, t) = 0,
\end{align*}
$$

which describes oscillations in a simplified suspension bridge configuration: the roadbed of length $L$ is modeled by a horizontal vibrating beam with both ends being simply supported; the supporting cable of length $L$ is modeled by a horizontal vibrating string with both ends being fixed; and the vertical stays connecting the roadbed to the supporting cable are modeled by one-sided springs which resist expansion but do not resist compression. In system (1.1), $u(x, t)$ and $w(x, t)$ denote the downward deflections of the cable and the roadbed from the unloaded state, respectively; $(w - u)^+ = \max\{w - u, 0\}$; $m_c$ and $m_b$ are the mass densities of the cable and the roadbed, respectively; $Q$ is the coefficient of cable tensile strength; $EI$ is the roadbed flexural rigidity; $K$ is the Hooke’s constant of the stays; $h_1$ and $h_2$ represent the external periodic aerodynamic forces; and, $\varepsilon$ is a parameter. We are interested in periodic oscillations in (1.1) which are symmetric about $x = L/2$,

$$
\begin{align*}
u(x, t + T) &= u(x, t), \quad w(x, t + T) = w(x, t), \quad 0 \leq x \leq L, \quad t > 0, \\
u(x, t) &= u(L - x, t), \quad w(x, t) = w(L - x, t), \quad 0 \leq x \leq L, \quad t > 0,
\end{align*}
$$

(1.2)
where $T$ is the period of periodic oscillations. By rescaling and translating $x$ and $t$, system (1.1) with (1.2) can be written in an equivalent form

\[
\begin{aligned}
&\frac{m_c u_{tt}}{\pi^2/2 < x < \pi/2, \ t > 0}, \\
&\frac{m_b w_{tt} + E I w_{xxxx} + K (w - u)^+}{\pi^2/2 < x < \pi/2, \ t > 0}, \\
&u(-\pi/2, t) = u(\pi/2, t) = 0, \ t > 0, \\
&w(-\pi/2, t) = w(\pi/2, t) = 0, \ w_{xx}(-\pi/2, t) = w_{xx}(\pi/2, t) = 0, \ t > 0, \\
&u(x, t + \pi) = u(x, t), \ w(x, t + \pi) = w(x, t), \ -\pi/2 \leq x \leq \pi/2, \ t > 0,
\end{aligned}
\tag{1.3}
\]

where $h_1(x, t)$ and $h_2(x, t)$ are $\pi$-periodic functions in $t$.

Nonlinear periodic oscillations in system (1.3) were studied first in [13], where $h_1$ and $h_2$ were assumed to be some special eigenfunctions. System (1.3) with $h_1$ and $h_2$ being any $H^2$-functions was further studied in [14] by applying the Mountain Pass Theorem to a dual variational formulation of the problem. It was proved in [14] that system (1.3) has at least two periodic solutions when $-\sigma_{10} < K < -\sigma_{20}$ ($\sigma_{mn}$ is to be defined in (2.2) and (2.4)). The objective of this paper is to study nonlinear periodic oscillations in system (1.3) by using a variational reduction method to obtain more periodic oscillations. The variational reduction method which reduces a critical point problem in an infinite-dimensional Hilber space to an equivalent finite-dimensional one was first introduced by Lazer et al. in [9] and was further studied by Amann in [15]. This method was applied to nonlinear elliptic boundary value problems with nonlinearities crossing multiple eigenvalues by Lazer and McKenna [16]. More recently, this method was applied to the single Lazer–McKenna suspension bridge equation by Choi et al. [4] and to the semilinear wave equation by Choi and Jung [17]. By applying the variational reduction method to the suspension bridge system (1.3), we show that system (1.3) admits at least three periodic oscillations when $-\sigma_{10} < K < -\sigma_{20}$.

The organization of this paper is as follows. In Section 2, we formulate an equivalent system of (1.3). In Section 3, we formulate the corresponding variational problem, and apply the variational reduction method to reduce the variational problem to an equivalent finite-dimensional one. In Section 4, we apply the Mountain Pass Theorem due to Ambrosetti and Rabinowitz [18] to the finite-dimensional variational problem and show that system (1.3) admits at least three periodic oscillations when $-\sigma_{10} < K < -\sigma_{20}$.
2. An equivalent system of (1.3)

To investigate the suspension bridge system (1.3), we assume throughout this paper that
\[
Q \leq m_c, \quad EI \leq m_b, \quad (2.1)
\]
which hold naturally for suspension bridges in civil engineering applications. In this section, we derive an equivalent formulation of system (1.3).

Define the wave operator \( \mathcal{L}_1 \) by
\[
\begin{aligned}
\mathcal{L}_1 u &= m_c u_{tt} - Qu_{xx}, \\
 u(-\pi/2, t) &= u(\pi/2, t) = 0, \\
 u(x, t) &= u(-x, t), \quad u(x, t + \pi) = u(x, t).
\end{aligned}
\]
Define the beam operator \( \mathcal{L}_2 \) by
\[
\begin{aligned}
\mathcal{L}_2 w &= m_b w_{tt} + EI w_{xxxx}, \\
 w(-\pi/2, t) &= w(\pi/2, t) = 0, \\
 w_{xx}(-\pi/2, t) &= w_{xx}(\pi/2, t) = 0, \\
 w(x, t) &= w(-x, t), \quad w(x, t + \pi) = w(x, t).
\end{aligned}
\]
Denote by \( \{\lambda_{mn}\} \) the eigenvalues of \( \mathcal{L}_1 \) and by \( \{\mu_{mn}\} \) the eigenvalues of \( \mathcal{L}_2 \). Then it follows from a direct calculation that
\[
\lambda_{mn} = Q(2n + 1)^2 - 4m_cm^2, \quad m, n = 0, 1, 2, \ldots, \\
\mu_{mn} = EI(2n + 1)^4 - 4mb_m^2, \quad m, n = 0, 1, 2, \ldots, \quad (2.2)
\]
The eigenfunctions of \( \mathcal{L}_1 \) corresponding to eigenvalue \( \lambda_{mn} \) are the same as that of \( \mathcal{L}_2 \) corresponding to eigenvalue \( \mu_{mn} \), which are given by
\[
\begin{aligned}
\phi_{0n}(x, t) &= \frac{\sqrt{2}}{\pi} \cos(2n + 1)x, \quad n \geq 0, \\
\phi_{mn}(x, t) &= \frac{2}{\pi} \cos(2n + 1)x \cos 2mt, \quad m \geq 1, \quad n \geq 0, \\
\psi_{mn}(x, t) &= \frac{2}{\pi} \cos(2n + 1)x \sin 2mt, \quad m \geq 1, \quad n \geq 0.
\end{aligned}
\]
Let \( \Omega = (-\pi/2, \pi/2) \times (-\pi/2, \pi/2) \), and \( H \) be the Hilbert space defined by
\[
H = \{ u \in L^2(\Omega) \mid u(-x, t) = u(x, t) \},
\]
with a norm \( \| \cdot \| \) being the usual \( L^2 \)-norm of \( L^2(\Omega) \). It is easy to check that the set of eigenfunctions \( \{\phi_{mn}, \psi_{mn}\} \) is an orthonormal basis of \( H \). Assume throughout this paper that the material parameters \( m_c, m_b, Q \) and \( EI \) are chosen such that
\[
\begin{aligned}
& \text{both } \sqrt{Q/m_c} \text{ and } \sqrt{EI/m_b} \text{ are rational numbers,} \\
& \lambda_{mn} = Q(2n + 1)^2 - 4m_cm^2 \neq 0, \\
& \mu_{mn} = EI(2n + 1)^4 - 4mb_m^2 \neq 0, \\
& \lambda_{mn} + \mu_{mn} \neq 0 \quad \text{for } m \geq 1, \quad n \geq 1. \quad (2.3)
\end{aligned}
\]
By the assumption (2.3), $L_1$, $L_2$ and $L_1 + L_2$ are invertible in $H$. The assumption of both $\sqrt{Q/mc}$ and $\sqrt{EI/m_b}$ being rational is necessary due to the known fact that certain number theoretical difficulties may be encountered [14]. Define
\[ A = L_2 L_1 (L_1 + L_2)^{-1}. \]
The eigenvalues of $A$ are given by
\[ \sigma_{mn} = \frac{\lambda_{mn} \mu_{mn}}{\lambda_{mn} + \mu_{mn}}, \] (2.4)
where the corresponding eigenfunctions are given by $\{\varphi_{mn}, \psi_{mn}\}$. Under assumption (2.3), the following mapping properties of $L_1$, $L_2$ and $A$ were proved in [14].

**Lemma 2.1.** Let $\beta \in \mathbb{R}$ and $\beta \neq -\sigma_{mn}$, and $s \geq 0$. Then

(a) $L_1^{-1}$ is a bounded linear operator from $H^s(\Omega) \cap H$ to $H^{s+1}(\Omega) \cap H$;
(b) $L_2^{-1}$ is a bounded linear operator from $H^s(\Omega) \cap H$ to $H^{s+2}(\Omega) \cap H$; and
(c) $(A + \beta)^{-1}$ is a bounded linear operator from $H^s(\Omega) \cap H$ to $H^{s+1}(\Omega) \cap H$.

By using the above notations and by restricting the domain of $(u, w)$ to $\Omega$, system (1.3) can be written as
\[
\begin{cases}
L_1 u - K(w - u)^+ = mc^2 g + \varepsilon h_1, \\
L_2 w + K(w - u)^+ = mb^2 g + \varepsilon h_2,
\end{cases}
\] (2.5)
where $h_1 \in H$ and $h_2 \in H$. From (2.5), one has
\[ L_1 u + L_2 w = (mc + mb)g + \varepsilon (h_1 + h_2). \]
By applying $L_1^{-1} L_2^{-1}$ to both sides of this equation, we have
\[ L_2^{-1} u + L_1^{-1} w = L_1^{-1} L_2^{-1} [(mc + mb)g + \varepsilon (h_1 + h_2)]. \]
Let $\tilde{w} = L_1^{-1} w$ and $\tilde{u} = L_2^{-1} u$; then $u = L_2 \tilde{u}$, $w = L_1 \tilde{w}$, and
\[ \tilde{w} + \tilde{u} = L_1^{-1} L_2^{-1} [(mc + mb)g + \varepsilon (h_1 + h_2)]. \]
By substituting them into the second equation of (2.5), we obtain
\[ L_2 L_1 \tilde{w} + K [(L_1 + L_2) \tilde{w} - (mc + mb)g L_1^{-1} (1) - \varepsilon L_1^{-1} (h_1 + h_2)]^+ = mb^2 g + \varepsilon h_2. \]
Let $v = (L_1 + L_2) \tilde{w}$, $f_0 = (mc + mb)g L_1^{-1} (1) \in H$ and $f_1 = L_1^{-1} (h_1 + h_2) \in H$; then the above equation can be written as
\[ A v + K [v - f_0 - \varepsilon f_1]^+ = mb^2 g + \varepsilon h_2. \] (2.6)
Note that the relation between $w - u$ and $v$ is given by
\[ w - u = L_1 \tilde{w} - L_2 \tilde{u} = v - f_0 - \varepsilon f_1. \] (2.7)
By substituting the above relation into (2.5), we obtain

\[
\begin{aligned}
    u &= L_1^{-1}\left[ K(v - L_1^{-1}[(m_c + m_b)g + \varepsilon(h_1 + h_2)])^+ + m_c g + \varepsilon h_1 \right], \\
    w &= L_2^{-1}\left[ -K(v - L_1^{-1}[(m_c + m_b)g + \varepsilon(h_1 + h_2)])^+ + m_b g + \varepsilon h_2 \right].
\end{aligned}
\] (2.8)

Thus, if \( v \in H \) is a solution of (2.6), then \((u, w) \in (H^1(\Omega) \cap H) \times (H^2(\Omega) \cap H)\) given by (2.8) is a solution of (2.5). Therefore, to study the multiple solutions of (2.5) becomes to study multiple solutions of (2.6).

Before studying multiple solutions of (2.6), we recall some known results on the equilibrium and near-equilibrium oscillations in system (1.3), which were discussed in detail in [14]. The equilibrium oscillation in system (1.3) is determined by the following equation:

\[
\begin{aligned}
    -Qu_{xx} - K(w - u)^+ &= m_c g, \quad -\pi/2 < x < \pi/2, \\
    EIw_{xxx} + K(w - u)^+ &= m_b g, \quad -\pi/2 < x < \pi/2, \\
    u(-\pi/2) &= u(\pi/2) = 0, \\
    w(-\pi/2) &= w(\pi/2) = 0, \quad w_{xx}(-\pi/2) = w_{xx}(\pi/2) = 0, \\
    u(-x) &= u(x), \quad w(-x) = w(x), \quad 0 \leq x \leq \pi/2.
\end{aligned}
\] (2.9)

**Theorem 2.1.** For any given \( K > 0 \), there exists a \( \mu_0 > 0 \), which depends only on \( K, Q \) and \( EI \), such that if \( m_c/m_b < \mu_0 \), then (2.9) admits a \( C^\infty \)-solution \((u_e, w_e)\) satisfying \( w'_e(-\pi/2) - u'_e(-\pi/2) > 0 \), \( w'_e(\pi/2) - u'_e(\pi/2) < 0 \), and \( w_e(x) - u_e(x) > 0 \) for \(-\pi/2 < x < \pi/2\).

For the near-equilibrium oscillation of (1.3) or (2.5), we have the following result.

**Theorem 2.2.** Let \( K \geq 0 \) and \( K \neq -\sigma_{mn} \), and the conditions in Theorem 2.1 be satisfied. If \((h_1, h_2) \in (H^2(\Omega) \cap H) \times (H^2(\Omega) \cap H)\), then there exists an \( \varepsilon_0 > 0 \), which depends only on \( K, (h_1, h_2) \) and \((u_e, w_e)\), such that if \( |\varepsilon| < \varepsilon_0 \), then the suspension bridge system (2.5) admits a near-equilibrium periodic solution \((u_0, w_0) \in (H^3(\Omega) \cap H) \times (H^4(\Omega) \cap H)\) satisfying \( w_0(x, t) - u_0(x, t) > 0 \) for \(|x| < \pi/2\), \((w_0)_x(-\pi/2, t) - (u_0)_x(-\pi/2, t) > 0 \), and \((w_0)_x(\pi/2, t) - (u_0)_x(\pi/2, t) < 0 \) for \(|t| \leq \pi/2\).

The proofs of Theorems 2.1 and 2.2, and the explicit expressions of \( \mu_0 \) and \((u_e, w_e)\) can be found in [14]. Let

\[
v_0 = w_0 - u_0 + f_0 + \varepsilon f_1.
\] (2.10)

By Theorem 2.2, Lemma 2.1, and (2.7), (2.8), \( v_0 \in H^3(\Omega) \cap H \) is a solution of (2.6) if \( |\varepsilon| < \varepsilon_0 \), and \((v_0 - f_0 - \varepsilon f_1)(x, t) > 0 \) on \( \Omega \).
Proof. Let $\varphi$ be the mapping from $H$ to $H$, and $\lambda$ be a variational formulation of (2.6). Then we focus on solutions of (2.6) when $-\sigma_{10} < K < -\sigma_{20}$ by defining a functional on an infinite-dimensional Hilbert space, whose critical points correspond to solutions of (2.6). We then apply the variational reduction method to the functional.

Under assumption (2.1), it is straightforward to check

$$\sigma_{20} < \sigma_{10} < 0 < \sigma_{00}, \quad 0 < \sigma_{00} < -\sigma_{10}.$$ 

Assume throughout this paper that

the only eigenvalue of $A$ in the interval $(\sigma_{20}, \sigma_{00})$ is $\sigma_{10}$. (3.1)

**Theorem 3.1.** If $0 < K < -\sigma_{10}$ and $(h_1, h_2) \in (H^2(\Omega) \cap H) \times (H^2(\Omega) \cap H)$, then Eq. (2.6) admits a unique solution in $H^2(\Omega) \cap H$ for any $\epsilon \geq 0$. Moreover, the unique solution of (2.6) is given by $\varphi = \varphi_0 \in H^3(\Omega) \cap H$ given in (2.10) if $|\epsilon| < \epsilon_0$, where $\epsilon_0$ is defined in Theorem 2.2.

**Proof.** Let $\delta = -(\sigma_{10} + \sigma_{00})/2$. By Lemma 2.1, $(A + \delta)^{-1}$ is a compact linear map from $H$ to $H$ with norm $2/(\sigma_{00} - \sigma_{10})$. Equation (2.6) can be written as

$$v = (A + \delta)^{-1}[(\delta - K)(v - f_0 + \epsilon_1) + - \delta(v - f_0 - \epsilon_1)]$$

$$+ \delta(f_0 + \epsilon_1) + m_{bg} + \epsilon h_2],$$

where we have used $u^- = \max(-u, 0)$ and $u = u^+ - u^-$. Note that

$$(\delta - K)[(v_1 - f_0 - \epsilon_1)^+ - (v_2 - f_0 - \epsilon_1)^+]$$

$$- \delta[(v_1 - f_0 - \epsilon_1)^- - (v_2 - f_0 - \epsilon_1)^-]$$

$$\leq \max\{|\delta - K|, |\delta|\} \|v_1 - v_2\|$$

$$< \frac{1}{2}(\sigma_{00} - \sigma_{10}) \|v_1 - v_2\|,$$ 

where we have used the inequality $|a_1^+ - a_2^+| + |a_1^- - a_2^-| \leq |a_1 - a_2|$ for any $a_1, a_2 \in \mathfrak{N}$. Thus the right-hand side of (3.2) defines a continuous contraction mapping from $H$ to $H$. Therefore, by the Banach fixed point theorem, (3.2) admits a unique solution $v \in H$. Since $f_0 \in C^\infty(\Omega) \cap H$ and $f_1 \in H^3(\Omega) \cap H$, one obtains $v \in H^2(\Omega) \cap H$ by applying the bootstrapping technique to (3.2). When $|\epsilon| < \epsilon_0$, where $\epsilon_0$ is defined in Theorem 2.2, then it follows from Theorem 2.2 that the unique solution of (2.6) is given by $v = \varphi_0$ defined by (2.10).

To study multiple solutions of (2.6) when $-\sigma_{10} < K < -\sigma_{20}$, we define a subspace $V$ of $H$ by

$$V = \left\{v \in H \mid v = \sum (a_{mn}\varphi_{mn} + b_{mn}\psi_{mn}), \sum |\sigma_{mn}|(a_{mn}^2 + b_{mn}^2) < \infty\right\}$$
with a norm
\[ \|v\|_V = \sqrt{\sum |\sigma_{mn}| \left( a_{mn}^2 + b_{mn}^2 \right)}. \]

Then it is easy to check that \( V \) is a complete normed space, and
\[ \|v\|_V \geq \sqrt{\sigma_{00}} \|v\|, \quad \forall v \in V. \]

If \( V \) is equipped with the inner product induced from \( \| \cdot \|_V \), then \( V \) is in fact a Hilbert space, and \( \{ \varphi_{nn}, \psi_{nn} \} \) is an orthogonal basis of \( V \). Note that \( 1 \in H \) but \( 1 \notin V \); thus \( V \) is a proper subspace of \( H \). Denote by \( V' \) the dual of \( V \), and identify \( H \) with its dual \( H' \). We have the following dense and continuous embeddings:
\[ V \subset H \subset V'. \]

By the definition of \( V \), it is easy to verify the following properties of \( A \) in \( V \):

(a) For any \( v \in V \), \( Av \in V' \) and \( \| Av \|_{V'} = \| v \|_V \); and
(b) if \( \beta \neq -\sigma_{nn} \), then \( (A + \beta)^{-1} \) is bounded from \( H \) to \( V \).

Define a functional \( J_K(v) \) on \( V \) by
\[
J_K(v) = \int_{\Omega} \left[ \frac{1}{2} (Av)v + \frac{K}{2} \left( [v - f_0 - \varepsilon f_1]^+ \right)^2 - m_b g v - \varepsilon h_2 v \right] dx \, dt.
\]

(3.3)

The following lemma states that the solutions of (2.6) coincide with the critical points of \( J_K \).

**Lemma 3.1.** Let \( h_1, h_2 \in V \) be given. Then \( J_K \) is continuous in \( V \), and Fréchet differentiable at each \( v \in V \) with
\[
J'_K(v)u = \int_{\Omega} (Av + K[v - f_0 - \varepsilon f_1]^+ - m_b g - \varepsilon h_2)u \, dx \, dt, \quad u \in V.
\]

Thus any critical point of \( J_K \) in \( V \) corresponds to a solution of (2.6) in \( V \), and vice versa.

**Proof.** Let \( u, v \in V \).

\[
J_K(v + u) - J_K(v) = \int_{\Omega} \left\{ (Av)u + \frac{1}{2} (Au)u + \frac{K}{2} \left( ([v + u - f_0 - \varepsilon f_1]^+)^2 \\
- ([v - f_0 - \varepsilon f_1]^+)^2 - m_b gu - \varepsilon h_2 u \right) \right\} dx \, dt.
\]
Let \( v = \sum (a_{mn}\varphi_{mn} + b_{mn}\psi_{mn}) \) and \( u = \sum (c_{mn}\varphi_{mn} + d_{mn}\psi_{mn}) \). Then
\[
\left| \int_{\Omega} A v \cdot u \, dx \, dt \right| \leq \|v\|_V \|u\|_V.
\]
By using the inequality \( |((a + b)^+) - (a^+)^2| \leq 2a^+|b| + b^2 \) for any \( a, b \in \mathbb{R} \), we have
\[
\int_{\Omega} \left| \left( [v + u - f_0 - \varepsilon f_1]^+ - [v - f_0 - \varepsilon f_1]^+ \right) \right| \, dx \, dt \leq \frac{2}{\sqrt{\sigma_{00}}} \|v - f_0 - \varepsilon f_1\|^2 \|u\|_V + \frac{1}{\sigma_{00}} \|u\|_V^2.
\]
Thus \( J_K \) is continuous in \( V \). By using the inequality \( 0 \leq ((a + b)^+) - (a^+)^2 - 2a^+b \leq b^2 \) for any \( a, b \in \mathbb{R} \), we have
\[
\left| J_K(v + u) - J_K(v) - J'_K(v)u \right| \leq \frac{1}{2} \|u\|_V^2 + \frac{K}{2} \|u\|_V^2 \leq \frac{1}{2} \|u\|_V^2 + \frac{K}{2\sigma_{00}} \|u\|_V^2.
\]
Thus \( J_K \) is Fréchet differentiable and
\[
J'_K(v)u = \int_{\Omega} (A v + K [v - f_0 - \varepsilon f_1]^+ - m_{bg} - \varepsilon h_2) u \, dx \, dt.
\]
Therefore, any critical point of \( J_K \) in \( V \) corresponds to a solution of (2.6) in \( V \), and vice versa.

\[\square\]

Next we will apply the variational reduction method [15] to reduce functional \( J_K \) on \( V \) to an equivalent one on a two-dimensional space \( V_0 \).

Let \( V_0 \) be the two-dimensional linear space spanned by \( \{\varphi_{10}, \psi_{10}\} \), which correspond to the eigenvalue \( \sigma_{10} \) of \( A \). Let \( U \) and \( W \) be the orthogonal complements of \( V_0 \) in \( H \) and \( V \), respectively. Since \( \{\varphi_{mn}, \psi_{mn}\} \) is an orthogonal basis of both \( H \) and \( V \), the orthogonal projections of \( H \) and \( V \) onto \( V_0 \) are the same. Denote by \( P \) the orthogonal projection of \( H \) or \( V \) onto \( V_0 \). Thus \( U = (I - P)H \) and \( W = (I - P)V \).

**Lemma 3.2.** If \( -\sigma_{00} < K < -\sigma_{20} \) and \( (h_1, h_2) \in H \times H \), then for each \( v \in V_0 \), the following equation admits a unique solution \( y \in W \):
\[
A y + (I - P)(K [v + y - f_0 - \varepsilon f_1]^+ - m_{bg} - \varepsilon h_2) = 0. \tag{3.4}
\]
Let \( y(v) = y \), where \( y \) is the solution of (3.4) with \( v \in V_0 \). Then \( y(v) \) is uniformly Lipschitz continuous on \( V_0 \) with respect to both the \( L^2 \)-norm \( \| \cdot \| \) on \( H \) and the norm \( \| \cdot \|_V \) on \( V \).
Proof. Let $\delta = -(\sigma_{20} + \sigma_{00})/2$. By assumption (2.1), it is easy to check

$$0 < -\sigma_{10} < \delta < -\sigma_{20}.$$ 

By assumption (3.1) and Lemma 2.1, $(A + \delta)$ is invertible in both $H$ and $V$. Let

$$p(\xi) = (\delta - K)[\xi - f_0 - \varepsilon f_1]^+ - \delta[\xi - f_0 - \varepsilon f_1]^-. $$

Then Eq. (3.4) can be rewritten as

$$y = (A + \delta)^{-1}(I - P)[p(y + v) + \delta(f_0 + \varepsilon f_1) + m_b g + \varepsilon h_2].$$  

By Lemma 2.1, $(A + \delta)^{-1}(I - P)$ is compact from $U$ into itself. Under assumption (3.1), the $L^2$-norm of $(A + \delta)^{-1}(I - P)$ is $2/(\sigma_{00} - \sigma_{20})$. Since

$$|p(\xi_1) - p(\xi_2)| \leq \max\{|\delta - K|, |\delta|\}|\xi_1 - \xi_2| < \frac{\sigma_{00} - \sigma_{20}}{2}|\xi_1 - \xi_2|,$$

the right-hand side of (3.5) defines a contraction mapping from $U$ into itself. By the Banach fixed-point theorem, (3.5) admits a unique solution $y \in U$ for given $v \in V_0$. Since $(A + \beta)^{-1}$ is bounded from $H$ to $V$, (3.5) admits a unique solution $y \in W$ for given $v \in V_0$. Let $y(v) = y$, where $y$ is the solution of (3.5) for given $v \in V_0$. Let

$$y = \frac{2}{\sigma_{00} - \sigma_{20}} \max\{|\delta - K|, |\delta|\}.$$ 

Then $0 < y < 1$. For any $v_1, v_2 \in V_0$, we have

$$\|y(v_1) - y(v_2)\| \leq \|(A + \delta)^{-1}(I - P)\| p(y(v_1) + v_1) - p(y(v_2) + v_2)\| \leq y(\|y(v_1) - y(v_2)\| + \|v_1 - v_2\|).$$

Thus

$$\|y(v_1) - y(v_2)\| \leq \frac{y}{1 - y}\|v_1 - v_2\|.$$ 

Under assumption (3.1), any $\sigma_{mn}$ except $\sigma_{10}$ satisfies

$$\frac{|\sigma_{mn}|}{(\sigma_{mn} + \delta)^2} \leq \frac{2}{|\sigma_{nn} + \delta|}.$$ 

Thus for any $u \in H$,

$$\|(A + \delta)^{-1}(I - P)u\|_V \leq \frac{2}{\sqrt{\sigma_{00} - \sigma_{20}}} \|u\|.$$ 

Then for any $v_1, v_2 \in V_0$, we have

$$\|y(v_1) - y(v_2)\|_V \leq \frac{2}{\sqrt{\sigma_{00} - \sigma_{20}}} \|p(y(v_1) + v_1) - p(y(v_2) + v_2)\| \leq \sqrt{\sigma_{00} - \sigma_{20}}(\|y(v_1) - y(v_2)\| + \|v_1 - v_2\|) \leq \frac{\sqrt{\sigma_{00} - \sigma_{20}}}{(1 - y)\sqrt{\sigma_{10}}} \|v_1 - v_2\|_V.$$
Therefore \( y(v) \) is uniformly Lipschitz continuous on \( V_0 \) with respect to both the \( L^2 \)-norm \( \| \cdot \| \) on \( H \) and the norm \( \| \cdot \|_V \) on \( V \). \( \Box \)

By using \( J_K \) and Lemma 3.2, we define the following functional on \( V_0 \),

\[
\tilde{J}_K(v) = J_K(v + y(v)),
\]

where \( y(v) \in W \) is the unique solution of (3.4).

**Lemma 3.3.** Assume \( -\sigma_{00} < K < -\sigma_{20} \) and \((h_1, h_2) \in H \times H\). Then \( \tilde{J}_K(v) \) defined by (3.6) is continuously Fréchet differentiable on \( V_0 \) with

\[
\tilde{J}_K'(v)u = J'_K(v + y(v))u, \quad u \in V_0.
\]

If \( v_0 \) is a critical point of \( \tilde{J}_K \) on \( V_0 \), then \( v_0 + y(v_0) \) is a critical point of \( J_K \) on \( V \).

On the other hand, if \( u \) is a critical point of \( J_K \) on \( V \), then \( Pu \) is a critical point of \( \tilde{J}_K \) on \( V_0 \).

**Proof.** Let \( v \in V_0 \) and \( y(v) \) be the solution of (3.4) in \( W \). By Lemma 3.2, we have

\[
\int_\Omega (Ay(v) + K[v + y(v) - f_0 - \varepsilon f_1]^+ - m_b g - \varepsilon h_2)w \, dx \, dt = 0,
\]

\( \forall w \in W \). Since \( AV_0 \subset V_0 \), we have

\[
\int_\Omega (Av)w \, dx \, dt = 0, \quad \forall w \in W.
\]

Thus we have

\[
J'_K(v + y(v))w = 0, \quad \forall w \in W. \tag{3.7}
\]

Let \( W_1 \) and \( W_2 \) be the subspaces of \( V \) defined by

\[
W_1 = \text{span}\{\phi_{mn}, \psi_{mn} \mid \sigma_{mn} \leq \sigma_{20} < 0\}
\]

and

\[
W_2 = \text{span}\{\phi_{mn}, \psi_{mn} \mid \sigma_{mn} \geq \sigma_{00} > 0\}.
\]

Under assumption (3.1), we have \( V = W_1 \oplus V_0 \oplus W_2 \). For a given \( v \in V_0 \), define a functional \( h : W_1 \times W_2 \to \mathfrak{N} \) by

\[
h(w_1, w_2) = J_K(v + w_1 + w_2), \quad w_1 \in W_1, \ w_2 \in W_2.
\]

By Lemma 3.1, \( h \) has Fréchet partial derivatives \( D_1 h \) and \( D_2 h \) with respect to its first and second variables, where

\[
\begin{cases}
D_1 h(w_1, w_2)u_1 = J'_K(v + w_1 + w_2)u_1, \quad u_1 \in W_1, \\
D_2 h(w_1, w_2)u_2 = J'_K(v + w_1 + w_2)u_2, \quad u_2 \in W_2.
\end{cases}
\]
Let $y(v) = y_1(v) + y_2(v)$ such that $y_1(v) \in W_1$ and $y_2(v) \in W_2$. Then by (3.7), we have
\[
\begin{align*}
D_1 h(y_1(v), y_2(v))u_1 &= 0, \quad u_1 \in W_1, \\
D_2 h(y_1(v), y_2(v))u_2 &= 0, \quad u_2 \in W_2.
\end{align*}
\] (3.8)

Let $q(\xi) = [\xi - f_0 - \varepsilon f_1]^+$. If $w_1 \in W_1$ and $u_2, w_2 \in W_2$, then
\[
\begin{align*}
[D_2 h(w_1, w_2) - D_2 h(w_1, u_2)](w_2 - u_2) &= [J_K'(v + w_1 + w_2) - J_K'(v + w_1 + u_2)](w_2 - u_2) \\
&= \int_\Omega [A(w_2 - u_2) + K(q(v + w_1 + w_2) - q(v + w_1 + u_2))]
\times (w_2 - u_2) \, dx \, dt \\
&= \int_\Omega [A(w_2 - u_2)(w_2 - u_2) + K(q(v + w_1 + w_2) - q(v + w_1 + u_2))]
\times (w_2 - u_2) \, dx \, dt.
\end{align*}
\]

Since $(q(\xi_1) - q(\xi_2))(\xi_1 - \xi_2) \geq 0$ for any $\xi_1, \xi_2 \in \mathcal{N}$, and since $\int_\Omega (Au)u \, dx \, dt = \|u\|^2_V$ for any $u \in W_2$, we have
\[
[D_2 h(w_1, w_2) - D_2 h(w_1, u_2)](w_2 - u_2) \geq \|w_2 - u_2\|^2_V.
\]

Thus $h(w_1, w_2)$ is strictly concave upward with respect to $w_2$. Similarly, since $(q(\xi_1) - q(\xi_2))(\xi_1 - \xi_2) \leq |\xi_1 - \xi_2|^2$ for any $\xi_1, \xi_2 \in \mathcal{N}$, and since $\int_\Omega (Au)u \, dx \, dt = -\|u\|^2_V$ for any $u \in W_1$, we obtain
\[
[D_1 h(w_1, w_2) - D_1 h(u_1, w_2)](w_1 - u_1) \leq -\|w_1 - u_1\|^2_V + K\|w_1 - u_1\|^2 \leq \left(1 - \frac{K}{\sigma_{20}}\right)\|w_1 - u_1\|^2_V,
\]
for any $u_1, w_1 \in W_1$. Thus, by the assumption $-\sigma_{10} < K < -\sigma_{20}$, $h(w_1, w_2)$ is strictly concave downward with respect to $w_1$. Thus it follows from (3.8) that
\[
\begin{align*}
h(y_1(v), y_2(v)) &\leq h(y_1(v), u_2), \quad \forall u_2 \in W_2, \\
h(y_1(v), y_2(v)) &\geq h(u_1, y_2(v)), \quad \forall u_1 \in W_1,
\end{align*}
\]
where the equalities are true if and only if $u_1 = y_1(v)$ and $u_2 = y_2(v)$. By Theorem 2.3 in [15] (a fundamental theorem for the variational reduction method), $\tilde{J}_K(v)$ is continuously differentiable, and
\[
\tilde{J}_K'(v)u = J_K'(v + y(v))u, \quad u \in V_0.
\] (3.9)

Suppose that $v_0 \in V_0$ is a critical point of $\tilde{J}_K$ on $V_0$, i.e., $\tilde{J}_K'(v_0)v = 0$, $\forall v \in V_0$. Then (3.7) and (3.9) imply that $J_K'(v_0 + y(v_0))v = 0$ for any $v \in V$. Thus $v_0 + y(v_0)$ is a critical point of $J_K$ on $V$. On the other hand, if $v_0$ is a critical
point of $J_K$ on $V$, i.e., $J_K'(v_0)v = 0$, $\forall v \in V$, then $v_0$ is a solution of (2.6) by Lemma 3.1:

$$A v_0 + K [v_0 - f_0 - \varepsilon f_1]^+ = m_b g + \varepsilon h_2.$$ 

Let $y = (I - P)v_0$; then $v_0 = P v_0 + y$. From the above equation, we have

$$A y + (I - P) [K P v_0 + y - f_0 - \varepsilon f_1]^+ - m_b g - \varepsilon h_2 = 0.$$ 

By Lemma 3.2, we have $y = y(Pv_0)$. Thus for any $v \in V_0$,

$$\tilde{J}_K'(Pv_0) v = J_K'(Pv_0 + y(Pv_0)) v = J_K'(v_0) v = 0.$$ 

Hence $P v_0$ is a critical point of $\tilde{J}_K$ on $V_0$. □

By Lemma 3.3, the study of critical points of $J_K(v)$ on $V$ becomes the study of critical points of $\tilde{J}_K(v)$ on $V_0$, where $V_0$ is a two-dimensional linear space. Before studying critical points of $\tilde{J}_K(v)$ on $V_0$ in the next section, we discuss properties of $v_0$ defined by (2.10). If $-\sigma_{10} < K < -\sigma_{20}$ and $(h_1, h_2) \in (H^2(\Omega) \cap W) \times (H^2(\Omega) \cap W)$, then Theorem 2.2 asserts that there exists a $\varepsilon_0 > 0$ such that if $|\varepsilon| < \varepsilon_0$, then $v_0$ defined by (2.10) is a solution of (2.6) and $v_0 \in H^3(\Omega) \cap H$. Furthermore, $(v_0 - f_0 - \varepsilon f_1)(x, t) > 0$ on $\Omega$.

We show next that $v_0 = y(0) \in W$ where, by Lemma 3.2, $y(0)$ is the unique solution of (3.4) with $v = 0$. Since $(A + K)^{-1}$ is bounded from $H$ to $V$, we have $v_0 \in V$. Since

$$1 = \sum_{n=0}^{\infty} \frac{2\sqrt{2}(-1)^n}{2n + 1} \varphi_0 n,$$

we have

$$f_0 = (m_c + m_b) g L_1^{-1}(1) = (m_c + m_b) g \sum_{n=0}^{\infty} \frac{2\sqrt{2}(-1)^n}{Q(2n + 1)^3} \varphi_0 n.$$ 

Then

$$\|f_0\|_V^2 = (m_c + m_b)^2 g^2 \sum_{n=0}^{\infty} \frac{8EI}{Q(Q(2n + 1)^2 + EI(2n + 1)^4)} < \infty.$$ 

Thus $f_0 \in W$. Under assumption (2.3), let $\sqrt{Q/m_c} = p/q$, then

$$|\lambda_{mn}| = \frac{m_c}{q^2} \left| p^2(2n + 1)^2 - 4q^2 m^2 \right| \geq \frac{m_c}{q^2} > 0.$$ 

Hence for any $(h_1, h_2) \in (H^2(\Omega) \cap W) \times (H^2(\Omega) \cap W)$,

$$\|f_i\|_V = \|L_1^{-1}(h_1 + h_2)\|_V \leq \frac{m_c}{q^2} \|h_1 + h_2\|_V \leq \frac{m_c}{q^2} (\|h_1\|_V + \|h_2\|_V).$$
Then $f_1 = L_1^{-1}(h_1 + h_2) \in W$. Therefore $v_0 - f_0 - \varepsilon f_1 \in V$. Since $v_0 - f_0 - \varepsilon f_1 > 0$ for $|x| < \pi/2$ and $v_0$ satisfies (2.6), we have

$$A v_0 + K [v_0 - f_0 - \varepsilon f_1] = m_b g + \varepsilon h_2.$$ 

Note that $1 \in U = (I - P) H$ and $h_2 \in W$. Let $z = P v_0$; then $z$ satisfies

$$A z + K z = 0.$$ 

Under assumptions (3.1) and $-\sigma_{10} < K < -\sigma_{20}$, we then have $z = 0$. Thus $v_0 \in W$, and hence $v_0 = y(0)$ by Lemma 3.2.

4. Multiple nonlinear oscillations

In this section, we apply the Mountain Pass Theorem due to Ambrosetti and Rabinowitz [18] to $\tilde{J}_K(v)$ on $V_0$ to show that $\tilde{J}_K$ has at least three critical points in $V_0$. Hence, by Lemmas 3.1 and 3.3, Eq. (2.6) admits at least three solutions in $V$.

The Mountain Pass Theorem [18,19] has been used to prove the existence of critical points of functionals satisfying a condition called the Palais–Smale (PS) condition, which occurs repeatedly in the critical point theory. We say that a functional $J$ satisfies the (PS) condition if any sequence $\{v_n\}$ for which $J(v_n)$ is bounded and $\frac{d}{d\lambda} J(v_{\lambda}) \to 0$ possesses a convergent subsequence.

**Mountain Pass Theorem.** Let $E$ be a real Banach space. $I \in C^1(E, \mathbb{R})$ satisfies the (PS) condition. Suppose

(a) there are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho(0)} \geq I(0) + \alpha$, where $B_\rho(0) = \{u \in E \mid \|u\|_E \leq \rho\}$; and

(b) there is an $e \in E \setminus \overline{B}_\rho$ such that $I(e) \leq I(0)$.

Then $I$ possesses a critical value $c \geq \alpha$. Moreover, $c$ can be characterized as

$$c = \inf_{g \in \Gamma} \max_{u \in g([0, 1])} I(u),$$

where $\Gamma = \{g \in C([0, 1], E) \mid g(0) = 0, g(1) = e\}$.

In the following several lemmas, we show that $\tilde{J}_K(v)$ satisfies all conditions in the Mountain Pass Theorem.

**Lemma 4.1.** Assume $-\sigma_{10} < K < -\sigma_{20}$ and $(h_1, h_2) \in (H^2(\Omega) \cap W) \times (H^2(\Omega) \cap W)$. If $|\varepsilon| < \varepsilon_0$, where $\varepsilon_0 > 0$ is a constant defined in Theorem 2.2, then $v = 0$ is a strict local minimum of $\tilde{J}_K(v)$ in $V_0$. 
Proof. If $|\varepsilon| < \varepsilon_0$, then $v_0 = y(0)$ is a solution of (3.4) with $v = 0$, satisfying $v_0 - f_0 - \varepsilon f_1 \in W$, $(v_0 - f_0 - \varepsilon f_1)(x, t) > 0$ for $|x| < \pi/2$, $(v_0 - f_0 - \varepsilon f_2)(\pm \pi/2, t) > 0$, $(v_0 - f_0 - \varepsilon f_2)(\pi/2, t) < 0$, and $(v_0 - f_0 - \varepsilon f_2)(\pm \pi/2, t) = 0$ for $|t| \leq \pi/2$. Then there exists a small neighborhood $B$ of 0 in $V_0$ such that for any $v \in B$, one has $(v + v_0 - f_0 - \varepsilon f_1)(x, t) > 0$ for $|x| < \pi/2$ and $(v + v_0 - f_0 - \varepsilon f_2)(\pm \pi/2, t) = 0$ for $|t| \leq \pi/2$. By Lemma 3.2, $y(0) = v_0$ is the solution of (3.4) for any $v \in B$. Hence $y(v) = y(0)$ for any $v \in B$. Let $y = y(0)$; then for any $v \in B$,

$$
\tilde{J}_K(v) = \int_\Omega \left[ \frac{1}{2} A\{v + y\}(v + y) + \frac{K}{2} (v + f_0 - \varepsilon f_1)^2 \right] \, dx \, dt
$$

$$
- m_y^2 (v + y) - \varepsilon h_2 (v + y) \right] \, dx \, dt
$$

$$
= \int_\Omega \left[ \frac{1}{2} (Ay)y + \frac{K}{2} y \right] \, dx \, dt
$$

$$
+ \int \left[ Ay + K (y - f_0 - \varepsilon f_1) - m_y^2 y - \varepsilon h_2 \right] \, dx \, dt
$$

$$
+ \frac{1}{2} \int \left[ (Av)v + Kv^2 \right] \, dx \, dt
$$

$$
= \tilde{J}_K(0) + \frac{1}{2} \int \left[ (Av)v + Kv^2 \right] \, dx \, dt.
$$

Note that $Av = \sigma_{10} v$, thus

$$
\tilde{J}_K(v) - \tilde{J}_K(0) = \frac{K + \sigma_{10}}{2} \int_\Omega v^2 \, dx \, dt.
$$

Since $-\sigma_{10} < K < -\sigma_{20}$, $v = 0$ is a strict local minimum of $\tilde{J}_K(v)$ in $V_0$. \qed

In order to prove that $\tilde{J}_K(v)$ satisfies the (PS) condition, we need the following lemma which was proved in [14].

**Lemma 4.2.** If $-\sigma_{00} < K < -\sigma_{20}$, then the equation

$$
Av + Kv^+ = 0
$$

(4.1)

admits only the trivial solution $v = 0$ in $H$.

**Lemma 4.3.** Let $(h_1, h_2) \in (H^2(\Omega) \cap W) \times (H^2(\Omega) \cap W)$ and $|\varepsilon| \leq 1$. If $-\sigma_{10} < K < -\sigma_{20}$, then $J_K$ satisfies the (PS) condition in $V_0$. 

Proof. Assuming \( \{v_n\} \subset V_0 \) such that \( \tilde{J}_K(v_n) \) is bounded and \( \tilde{J}'_K(v_n) \to 0 \), we need to show that \( \{v_n\} \) has a convergent subsequence in \( V_0 \). Since \( V_0 \) is a two-dimensional linear space, we only need to show that \( \{v_n\} \) is bounded in \( V_0 \).

Assume the contrary, \( \|v_n\| \to \infty \) as \( n \to \infty \). Let \( y(v_n) \) be the solution of (3.4) with \( v = v_n \) and \( z_n = v_n + y(v_n) \). \( \tilde{J}'_K(v_n) \to 0 \) and (3.4) imply \( J'_K(z_n) = J'_K(v_n + y(v_n)) \to 0 \); i.e.,

\[
A\bar{z}_n + K[\bar{z}_n - f_0 - \epsilon f_1]^+ - m_b g - \epsilon h_2 \to 0.
\]

Since \( v_n \) is orthogonal to \( y(v_n) \), we have \( \|z_n\| \geq \|v_n\| \). Thus \( z_n \to \infty \). Let \( \bar{z}_n = z_n/\|z_n\| \); then

\[
A\bar{z}_n + K\left[\bar{z}_n - \frac{f_0}{\|z_n\|} - \epsilon \frac{f_1}{\|z_n\|}\right]^+ - \frac{m_b g}{\|z_n\|} - \epsilon \frac{h_2}{\|z_n\|} \to 0. \tag{4.2}
\]

Since \( \{\bar{z}_n\} \) is bounded in \( H \), there exists a subsequence of \( \{\bar{z}_n\} \), denoted again by \( \{\bar{z}_n\} \), such that

\[
\lim_{n \to \infty} \bar{z}_n = z_0 \quad \text{weakly in } H.
\]

By Lemma 2.1, \( A^{-1} \) is compact in \( H \). Thus by applying \( A^{-1} \) to (4.2), we have that \( \bar{z}_n \) converges to \( z_0 \) strongly in \( H \). (4.2) implies

\[
A\bar{z}_0 + K[z_0]^+ = 0.
\]

By Lemma 4.2, we have \( z_0 = 0 \) which contradicts to \( \|z_0\| = 1 \). Therefore \( \{v_n\} \) is bounded in \( V_0 \). \( \square \)

In order to show that \( \tilde{J}_K \) satisfies condition (b) in the Mountain Pass Theorem, we define a functional \( I \) on \( V \) by

\[
I(v) = \int_\Omega \left( \frac{1}{2} (Av)^2 + \frac{K}{2} (v^+)^2 \right) \, dx \, dt.
\]

Note that \( I \) is a special case of \( J_K \) defined in (3.3). By Lemma 3.1, critical points of \( I \) in \( V \) correspond to solutions of the following equation in \( V \):

\[
Av + Kv^+ = 0.
\]

By Lemma 4.2, this equation admits only the trivial solution \( v = 0 \) in \( V \) when \( -\sigma_{00} < K < -\sigma_{20} \). Hence \( I \) has only one critical point \( v = 0 \) in \( V \). Let \( -\sigma_{00} < K < -\sigma_{20} \). For any given \( v \in V_0 \), consider the following equation:

\[
A\bar{z} + K(I - P)(v + \bar{z})^+ = 0. \tag{4.3}
\]

Note that (4.3) is a special case of (3.4). By Lemma 3.2, (4.3) admits a unique solution in \( W \), denoted by \( z = z(v) \). For any \( v \in V_0 \), define \( \bar{I}(v) = I(v + z(v)) \).

By repeating the same argument as that in the proof of Lemma 3.3, we obtain that \( \bar{I} \) has only one critical point \( v = 0 \) in \( V_0 \) if \( -\sigma_{00} < K < -\sigma_{20} \). Furthermore, we have the following lemma.
Lemma 4.4. Let $-\sigma_{10} < K < -\sigma_{20}$. Then $\tilde{I}(v) < 0$ for any $v \in V_0$ and $v \neq 0$.

Proof. For any $c > 0$ and $v \in V_0$, it is easy to show that $z(cv) = cz(v)$ and $\tilde{I}(cv) = c^2 \tilde{I}(v)$. Thus we only need to show that $\tilde{I}(v) < 0$ for any $v \in V_0$ and $\|v\|_V = 1$.

For any $v \in V_0$ and $0 \leq s \leq 1$, consider the following equation:

$$A_z + sK(I - P)[v + z]^+ = 0. \quad (4.4)$$

Note that $0 \leq sK \leq -\sigma_{20}$. By Lemma 3.2, (4.4) admits a unique solution in $W$, denoted by $z = z(s, v)$. Note that $z(1, v) = z(v)$ for any $v \in V_0$ and $z(s, 0) = 0$ for $0 \leq s \leq 1$. Let $\delta$ and $\gamma$ be defined as in the proof of Lemma 3.2. For any $v \in V_0$ and $\|v\| \leq 1$,

$$\|z(s, v)\| = \|(A + \delta)^{-1}(I - P)(-sK[z(s, v) + v]^+ + \delta(z(s, v) + v))\|$$

$$\leq \frac{2}{\sigma_{00} - \sigma_{20}} \|\delta - sK([z(s, v) + v]^+ + \delta[z(s, v) + v]^+)\|$$

$$\leq \gamma(\|z(s, v)\| + \|v\|).$$

Thus $\|z(s, v)\| \leq \gamma/(1 - \gamma)$ for any $v \in V_0$ and $\|v\| \leq 1$. For any $0 \leq s_1, s_2 \leq 1$ and for any $v_1, v_2 \in V_0$ with $\|v_1\| \leq 1$ and $\|v_2\| \leq 1$, let $z_1 = z(s_1, v_1)$ and $z_2 = z(s_2, v_2)$. Then

$$\|z_1 - z_2\| \leq \frac{2}{\sigma_{00} - \sigma_{20}} \|s_1 - s_2\| \|z_1 + v_1|^+$$

$$+ (\delta - s_2 K)([z_1 + v_1]^+ - [z_2 + v_2]^+)$$

$$- \delta([z_1 + v_1]^+ - [z_2 + v_2]^+)\|$$

$$\leq \frac{2K}{(\sigma_{00} - \sigma_{20})(1 - \gamma)}|s_1 - s_2| + \gamma(\|z_1 - z_2\| + \|v_1 - v_2\|),$$

where we have used $\|z(s, v)\| \leq \gamma/(1 - \gamma)$ and $|a_1^+ - a_2^+| + |a_1^- - a_2^-| \leq |a_1 - a_2|$ in deriving the above second inequality. Thus

$$\|z_1 - z_2\| \leq \frac{1}{1 - \gamma}\left[\frac{2K}{(\sigma_{00} - \sigma_{20})(1 - \gamma)}|s_1 - s_2| + \gamma\|v_1 - v_2\|\right].$$

Therefore $z(s, v)$ is continuous in $s$ and $v$.

For any $0 \leq s \leq 1$, define a functional $\hat{I}_s$ on $V_0$ by

$$\hat{I}_s(v) = \int_{\Omega} \left(\frac{1}{2}A(z(s, v) + v)(z(s, v) + v) + \frac{sK}{2}(z(s, v) + v)^+\right) dx \; dt,$$

where $z(s, v)$ is the unique solution of (4.4). Note that $\hat{I}_1(v) = \tilde{I}(v)$. By using (4.4) and the orthogonality between $V_0$ and $W$, one obtains

$$\hat{I}_s(v) = \frac{\sigma_{10}}{2} \int_{\Omega} v^2 dx \; dt + \frac{sK}{2} \int_{\Omega} v[z(s, v) + v]^+ dx \; dt.$$
Note that \( \tilde{I}_0(v) < 0 \) for any \( v \in V_0 \) with \( \|v\| = 1 \). Since \( z(s,v) \) is continuous in \( s \) and \( v \), \( \tilde{I}_s(v) \) is continuous in \( s \) and \( v \). For each \( v \in V_0 \) with \( \|v\| = 1 \), define
\[
t(v) = \sup \{ t \mid \tilde{I}_s(v) \leq 0, \ s \in [0,t] \}.
\]
Then \( 0 < t(v) \leq 1 \). Since \( \tilde{I}_s(v) \) is continuous in \( s \) and \( v \), \( t(v) \) is a continuous function of \( v \) in \( V_0 \). Define
\[
t^* = \inf \{ t(v) \mid v \in V_0, \ \|v\| = 1 \}.
\]
There are only three possible cases for \( t^* \): (i) \( t^* = 0 \); (ii) \( 0 < t^* < 1 \); and (iii) \( t^* = 1 \). For case (i), there exists a sequence \( \{v_n\} \subset V_0 \) with \( \|v_n\| = 1 \) such that \( t(v_n) \to 0 \). Since \( V_0 \) is a two-dimensional space, there exists a subsequence of \( \{v_n\} \), denoted again by itself, such that \( v_n \to v^* \in V_0 \) and \( \|v^*\| = 1 \). Thus
\[
t(v^*) = 0 \quad \text{which contradicts to the fact that} \quad t(v) > 0 \quad \text{for any} \quad v \in V_0 \quad \text{with} \quad \|v\| = 1.
\]
Therefore case (i) is impossible. For case (ii), it follows from a similar argument as in case (i) that there is a \( v^* \in V_0 \) with \( \|v^*\| = 1 \) such that \( 0 < t(v^*) = t^* < 1 \). Then \( \tilde{I}_{t^*} (v^*) = 0 \) and \( \tilde{I}_{t^*} (v) \leq 0 \) for any \( v \in V_0 \) with \( \|v\| = 1 \). Since \( \tilde{I}_{t^*} (cv) = c^2 \tilde{I}_{t^*} (v) \), \( v^* \) is a maximum point of \( \tilde{I}_{t^*} (v) \) on \( V_0 \). Thus \( \tilde{I}_{t^*} (v^*) v = 0 \) for any \( v \in V_0 \); that is,
\[
\int_\Omega \left[ A(z(t^*, v^*) + v^*) + t^* K [z(t^*, v^*) + v^*]^+ \right] v \, dx \, dt = 0, \quad \forall v \in V_0.
\]
Thus
\[
A v^* + t^* K P [z(t^*, v^*) + v^*]^+ = 0.
\]
Note that
\[
A z(t^*, v^*) + t^* K (I - P) [z(t^*, v^*) + v^*]^+ = 0.
\]
Therefore
\[
A (z(t^*, v^*) + v^*) + t^* K [z(t^*, v^*) + v^*]^+ = 0.
\]
By Lemma 4.2, we then obtain \( z(t^*, v^*) + v^* = 0 \). Since \( z(t^*, v^*) \in W \) and \( v^* \in V_0 \), we have \( v^* = 0 \) which contradicts to \( \|v^*\| = 1 \). Thus case (ii) is also impossible. Hence case (iii) is true. Then \( 1 \geq t(v) \geq t^* = 1 \). Therefore \( \tilde{I} (v) = \tilde{I}_1 (v) \leq 0 \) for any \( v \in V_0 \). Since \( v = 0 \) is the only critical point of \( \tilde{I} (v) \) on \( V_0 \), we have \( \tilde{I} (v) < 0 \) for any \( v \in V_0 \) with \( v \neq 0 \). \( \square \)

**Lemma 4.5.** Let \( -\sigma_{10} < K < -\sigma_{20} \) and \( v \in V_0 \) with \( \|v\| = 1 \). Then
\[
\lim_{c \to \infty} \tilde{J}_K (cv) = -\infty.
\]

**Proof.** Suppose that \( \lim_{c \to \infty} \tilde{J}_K (cv) = -\infty \) is not true. Then there exists a sequence \( \{c_n\} \subset \mathbb{R}_+ \) and a number \( b \) such that \( c_n \to \infty \) and
\[
\tilde{J}_K (c_n v) = J_K (c_n v + y_n) \geq b, \quad (4.5)
\]
where $J_K$ is defined by (3.3), and $y_n$ denotes, by Lemma 3.2, the unique solution of
\[ Ay_n + (I - P)(K[c_n v + y_n - f_0 - \varepsilon f_1] + m bg - \varepsilon h_2) = 0. \] (4.6)

Let $z_n = y_n/c_n$. By Lemma 3.2, $\{z_n\}$ is a bounded sequence in $H$. Hence there is a weakly convergent subsequence of $\{z_n\}$ in $H$, denoted again by itself, and $z^* \in H$ such that $\{z_n\}$ converges to $z^*$ weakly in $H$. Rewrite Eq. (4.6) as
\[ z_n = -A^{-1}(I - P)\left(K\left[v + z_n - \frac{f_0 + \varepsilon f_1}{c_n}\right]^+ - \frac{m bg + \varepsilon h_2}{c_n}\right). \] (4.7)

Since $A^{-1}$ is a compact operator in $H$ and $c_n \to \infty$, (4.7) implies $\{z_n\}$ converges to $z^*$ in $H$. By applying Lemma 2.1, we obtain that $z^* \in V$ satisfies (4.3) and $\{z_n\}$ converges to $z^*$ strongly in $V$. By using (3.3) and by dividing inequality (4.5) by $c_n^2$, we have
\[ \int_\Omega \left[ \frac{1}{2} A(v + z_n)(v + z_n) + \frac{K}{2} \left(\left[v + z_n - \frac{f_0 + \varepsilon f_1}{c_n}\right]^+\right)^2 \right. \\
- \left( v + z_n \right) \frac{m bg + \varepsilon h_2}{c_n} \bigg] \, dx \, dt \geq \frac{b}{c_n^2}. \]

By letting $n \to \infty$ in the above inequality, we then obtain
\[ \int_\Omega \left[ \frac{1}{2} A(v + z^*)(v + z^*) + \frac{K}{2} \left(\left[v + z^*\right]^+\right)^2 \right] \, dx \, dt \geq 0. \]

Note that $z^*$ satisfies (4.3). By the definitions of $I(v)$ and $\tilde{I}(v)$, the above inequality is exactly $\tilde{I}(v) \geq 0$, which contradicts to Lemma 4.4. Thus $\lim_{c \to \infty} \tilde{J}_K(cv) = -\infty$. □

By using Lemmas 4.1, 4.3 and 4.5, we are ready to apply the Mountain Pass Theorem to $\tilde{J}_K$ and to prove the following important theorem.

**Theorem 4.1.** Let $-\sigma_{10} < K < -\sigma_{20}$ and $(h_1, h_2) \in (H^2(\Omega) \cap W) \times (H^2(\Omega) \cap W)$. Let $\varepsilon_0 > 0$ be a constant defined in Theorem 2.2. If $|\varepsilon| < \varepsilon_0$, then $\tilde{J}_K(v)$ has at least three critical points in $V_0$.

**Proof.** By Lemma 4.1, if $|\varepsilon| < \varepsilon_0$, where $\varepsilon_0 > 0$ is a constant defined in Theorem 2.2, then there is a small neighborhood $B$ of 0 in $V_0$ such that $v = 0$ is a strict local minimum of $\tilde{J}_K$ in $B$. By Lemma 4.5, $\lim_{c \to \infty} \tilde{J}_K(cv) = -\infty$ for any $v \in V_0$ with $\|v\| = 1$. Since $V_0$ is a two-dimensional space, one has $\lim_{\|v\| \to \infty} \tilde{J}_K(v) = -\infty$ in $V_0$. Thus $\tilde{J}_K$ has a critical point $v_1$ in $V_0$ such that
\[ \tilde{J}_K(v_1) = \sup_{v \in V_0} \tilde{J}_K(v). \] (4.8)
Let $N$ be a small neighborhood of $v_1$ in $V_0$ such that $N \cap B = \emptyset$. Choose a $v_0 \in V_0 \setminus (N \cup B)$ such that $\tilde{J}_K(v_0) = 0$. By Lemma 4.3, $\tilde{J}_K$ satisfies the (PS) condition. Thus $\tilde{J}_K$ satisfies all conditions in the Mountain Pass Theorem. Then $\tilde{J}_K$ has a critical value $c > \tilde{J}_K(0)$ such that

$$c = \inf_{\gamma \in \Gamma} \sup_{v \in \gamma([0,1])} \tilde{J}_K(v),$$  

(4.9)

where $\Gamma = \left\{ \gamma \in \mathcal{C}([0,1], V_0) \mid \gamma(0) = 0, \gamma(1) = v_0 \right\}$.

If $\tilde{J}_K(v_1) \neq c$, then there exists a critical point $v_2 \in V_0$ of $\tilde{J}_K$ such that $\tilde{J}_K(v_2) = c$. Therefore $\tilde{J}_K$ has at least three critical points in $V_0$: $0$, $v_1$, and $v_2$. If $\tilde{J}_K(v_1) = c$, then we will show that there exists a critical point $v_2 \in V_0$ of $\tilde{J}_K$ such that $v_2 \neq v_1$ and $\tilde{J}_K(v_2) = c$. Assume by the contradiction $K_c \equiv \{ v \in V_0 \mid \tilde{J}_K(v) = c, \tilde{J}_K(v) = 0 \} = \{ v_1 \}$. Let $A_s = \{ v \in V_0 \mid \tilde{J}_K(v) \leq s \}$ and let $\bar{\epsilon} = c - \tilde{J}_K(0) > 0$ be given. By the Deformation Theorem [19, Theorem A.4], there exists an $\epsilon \in (0, \bar{\epsilon})$ and $\eta \in \mathcal{C}([0,1] \times V_0, V_0)$ such that

(i) $\eta(t, v) = v$ for all $t \in [0,1]$ if $\tilde{J}_K(v) \notin [c - \epsilon, c + \epsilon]$; and

(ii) $\eta(1, A_{c+\epsilon} \setminus N) \subset A_{c-\epsilon}$.

Choose a $g \in \Gamma'$ such that $g([0,1]) \cap N = \emptyset$. Then $\eta(1, \cdot) \circ g \in \Gamma$. By (4.8), we have $g([0,1]) \subset A_{c+\epsilon} \setminus N$. Then $\eta(1, \cdot) \circ g([0,1]) \subset A_{c-\epsilon}$. Thus

$$c = \inf_{\gamma \in \Gamma} \sup_{v \in \gamma([0,1])} \tilde{J}_K(v) \leq \sup_{v \in \eta(1, \cdot) \circ g([0,1])} \tilde{J}_K(v) \leq c - \epsilon,$$

which is a contradiction. Therefore, if $\tilde{J}_K(v_1) = c$, then there exists a critical point $v_2 \in V_0$ of $\tilde{J}_K$ such that $v_2 \neq v_1$ and $\tilde{J}_K(v_2) = c$. Hence, $\tilde{J}_K$ has at least three critical points in $V_0$: $0$, $v_1$, and $v_2$. $\square$

By Lemmas 3.1 and 3.3, and by Theorem 4.1, we have the following theorem.

**Theorem 4.2.** Assume assumptions (2.1) and (2.3) hold. If $-\sigma_{10} < K < -\sigma_{20}$ and $(h_1, h_2) \in (H^2(\Omega) \cap W) \times (H^2(\Omega) \cap W)$, then there exists $\epsilon_0 > 0$ such that Eq. (2.6) admits at least three solutions in $V$ when $|\epsilon| < \epsilon_0$.

By Lemma 2.1, (2.7) and (2.8), and by applying the bootstrapping technique, we obtain the following main theorem from Theorem 4.2.

**Theorem 4.3.** Assume assumptions (2.1) and (2.3) hold. If $-\sigma_{10} < K < -\sigma_{20}$ and $(h_1, h_2) \in (H^2(\Omega) \cap W) \times (H^2(\Omega) \cap W)$, then there exists $\epsilon_0 > 0$ such that the suspension bridge system (1.3) admits at least three nonlinear periodic oscillations $(w, u)$ in $(H^2(\Omega) \cap H) \times (H^3(\Omega) \cap H)$ when $|\epsilon| < \epsilon_0$.

As a final remark, we wish to point out that assumption (2.1), which is the starting point of this work, can be relaxed certainly. However, a relaxation of
(2.1) will cause some tedious calculations particularly in proving Lemma 4.2, which plays a key role in proving Lemma 4.3 that $\tilde{J}_K$ satisfies the Palais–Smale condition. Such a relaxation will create no essential difficulty and difference in the work presented here.

References