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Boundary Layers on Sobolev–Besov Spaces and Poisson's Equation for the Laplacian in Lipschitz Domains

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We study inhomogeneous boundary value problems for the Laplacian in arbitrary Lipschitz domains with data in Sobolev–Besov spaces. As such, this is a natural continuation of work in [Jerison and Kenig, *J. Funct. Anal.* (1995), 16–219] where the inhomogeneous Dirichlet problem is treated via harmonic measure techniques. The novelty of our approach resides in the systematic use of boundary integral methods. In this regard, the key results are establishing the invertibility of the classical layer potential operators on scales of Sobolev–Besov spaces on Lipschitz boundaries for optimal ranges of indices. Applications to L^p-based Helmholtz type decompositions of vector fields in Lipschitz domains are also presented. © 1998 Academic Press

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1. INTRODUCTION

In the class of smooth domains $\Omega \subseteq \mathbb{R}^n$, the Poisson equation for the Laplacian

$$\Delta u = f \qquad \text{in } \Omega \tag{0.1}$$

with Dirichlet boundary conditions is well understood for data in Sobolev–Besov spaces. Problems as such are regular elliptic and a general approach,

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based on pseudodifferential operators and Calderón–Zygmund theory, can be found in [AgDoNi]. In particular, if $f \in L^p_{s+(1/p)-2}(\Omega)$ then (0.1) has a unique solution $u \in L^p_{s+1/p,\,0}(\Omega)$ for any $1 < p < \infty$ and any $s > 0$ (for definitions see Section 1).

The situation is radically different in less smooth domains. For instance, Dahlberg [Dah] has constructed a C^1-domain Ω and $f \in C^\infty(\bar{\Omega})$ such that the unique variational solution $u \in L^2_{1,\,0}(\Omega)$ to (0.1) has $\partial^2 u/\partial x_j\,\partial x_k \notin L^p(\Omega)$ for any $1 < p < \infty$, $j, k = 1, 2, ..., n$.

Turning to positive results, fairly recently Jerison and Kenig [JeKe] have been able to identify the optimal range of solvability of the Poisson equation with homogeneous Dirichlet boundary conditions on scales of Sobolev–Besov spaces for arbitrary Lipschitz domains. One of their main results is as follows. Given $1 \geqslant \varepsilon > 0$, consider the following three conditions for the parameters s, p:

(1) $2/(1+\varepsilon) < p < 2/(1-\varepsilon)$ and $0 < s < 1$;

(2) $1 < p \leqslant 2(1+\varepsilon)$ and $(2/p) - 1 - \varepsilon < s < 1$;

(3) $2/(1-\varepsilon) \leqslant p < \infty$ and $0 < s < (2/p) + \varepsilon$.

Then, for any bounded Lipschitz domain Ω, there exists $\varepsilon = \varepsilon(\Omega) \in (0, 1]$ such that (0.1) has a unique solution $u \in L^p_{s+1/p,\,0}(\Omega)$ for any $f \in L^p_{s+(1/p)-2}(\Omega)$, granted that s, p satisfy one of the conditions (1)–(3) listed above.

It should be noted, however, that the approach in [JeKe] makes essential use of fine estimates for the harmonic measure in Ω and, hence, does not extend to treating similar problems for Neumann type boundary conditions or systems of equations. In fact, the former issue is also singled out as an open problem in Kenig's book [Ke; cf. Problem 3.2.21].

The major aim of this paper is to develop an alternative approach to these (and related) problems in which the main emphasis falls on the functional analytic properties of boundary layer potential operators on scales of Sobolev–Besov spaces. This is a unified approach which does not differentiate, in principle, between Dirichlet and Neumann type boundary conditions or between a single equation and a system of equations.

At the level of boundary operators, we prove that if

$$Kf(P) := \lim_{\varepsilon \to 0} \frac{1}{\omega_n} \int_{|P-Q| > \varepsilon} \frac{\langle Q-P, N(Q) \rangle}{|P-Q|^n} f(Q)\, d\sigma(Q), \qquad P \in \partial\Omega, \qquad (0.2)$$

denotes the usual double layer potential operator on $\partial\Omega$, and if $B^p_s(\partial\Omega)$, $1 \leqslant p \leqslant \infty$, $0 < s < 1$, stands for the scale of Besov spaces on the Lipschitz manifold $\partial\Omega$, then there exists $\varepsilon > 0$ depending on $\partial\Omega$ such that

$$\tfrac{1}{2}I + K \colon B^p_s(\partial\Omega) \to B^p_s(\partial\Omega) \qquad \text{is an isomorphism} \qquad (0.3)$$

for any s, p satisfying any of the conditions (1)–(3) above. This result is sharp in the class of Lipschitz domains (but ε can be taken to be 1 if $\partial\Omega \in C^1$). In particular, it allows us to recover the main result in [JeKe] and to augment it with an integral representation formula.

Dualizing (0.3) allows for a complete solution to the Poisson equation with Neumann boundary conditions in arbitrary Lipschitz domains for the "natural" range of indices. More concretely, we show that for any bounded Lipschitz domain Ω, the boundary problem

$$\begin{cases} \Delta u = f \in L^q_{(1/q)-s-1,\,0}(\Omega), \\ \dfrac{\partial u}{\partial N} = g \in B^q_{-s}(\partial\Omega), \\ u \in L^q_{1-s+(1/q)}(\Omega), \end{cases} \tag{0.4}$$

subject to the compatibility condition $\langle f, 1\rangle = \langle g, 1\rangle$, has a unique (modulo additive constants) solution if, with p standing for the Hölder conjugate exponent of q, the pair s, p satisfies one of the conditions (1)–(3) for some $\varepsilon = \varepsilon(\Omega) > 0$. In the class of Lipschitz domains this range is in the nature of best possible.

It is important to point out that, as will transpire from our analysis, there is no immediate impediment to extending our approach to other cases of interests like, for instance, the three-dimensional Lamé system or the heat operator. However, the discussion of these themes is postponed for a separate occasion. One important application which we do present here deals with Helmholtz type decompositions of vector fields in Lipschitz domains.

Specifically, given a vector field u (with components) in $L^p(\Omega)$, one would like to find a divergence-free vector field $v \in L^p(\Omega)$ with vanishing normal component and a scalar-valued function $\psi \in L^p_1(\Omega)$, such that

$$u = \nabla\psi + v \qquad \text{in } \Omega \tag{0.5}$$

and such that the estimate

$$\|v\|_{L^p(\Omega)} + \|\nabla\psi\|_{L^p(\Omega)} \leqslant C\,\|u\|_{L^p(\Omega)} \tag{0.6}$$

holds for a positive constant $C = C(\Omega, p)$. It is known (cf. [FuMo]) that if $\partial\Omega$ is smooth then the Helmholtz decomposition (0.5)–(0.6) holds for any $1 < p < \infty$. Also, even if $\partial\Omega$ is Lipschitz, the case $p = 2$ easily follows from well known variational techniques. Nonetheless, in the non-smooth context, the case $p \neq 2$ is more subtle and the question of investigating the validity of (0.5)–(0.6) in this setting has been posed to us by M. E. Taylor. Based on our solution to the Poisson equation with Neumann boundary conditions, we are able to show that for a given Lipschitz domain the

Helmholtz decomposition holds for $\frac{3}{2}-\varepsilon<p<3+\varepsilon$ for some $\varepsilon=\varepsilon(\Omega)>0$. By means of counterexamples we show that this range is sharp in the class of Lipschitz domains.

The layout of the paper is as follows. Section 1 contains basic terminology, notation, and results. Mapping properties of the Newtonian potential operator are studied in Section 2, while a similar analysis is carried out for the single and double layer potential operators in Sections 3 and 4, respectively. Section 5 contains a discussion of the Banach envelope of a quasi-Banach space which is primarily applied to the atomic Hardy spaces $H^p(\partial\Omega)$ in the sequel. Two endpoint boundary problems are treated separately in Sections 6–7 and general invertibility results for boundary layer operators are proved in Section 8. The Poisson problem with Neumann and Dirichlet boundary conditions are formulated and solved in Sections 9 and 10, while Helmholtz type decompositions are established in Section 11. Finally, certain counterexamples are discussed in Section 12.

PRELIMINARIES

In this section we summarize basic concepts, notation, and results that will be used throughout the paper.

By a bounded Lipschitz domain in $\mathbb{R}^n$ we mean a bounded domain $\Omega\subseteq\mathbb{R}^n$ such that for every $P\in\partial\Omega$ one can find a system of coordinates (isometric to the original one) with origin at P, a cylinder $Z(P,r)=\{x'\in\mathbb{R}^{n-1};\, |x'|<r\}\times(-r,r)$, and a Lipschitz function $\varphi\colon \mathbb{R}^{n-1}\to\mathbb{R}$ so that $\Omega\cap Z(P,r)=\{(x,y)\colon y>\varphi(x)\}\cap Z(P,r)$ and $\partial\Omega\cap Z(P,r)=\{(x,\varphi(x));\, x\in\mathbb{R}^{n-1}\}\cap Z(P,r)$. We also set $\Phi\colon \{x'\in\mathbb{R}^{n-1};\; |x'|<r/2\}\to\partial\Omega\cap Z(P,r)$ by $\Phi(x'):=(x',\varphi(x'))$. Corresponding to a fixed, finite open covering of $\partial\Omega$ by coordinate cylinders as described above, $(Z_i)_{i\in I}$, we select a partition of unity $(\xi_i)_{i\in I}$ subordinated to it and denote the coordinate functions by $(\Phi_i)_{i\in I}$. We also denote by $d\sigma$ the canonical surface measure on $\partial\Omega$ and by N the outward unit normal defined $d\sigma$-a.e. on $\partial\Omega$. Unless otherwise specified, we shall always assume that Ω has a connected boundary.

For $1\leqslant p<\infty$ and $0<s<1$, we define the Besov space $B^p_s(\partial\Omega)$ as the collection of all measurable functions f on $\partial\Omega$ such that

$$\|f\|_{B^p_s(\partial\Omega)}=:\|f\|_{L^p(\partial\Omega)}+\left(\int_{\partial\Omega}\int_{\partial\Omega}\frac{|f(P)-f(Q)|^p}{|P-Q|^{n-1+sp}}\,d\sigma(P)\,d\sigma(Q)\right)^{1/p}<\infty. \tag{1.1}$$

The case $p=\infty$ corresponds to the non-homogeneous version of the space of Hölder continuous functions on $\partial\Omega$. More precisely, $B^\infty_s(\partial\Omega)$, $0<s\leqslant 1$, is defined as the Banach space of measurable functions on f such that

$$\|f\|_{B^\infty_s(\partial\Omega)} := \|f\|_{L^\infty(\partial\Omega)} + \sum_{\substack{P,\,Q\in\partial\Omega\\ P\neq Q}} \frac{|f(P)-f(Q)|}{|P-Q|^s} < \infty. \tag{1.2}$$

Occasionally, we shall write $\mathrm{Lip}(\partial\Omega)$ for $B^\infty_1(\partial\Omega)$. Also, recall that $B^p_{-s}(\partial\Omega) := (B^q_s(\partial\Omega))^*$ for each $0<s<1$, $1<p\leqslant\infty$ and $q=(1-1/p)^{-1}$.

The following lemma allows for transferring many results about Besov spaces originally proved in the flat, Euclidean setting to the case when the underlying manifold is the boundary of a Lipschitz domain.

Lemma 1.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and let $(\Phi_i)_{i\in I}$, $(\xi_i)_{i\in I}$ be as before. Also, for a measurable function f on $\partial\Omega$ denote by F_i the extension by zero outside its support of $(f\xi_i)\circ\Phi_i$, for each $i\in I$. Then, for any $1\leqslant p\leqslant\infty$ and $0<s<1$, the following two statements are equivalent*:

(1) $f\in B^p_s(\partial\Omega)$;

(2) $F_i\in B^p_s(\mathbb{R}^{n-1})$ *for each* $i\in I$.

Moreover, if any of the above conditions holds, then

$$\|f\|_{B^p_s(\partial\Omega)} \approx \sum_{i\in I} \|F_i\|_{B^p_s(\mathbb{R}^{n-1})}. \tag{1.3}$$

To illustrate the point made before the statement of this result we note that from well known interpolation results in $\mathbb{R}^n$ the following can be proved. First, recall that $[\cdot,\cdot]_\theta$ and $(\cdot,\cdot)_{\theta,p}$ denote, respectively, the brackets for the complex and real interpolation methods.

Proposition 1.2. *For $0<\theta<1$, $1\leqslant p_1, p_2\leqslant\infty$, and $0<s_1, s_2<1$, there holds*

$$[B^{p_1}_{s_1}(\partial\Omega), B^{p_2}_{s_2}(\partial\Omega)]_\theta = B^p_s(\partial\Omega) \tag{1.4}$$

where $1/p := (1-\theta)/p_1+\theta/p_2$ and $s := (1-\theta)\,s_1+\theta s_2$. A similar result is valid for $-1<s_1, s_2<0$, and for the real method of interpolation.

Another result of interest for us is an atomic characterization of the Besov space $B^1_s(\partial\Omega)$. First, we shall need a definition. For $0<s<1$, a $B^1_s(\partial\Omega)$-atom is a function $a\in L^\infty_1(\partial\Omega)$ with support contained in a surface ball $B(P,r)\cap\partial\Omega$, $P\in\partial\Omega$, $r>0$, and satisfying the normalization conditions

$$\|a\|_{L^\infty(\partial\Omega)} \leqslant r^{s-n+1}, \qquad \|\nabla_{\tan} a\|_{L^\infty(\partial\Omega)} \leqslant r^{s-n}. \tag{1.5}$$

Here $\nabla_{\tan}$ stands for the tangenital gradient operator on $\partial\Omega$. Now the atomic theory of [FrJa] lifted to $\partial\Omega$ gives the following.

THEOREM 1.3. *Let $0<s<1$ and $f \in B^1_s(\partial\Omega)$. Then there exist a sequence of $B^1_s(\partial\Omega)$-atoms $\{a_k\}_k$ and a sequence of scalars $\{\lambda_k\}_k \in \ell^1$ such that*

$$f = \sum_{k=0}^{\infty} \lambda_k a_k, \tag{1.6}$$

with convergence in $B^1_s(\partial\Omega)$, and

$$\|f\|_{B^1{}_s(\partial\Omega)} \approx \inf\left\{ \sum_{k=0}^{\infty} |\mu_k|;\ f = \sum_{k=0}^{\infty} \mu_k b_k, \quad b_k\text{'s} \right.$$
$$\left. \text{are } B^1_s(\partial\Omega)\text{-atoms},\ (\mu_k)_k \in \ell^1 \right\}. \tag{1.7}$$

In the second part of this section we include a brief discussion of Sobolev–Besov scales of spaces in the interior of a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$. First, for $1 \leqslant p,\ q \leqslant \infty,\ s>0$, the Besov space $B^{p,q}_s(\Omega)$ is defined as the collection of restrictions to Ω of functions from $B^{p,q}_s(\mathbb{R}^n)$ (for the latter see, e.g., [Pe, BeLö, BeSh, Tr, JoWa]). This is equipped with the natural norm defined by taking the infimum of the $\|\cdot\|_{B^{p,q}_s(\Omega)}$-norms of all possible extensions to $\mathbb{R}^n$. The spaces $B^{p,p}_s(\Omega)$ will be abbreviated as $B^p_s(\Omega)$.

Using Stein's extension operator and then invoking well known real interpolation results in $\mathbb{R}^n$ ([BeLö]), it follows that for any bounded Lipschitz domain Ω,

$$(B^{p_1}_{s_1}(\Omega), B^{p_2}_{s_2}(\Omega))_{\theta,p} = B^p_s(\Omega)$$

if $1/p=(1-\theta)/p_1+\theta/p_2,\ \ s=(1-\theta)\,s_1+\theta s_2,\ \ 0<\theta<1,\ \ 1 \leqslant p_1,\ \ p_2 \leqslant \infty$, $s_1 \neq s_2,\ s_1,\ s_2>0$.

A similar discussion applies to the Sobolev spaces $L^p_s(\Omega)$, this time starting with the potential spaces $L^p_s(\mathbb{R}^n)$ (cf., e.g., [St, BeLö]). For $s \in \mathbb{R}$ the space $L^p_{s,0}(\Omega)$ constists of distributions in $L^p_s(\mathbb{R}^n)$ supported in $\bar{\Omega}$ (with the norm inherited from $L^p_s(\mathbb{R}^n)$). It is known that $C^\infty_{\text{comp}}(\Omega)$ is dense in $L^p_{s,0}(\Omega)$ for all values of s and p.

Moreover, if $1<p<\infty$ and $1/p<s<1+1/p$, the space $L^p_{s,0}(\Omega)$ is the kernel of the trace operator Tr acting on $L^p_s(\Omega)$; cf. [JeKe; Proposition 3.3]. In fact, for the same range of indices, $L^p_{s,0}(\Omega)$ is the closure of $C^\infty_{\text{comp}}(\Omega)$ in the $L^p_{s,0}(\Omega)$ norm.

For positive s, $L^p_{-s}(\Omega)$ is defined as the space of linear functionals on test functions in Ω equipped with the norm

$$\|f\|_{L^p_{-s}(\Omega)} := \sup\{|\langle f, g\rangle|;\ g \in C^\infty_{\mathrm{comp}}(\Omega),\ \|\tilde{g}\|_{L^q_s(\mathbb{R}^n)} \leqslant 1\} \tag{1.8}$$

where tilde denotes the extension by zero outside Ω and $1/p + 1/q = 1$. For all values of p and s, $C^\infty(\bar{\Omega})$ is dense in $L^p_s(\Omega)$. Finally, for $s \geqslant 0$,

$$L^q_{-s,0}(\Omega) = (L^p_s(\Omega))^* \qquad \text{and} \qquad L^p_{-s}(\Omega) = (L^q_{s,0}(\Omega))^*. \tag{1.9}$$

See, e.g., [JeKe] for a discussion. In fact, so we claim, the formulas (1.9) are valid for arbitrary $s \in \mathbb{R}$. For example, $L^q_{s,0}(\Omega)$ is reflexive (as a closed subspace of the reflexive Banach space $L^q_s(\mathbb{R}^n)$) and, hence, $(L^p_s(\Omega))^* = (L^q_{-s,0}(\Omega))^{**} = L^q_{-s,0}(\Omega)$ for any $s \leqslant 0$. This proves (the first) half of the claim. The remaining assertion is also easily seen from an application of the Hahn–Banach theorem.

It is well known that $L^p_s(\Omega)$, $L^p_{s,0}(\Omega)$, $L^p_{-s}(\Omega)$, $L^p_{-s,0}(\Omega)$ are complex interpolation scales for positive s and $1 < p < \infty$. Also, the Besov and Sobolev spaces on the domain are related via real interpolation. For instance, we have the formula

$$(L^p(\Omega), L^p_k(\Omega))_{s,q} = B^{p,q}_{sk}(\Omega) \tag{1.10}$$

when $0 < s < 1$, $1 < p < \infty$, and k is a nonnegative integer. A more detailed discussion and further properties of these spaces, as well as proofs for some of the statements in this paragraph can be found in [BeLö, BeSh, JeKe].

We conclude this section with a version of a result proved in [JeKe] regarding global interior estimates for harmonic functions in Sobolev–Besov spaces. Recall that $\nabla^k u$ stands for the vector consisting of all partial derivatives of u of order $\leqslant k$.

THEOREM 1.4. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and denote by $\delta(X)$ the distance from $X \in \Omega$ to $\partial\Omega$. Also, fix $1 \leqslant p \leqslant \infty$, k a nonnegative integer and $0 < s < 1$. For a C^2 function u in Ω consider the following four statements*:

(1) $u \in B^p_{k+s}(\Omega)$;

(2) $\delta^{1-s}\,|\nabla^{k+1} u| + |\nabla^k u| + |u| \in L^p(\Omega)$;

(3) $\delta^{-s}\,|\nabla^k u| + |u| \in L^p(\Omega)$;

(4) $u \in L^p_{k+s}(\Omega)$.

Then, the assertions below are valid.

[i] *For any function u, (2) $\Rightarrow$ (1).*

[ii] *If $0 < s < 1/p < 1$, then (1) $\Rightarrow$ (3) for any u.*

[iii] *If u is harmonic, then* $(3) \Rightarrow (2) \Leftrightarrow (1)$.

[iv] *If u is harmonic and* $1 < p < \infty$, *then* $(1) \Leftrightarrow (4)$.

Moreover, naturally accompanying estimates are valid in each case.

Proof. The assertions [i], [iv] as well as the equivalence in [iii] are proved in [JeKe]. Also, [ii] follows by observing that (1) implies $\nabla^k u \in B^p_s(\Omega)$ and then invoking [Gr, Theorems 1.4.2.4 and 1.4.4.4]. Thus, we are left with proving the first implication in [iii]. To this effect, let u be a harmonic function in Ω which satisfies (3). Then, for each $X \in \Omega$,

$$\delta(X)^{1-s} |\nabla^{k+1} u(X)| \leqslant C\, \delta(X)^{-n-s} \int_{B(X,\, \delta(X)/2)} |\nabla^k u(Y)|\, dY. \tag{1.11}$$

Let χ_E stand for the characteristic function of the set E. Now, on account of the easily verified inequality

$$\chi_{B(X,\, \delta(X)/2)}(Y) \leqslant \chi_{B(Y,\, \delta(Y)/2)}(X), \qquad \forall X, Y \in \Omega, \tag{1.12}$$

we may conclude that

$$\|\delta^{1-s} \nabla^{k+1} u\|^p_{L^p(\Omega)} \leqslant C \int_\Omega \int_\Omega \delta(X)^{-np-sp} |\nabla^k u(Y)|^p \chi_{B(X,\, \delta(X)/2)}(Y)\, dY\, dX$$

$$\leqslant C \int_\Omega |\nabla^k u(Y)|^p\, \delta(Y)^{-sp}\, dY < +\infty,$$

where the next to the last inequality follows immediately by using (1.12) and reversing the order of integration. This conclude the proof of the theorem. ∎

Remark. For another proof of the implication $(1) \Rightarrow (2)$, valid for harmonic functions, as well as for various related results see also [DaDe].

2. THE NEWTONIAN POTENTIAL

Here we study the mapping properties on scales of Sobolev–Besov spaces for the Newtonian potential operator Π: $\mathscr{E}'(\mathbb{R}^n) \to \mathscr{D}'(\mathbb{R}^n)$, given by the convolution with the locally integrable function

$$\Gamma(X) := \frac{1}{(2-n)\, \omega_n\, |X|^{n-2}}, \qquad X \in \mathbb{R}^n \backslash \{0\} \tag{2.1}$$

(where ω_n is the surface area of the unit ball in $\mathbb{R}^n$, $n \geqslant 3$), together with a localized version of it. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and denote by $\mathscr{R}_\Omega$ the operator restricting distributions in $\mathbb{R}^n$ to Ω. Also, let $u \mapsto \tilde{u}$ be its formal transpose, mapping $\mathscr{E}'(\Omega)$ into $\mathscr{E}'(\mathbb{R}^n)$ by

$$\langle \tilde{u}, \psi \rangle = \langle u, \mathscr{R}_\Omega(\psi) \rangle \qquad \text{for any} \quad \psi \in C^\infty(\mathbb{R}^n). \tag{2.2}$$

This extends to a bounded mapping from $(L^p_s(\Omega))^* = L^q_{-s,0}(\Omega)$ into $L^q_{-s}(\mathbb{R}^n) = (L^p_s(\mathbb{R}^n))^*$.

In the sequel, we shall find it convenient to work with the following localized version of the Newtonian potential operator

$$\Pi_\Omega : \mathscr{E}'(\Omega) \to \mathscr{D}'(\Omega), \qquad \Pi_\Omega(u) := \mathscr{R}_\Omega \Pi(\tilde{u}). \tag{2.3}$$

PROPOSITION 2.1. *The linear operator Π_Ω maps $(L^q_{s+1}(\Omega))^*$ boundedly into $L^p_{1-s}(\Omega)$, for all $-1 \leqslant s \leqslant 2$ and $1 < p, q < \infty$, $1/p + 1/q = 1$. It also maps $(B^q_{s+1}(\Omega))^*$ linearly and boundedly into $B^p_{1-s}(\Omega)$ for all $-1 < s < 2$ and $1 < p, q < \infty$, $1/p + 1/q = 1$.*

Proof. On the scale of Sobolev spaces, the assertion corresponding to $s = -1$ is simply the classical Calderón–Zygmund inequality. The range $-1 \leqslant s \leqslant 1$ is then easily seen from this, duality, and interpolation (note that Π_Ω and $(\Pi_\Omega)^*$ coincide on test functions in Ω). Next, we claim that Π_Ω maps $L^p_{s,0}(\Omega)$ boundedly into $L^p_{s+2}(\Omega)$ for each $0 \leqslant s \leqslant 1$ and $1 < p < \infty$. Dualizing the claim yields the conclusion in the proposition for $1 \leqslant s \leqslant 2$, as desired.

To see the claim, note first that $s = 0$ corresponds once again to the Calderón–Zygmund inequality. Going further, the case $s = 1$ is proved integrating by parts and then invoking the Calderón–Zygmund inequality. Finally, the full claim follows by complex interpolation.

The similar assertion on Besov spaces is a corollary of the preceding result and repeated applications of the method of real interpolation (together with the corresponding duality and reiteration theorems). ∎

3. THE SINGLE LAYER POTENTIAL

Consider Ω a bounded Lipschitz domain in $\mathbb{R}^n$, fixed for the duration of this section. Recall the (radial) fundamental solution for the Laplacian $\Gamma(X)$ in (2.1) and the single layer potential operator

$$\mathscr{S} : (\mathrm{Lip}(\partial\Omega))^* \to C^\infty(\Omega) \tag{3.1}$$

defined by

$$(\mathscr{S}f)(X) := \langle f, \Gamma(X - \cdot) \rangle, \qquad X \in \Omega, \tag{3.2}$$

for each $f \in (\mathrm{Lip}(\partial\Omega))^*$. Our main result in this section summarizes the main mapping properties of the operator (3.1) on scales of Sobolev–Besov spaces.

THEOREM 3.1. *For any* $1 \leqslant p \leqslant \infty$ *and* $0 < s < 1$, *the single layer potential* $\mathcal{S}$ *is a bounded linear map from* $B^p_{-s}(\partial\Omega)$ *into* $B^p_{1+(1/p)-s}(\Omega)$. *In fact, if* $1 < p < \infty$, *then* $\mathcal{S}$ *is a bounded linear map from* $B^p_{-s}(\partial\Omega)$ *into* $B^p_{1+(1/p)-s}(\Omega) \cap L^p_{1+(1/p)-s}(\Omega)$, *i.e.*

$$\max\{\|\mathcal{S}f\|_{B^p_{1+(1/p)-s}(\Omega)}, \|\mathcal{S}f\|_{L^p_{1+(1/p)-s}(\Omega)}\} \leqslant C\, \|f\|_{B^p_{-s}(\partial\Omega)} \tag{3.3}$$

uniformly for $f \in B^p_{-s}(\partial\Omega)$.

In particular, if Tr stands for the trace map on $\partial\Omega$, *the operator* $S := Tr\mathcal{S}$ *maps* $B^p_{-s}(\partial\Omega)$ *into* $B^p_{1-s}(\partial\Omega)$ *for each* $1 \leqslant p \leqslant \infty$ *and* $0 < s < 1$.

It is possible to obtain the $1 < p < \infty$ part of the above result from Theorem 2.1 and a duality argument. However, in order to treat the full range of p's we chose to first study the action of the operator $\mathcal{S}$ on the spaces $(B^\infty_s(\partial\Omega))^*$ and $B^\infty_{-s}(\partial\Omega)$ and then use A. P. Calderón's complex interpolation method [Ca, BeLö]. These end-point cases are treated separately in Lemmas 3.2 and 3.3 below.

LEMMA 3.2. *For each* $0 < s < 1$, *the single layer potential is a bounded linear map from* $B^\infty_{-s}(\partial\Omega)$ *into* $B^\infty_{1-s}(\Omega)$.

Proof. By Theorem 1.4 it suffices to show that

$$\sup_{X \in \Omega} |\delta(X)^s\, \nabla(\mathcal{S}f)(X)| \leqslant C\, \|f\|_{B^\infty_{-s}(\partial\Omega)} \tag{3.4}$$

uniformly for $f \in B^\infty_{-s}(\partial\Omega) := (B^1_s(\partial\Omega))^*$. In turn, this is a direct consequence of the estimate

$$\|\nabla\Gamma(X - \cdot)\|_{B^1_s(\partial\Omega)} \leqslant C(\Omega)\, \delta(X)^{-s}, \qquad \forall X \in \Omega. \tag{3.5}$$

The remainder of the proof consists of a verification of (3.5). The problem localizes and, since the only nontrivial case is when X is close to the boundary, it suffices to prove the estimate

$$\left| \int_{\partial\Omega} \int_{\partial\Omega} g(P)\, \frac{\nabla\Gamma(X-P) - \nabla\Gamma(X-Q)}{|P-Q|^{n-1+s}}\, d\sigma(P)\, d\sigma(Q) \right| \leqslant C(\Omega)\, \delta(X)^{-s} \tag{3.6}$$

uniformly for $g \in L^\infty_{\mathrm{comp}}(\partial\Omega)$ with $\|g\|_{L^\infty(\partial\Omega)} \leqslant 1$, in the case when Ω is the domain above the graph of a Lipschitz function $\phi: \mathbb{R}^{n-1} \to \mathbb{R}$.

Now, for a fixed, sufficiently large $C>0$, the inner integral can be expressed as the difference between

$$\left(\int_{|X-P|<C|P-Q|}+\int_{|X-P|>C|P-Q|}\right) g(P)\,\frac{\nabla\Gamma(X-P)}{|P-Q|^{n-1+s}}\,d\sigma(P)=: I+II, \tag{3.7}$$

and

$$\left(\int_{|X-Q|<C|P-Q|}+\int_{|X-Q|>C|P-Q|}\right) g(P)\,\frac{\nabla\Gamma(X-Q)}{|P-Q|^{n-1+s}}\,d\sigma(P)=: I'+II', \tag{3.8}$$

The plan is to estimate I and I' separately and then use the mean value theorem to bound the difference II–II'. To this end, note first that a change of variables based on the representations $X=(x,\varphi(x)+t)$, $P=(y,\varphi(y))$, and $Q=(z,\varphi(z))$ gives

$$\begin{aligned}|I| &\leqslant C\int_{|x-x|+|t+\varphi(x)-\varphi(y)|<C|y-z|}\\ &\qquad\times\frac{dy}{|z-y|^{n-1+s}\,(|t+\varphi(x)-\varphi(y)|^2+|x-y|^2)^{(n-1)/2}}\\ &\leqslant C\int_{|x-y|+t<C|y-z|}\frac{dy}{|z-y|^{n-1+s}\,(t+|x-y|)^{n-1}}.\end{aligned}$$

Substituting $x-y=ht$ in the last integral above and then integrating against $\int_{\mathbb{R}^{n-1}} dz$ further yields

$$\int_{\partial\Omega}|I|\,d\sigma\leqslant C\int_{\mathbb{R}^{n-1}}\left(\int_{|h|+1\leqslant C|x-th-z|/t}\frac{dh}{(|h|+1)^{n-1}\,|z+ht-x|^{n-1+s}}\right)dz. \tag{3.9}$$

Substituting again, this time first $x-z=wt$ and then $w-h=r\omega$, $r>0$, $\omega\in S^{n-2}$, we may further bound the last integral in (3.9) by

$$\begin{aligned}&\int_{\mathbb{R}^{n-1}}\int_{|h|+1\leqslant C|w-h|}\frac{dh\,dw}{(|h|+1)^{n-1}\,t^s\,|w-h|^{n-1+s}}\\ &\qquad=\frac{C}{t^s}\int\frac{1}{(|h|+1)^{n-1}}\left(\int_{|h|+1}^{\infty}\frac{dr}{r^{s+1}}\right)dh=C_{n,s}t^{-s},\end{aligned}$$

which is a bound of the right order for $\int_{\partial\Omega} |I|\, d\sigma$. Essentially the same arguments work for I' and we shall not repeat them here. Going further, we have

$$\begin{aligned}
&\int_{\partial\Omega} (II - II')\, d\sigma(Q) \\
&\quad = \int_{\partial\Omega} \int_{|X-P|>C\,|P-Q|} \frac{\nabla\Gamma(X-P) - \nabla\Gamma(X-Q)}{|P-Q|^{n-1+s}}\, g(P)\, d\sigma(P)\, d\sigma(Q) \\
&\qquad + \int_{\partial\Omega} \left(\int_{|X-P|>C\,|P-Q|} - \int_{|X-Q|>C\,|P-Q|} \right) \\
&\qquad \times \frac{\nabla\Gamma(X-Q)}{|P-Q|^{n-1+s}}\, g(P)\, d\sigma(P)\, d\sigma(Q) \\
&\quad =: III + IV.
\end{aligned}$$

Let us estimate III above. Since $|X-Z| \geqslant C\,|X-P|$ uniformly for $Z \in [P, Q]$, we get

$$|III| \leqslant C \int_{\partial\Omega} \int_{|X-P|>C\,|P-Q|} \frac{1}{|P-Q|^{n-2+s}\, |X-P|^n}\, d\sigma(P)\, d\sigma(Q). \tag{3.10}$$

As before, pull-backing everything to $\mathbb{R}^{n-1}$, it is enough to bound

$$\int_{\mathbb{R}^{n-1}} \int_{|x-y|+t>C\,|z-y|} \frac{1}{|z-y|^{n-2+s}\, (|x-y|+t)^n}\, dx\, dz. \tag{3.11}$$

Substituting $x - y = th$ and then $x - z = tw$, the above integral can be transformed as

$$\begin{aligned}
&\int_{\mathbb{R}^{n-1}} \left(\int_{|h|+1>C\,|z-y|/t} \frac{1}{t(|h|+1)^n\, |z-x+ht|^{n-2+s}}\, dh \right) dz \\
&\quad = \frac{1}{t^s} \int_{\mathbb{R}^{n-1}} \frac{1}{(|h|+1)^n} \left(\int_{|h|+1>C\,|w-h|} \frac{dw}{|w-h|^{n-2+s}} \right) dh \\
&\quad = C_{n,s}\, t^{-s},
\end{aligned} \tag{3.12}$$

as desired. This completes the treatment of III.

The final step deals with estimating IV. First, clearly,

$$
\begin{aligned}
|IV| \leqslant C \int_{\partial\Omega} \int_{|X-P| > C\,|P-Q| > C'\,|X-Q|} \frac{|\nabla\Gamma(X-Q)|}{|P-Q|^{n-1+s}}\, d\sigma(P)\, d\sigma(Q) \\
+ C \int_{\partial\Omega} \int_{|X-Q| > C\,|P-Q| > C'\,|X-P|} \frac{|\nabla\Gamma(X-Q)|}{|P-Q|^{n-1+s}}\, d\sigma(P)\, d\sigma(Q).
\end{aligned}
\tag{3.13}
$$

At this point, each integral in the right side can be estimated essentially in the same way as I and I' above, by producing upper bounds of the same order. Summing up these various partial estimates, (3.5) finally follows. ∎

Next, we turn attention to the other endpoint case, contained in the next lemma. Let us parenthetically note that this is actually an improvement over the formal dual statement of Lemma 3.2.

LEMMA 3.3. *For each* $0 < s < 1$, *the operator* $\mathscr{S}$ *is a bounded linear map from* $(B_s^{\infty}(\partial\Omega))^*$ *into* $B_{2-s}^{1}(\Omega)$.

Proof. Invoking once again Theorem 1.4, it is enough to prove that

$$
\|\delta^{s-1}\,|\nabla\mathscr{S}f|\,\|_{L^1(\Omega)} + \|\mathscr{S}f\|_{L^1(\Omega)} \leqslant C\,\|f\|_{(B_s^{\infty}(\partial\Omega))^*} \tag{3.14}
$$

uniformly for $f \in (B_s^{\infty}(\partial\Omega))^*$.

We shall only indicate how the first term in the left side of (3.14) can be estimated and leave the second (simpler) term to the interested reader. To this end, we claim that it suffices to show that for each $g \in L^{\infty}_{\mathrm{comp}}(\Omega)$ with $\|g\|_{L^{\infty}(\Omega)} \leqslant 1$, the inequality

$$
\int_{\Omega} g(X)\,\delta(X)^{s-1}\,\nabla(\mathscr{S}f)(X)\,dX \leqslant C\,\|f\|_{(B_s^{\infty}(\partial\Omega))^*} \tag{3.15}
$$

holds uniformly for $f \in (B_s^{\infty}(\partial\Omega))^*$. Indeed, the claim is more or less an immediate consequence of the fact that $(L^1(\Omega))^* = L^{\infty}(\Omega)$ and Lebesgue's monotone convergence theorem.

To prove (3.15), observe that

$$
\int_{\Omega} g(X)\,\delta(X)^{s-1}\,\nabla(\mathscr{S}f)(X)\,dX = \left\langle f, \int_{\Omega} g(X)\,\delta(X)^{s-1}\,\nabla\Gamma(X-\cdot)\,dX \right\rangle. \tag{3.16}
$$

and, hence, everything will follow from

$$
\int_{\Omega} g(X)\,\delta(X)^{s-1}\,\nabla\Gamma(X-\cdot)\,dX \in B_s^{\infty}(\partial\Omega). \tag{3.17}
$$

Clearly, the only non-trivial contribution comes from integrating near the boundary so we will only show (3.17) with Ω replaced by $\Omega_\varepsilon := \{X \in \Omega;\ \delta(X) < \varepsilon\}$, for some fixed, small $\varepsilon > 0$. Therefore, we shall focus on establishing the estimate

$$\sup_{P \neq Q} \frac{1}{|P-Q|^s} \left| \int_{\Omega_\varepsilon} g(X)\, \delta^{s-1}(X)(\nabla\Gamma(X-P) - \nabla\Gamma(X-Q))\, dX \right| \leqslant C < +\infty. \tag{3.18}$$

Now, fix two arbitrary points $P, Q \in \partial\Omega$ and decompose the integral in a way similarly to what was done in the proof of Lemma 3.2, i.e., into

$$\begin{aligned}
&\int_{(|X-P| < C\,|P-Q|) \cap \Omega_\varepsilon} g(X)\, \delta(X)^{s-1}\, \nabla\Gamma(X-P)\, dX \\
&\quad + \int_{(|X-Q| < C\,|P-Q|) \cap \Omega_\varepsilon} g(X)\, \delta(X)^{s-1}\, \nabla\Gamma(X-Q)\, dX \\
&\quad + \int_{(|X-P| > C\,|P-Q|) \cap \Omega_\varepsilon} g(X)\, \delta(X)^{s-1}\, (\nabla\Gamma(X-P) - \nabla\Gamma(X-Q))\, dX \\
&\quad + \int_{(|X-P| > C\,|P-Q| > C'\,|X-Q|) \cap \Omega_\varepsilon} \delta(X)^{s-1}\, \nabla\Gamma(X-Q)\, dX \\
&\quad =: I + II + III + IV.
\end{aligned} \tag{3.19}$$

As before, to deal with I, by localizing and pulling back to $\mathbb{R}^n_+$, it suffices to consider

$$\int_{|x-y| + |t+\varphi(x)-\varphi(y)| < C\,|z-y|} \int_0^\varepsilon \frac{1}{t^{1-s}(|x-y| + |t+\varphi(x)-\varphi(y)|)^{n-1}}\, dt\, dx \tag{3.20}$$

where $\varphi\colon \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function. Now, the above integral is majorized by

$$\begin{aligned}
&C \int_{|x-y| + t < C\,|z-y|} \int_0^\varepsilon \frac{1}{t^{1-s}(|x-y|+t)^{n-1}}\, dt\, dx \\
&\quad \leqslant C \int_0^{C|y-z|} \frac{1}{t^{1-s}} \left(\int_{t < |x-y| < C\,|y-z|} + \int_{|x-y| < t} \right) \frac{dx}{(|x-y|+t)^{n-1}}\, dt \\
&\quad \leqslant C \int_0^{C|y-z|} \frac{1}{t^{1-s}} \left(\int_{1 < |h| < C\,|y-z|/t} \frac{dh}{(|h|+1)^{n-1}} + \int_{|h|<1} \frac{dh}{(|h|+1)^{n-1}} \right) dt \\
&\quad \leqslant C \int_0^{C\,|y-z|} \frac{1}{t^{1-s}} \left(\int_1^C \frac{dr}{r} + C \right) dt \\
&\quad \leqslant C_{n,s}\, |y-z|^s,
\end{aligned} \tag{3.21}$$

and the last bound has the right order. Similar arguments also apply to II and IV in (3.19) so we are left with estimating III. For this, an application of the mean-value theorem together with a change of variables allow us to write

$$\begin{aligned}
III &\leqslant C \int_{|x-y|+(t+\varphi(x)-\varphi(y))>C|y-z|} \frac{|y-z|}{t^{1-s}(|x-y|+|t+\varphi(x)-\varphi(y)|)^n}\,dx\,dt \\
&\leqslant C \int_{|x-y|+t>C|y-z|} \frac{|y-z|}{t^{1-s}(|x-y|+t)^n}\,dx\,dt \\
&= C\Bigg(\int_{\substack{|x-y|>(C/2)|y-z| \\ t>(C/2)|y-z|}} + \int_{\substack{|x-y|>(C/2)|y-z| \\ t>(C/2)|y-z|}} \\
&\quad + \int_{\substack{|x-y|<(C/2)|y-z| \\ t<(C/2)|y-z|}}\Bigg) \frac{|y-z|}{(|x-y|+t)^n}\,\frac{dx\,dt}{t^{1-s}} \\
&=: III_a + III_b + III_c.
\end{aligned}$$

The first term in the right-side of the last equality above is estimated as follows:

$$\begin{aligned}
|III_a| &\leqslant C \int_{C>|x-y|>C'|y-z|} \int_{C|y-z|}^{C''} \frac{|y-z|}{t^{1-s}(|x-y|+t)^n}\,dx\,dt \\
&= C\,|y-z| \int_{C|y-z|}^{C''} \frac{1}{t^{2-s}} \int_{C|y-z|<|h|\,t<C'} \frac{1}{(|h|+1)^n}\,dh\,dt \\
&\leqslant C\,|y-z| \int_{C|y-z|}^{C''} \frac{1}{t^{2-s}} \left(\int_{C<|h|<C'/t} + \int_{|h|<C}\right) \frac{dh}{(|h|+1)^n}\,dt \\
&= C\,|y-z|\,[(C'-|y-z|^{s-1})+(C''-|y-z|^s)] \\
&\leqslant C\,|y-z|^s. \qquad (3.22)
\end{aligned}$$

Further, for the next term we have

$$\begin{aligned}
|III_b| &\leqslant C\,|y-z| \int_0^{C|y-z|} \frac{1}{t^{1-s}} \int_{C>|x-y|>C'|y-z|} \frac{1}{(|x-y|+t)^n}\,dx\,dt \\
&= C\,|y-z| \int_0^{C|y-z|} \frac{1}{t^{2-s}} \int_{\mathbb{R}^{n-1}} \frac{1}{(|h|+1)^n}\,dh\,dt \\
&\leqslant C\,|y-z| \int_0^{|y-z|} \frac{dt}{t^{2-s}} \\
&= C\,|y-z|^s. \qquad (3.23)
\end{aligned}$$

Finally, the last term of interest can be handled by writing

$$
\begin{aligned}
|III_c| &\leqslant C \int_{C|y-z|}^{C'} \frac{1}{t^{1-s}} \int_{|x-y|<C|y-z|} \frac{|y-z|}{(|x-y|+t)^n}\, dx\, dt \\
&\leqslant C \int_{C|y-z|}^{C'} \frac{1}{t^{1-s}} \int_{|x-y|<Ct} \frac{|y-z|}{(|x-y|+t)^n}\, dx\, dt \\
&\leqslant C\,|y-z|\,(C' - C''\,|y-z|^{s-1}) \\
&\leqslant C\,|y-z|^s. \qquad (3.24)
\end{aligned}
$$

This proves (3.17) and, consequently, finishes the proof of Lemma 3.2. ▮

Finally, we are now in a position to present the

Proof of Theorem 3.1. Invoking a duality theorem and interpolating by the real or the complex method between the results of Lemmas 3.2 and 3.3 gives that the single layer potential is a bounded map from $B^p_{-s}(\partial\Omega)$ into $B^p_{1-s+1/p}(\Omega)$ for any $0<s<1$ and $1 \leqslant p \leqslant \infty$. This takes care of the first part of the theorem. The second part follows from this and Theorem 1.4 whenever $1-s+1/p \neq 1$. However, the fact that $\mathscr{S}\colon B^p_{-1/p}(\partial\Omega) \to L^p_1(\Omega)$ is known; see, e.g., Lemma 3.1 in [MiMiPi] (alternatively, this also follows from what we have proved so far and reiteration). Finally, the last part follows from what we have proved so far and the continuity of the trace operator. The proof of Theorem 3.1 is now complete. ▮

4. THE DOUBLE LAVER POTENTIAL ON SOBOLEV–BESOV SPACES

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ whose outward unit normal to $\partial\Omega$ we denote by N. Recall that the double layer potential of a density f in, say, $L^1(\partial\Omega)$ is defined by

$$\mathscr{D}f(X) := \int_{\partial\Omega} \frac{\partial\Gamma}{\partial N}(X-Q)\, f(Q)\, d\sigma(Q), \qquad X \in \Omega, \tag{4.1}$$

and its (nontangential) boundary trace is $\mathscr{D}f|_{\partial\Omega} = (\frac{1}{2}I + K)\, f$ where

$$Kf(X) := \lim_{\varepsilon \to 0} \int_{\substack{|X-Q| \geqslant \varepsilon \\ Q \in \partial\Omega}} \frac{\partial\Gamma}{\partial N}(X-Q)\, f(Q)\, d\sigma(Q), \qquad X \in \partial\Omega.$$

The main result of this section describes the action of the operators (4.1)–(4.2) on scales of Sobolev–Besov spaces.

THEOREM 4.1. *For $1 \leqslant p \leqslant \infty$ and $0 < s < 1$, the operator $\mathscr{D}$ extends to a bounded linear map from $B_s^p(\partial\Omega)$ into $B_{s+1/p}^p(\Omega)$. In fact, if $p \neq 1$ and $p \neq \infty$, then $\mathscr{D}$ also extends as a bounded linear map from $B_s^p(\partial\Omega)$ into $B_{s+1/p}^p(\Omega) \cap L_{s+1/p}^p(\Omega)$. In other words, for any $1 < p < \infty$ and $0 < s < 1$, the estimate*

$$\max\{\|\mathscr{D}f\|_{L_{s+1/p}^p(\Omega)}, \|\mathscr{D}f\|_{B_{s+1/p}^p(\Omega)}\} \leqslant C\,\|f\|_{B_s^p(\partial\Omega)} \tag{4.3}$$

holds uniformly for $f \in B_s^p(\partial\Omega)$.

Furthermore, $Tr\mathscr{D} = \frac{1}{2}I + K$ so that, in particular, $K\colon B_s^p(\partial\Omega) \to B_s^p(\partial\Omega)$ is well defined and bounded for any $1 \leqslant p \leqslant \infty$, $0 < s < 1$.

The proof is similar in spirit to that of Theorem 3.1. We first establish the corresponding results for the end-point cases $p = 1$ and $p = \infty$ and then use interpolation in order to conclude. We debut with an essentially well-known lemma, corresponding to $p = \infty$ (for this we lack a precise reference; a short proof is included in [KaMi])

LEMMA 4.2. *For each $0 < s < 1$, $\mathscr{D}$ is a bounded linear map from $B_s^\infty(\partial\Omega)$ into $B_s^\infty(\Omega)$.*

Next, we consider the case $p = 1$.

LEMMA 4.3. *For each $0 < s < 1$, the operator $\mathscr{D}$ maps $B_s^1(\partial\Omega)$ linearly and boundedly into $B_{s+1}^1(\Omega)$.*

Proof. Since $\mathscr{D}f$ is harmonic in Ω, by Theorem 1.4 it suffices to show that

$$\int_\Omega \delta(X)^{-s}\,|\nabla\mathscr{D}f(X)|\,dX \leqslant C\,\|f\|_{B_s^1(\partial\Omega)}, \tag{4.4}$$

uniformly for $f \in B_s^1(\partial\Omega)$. Fix an arbitrary $f \in B_s^1(\partial\Omega)$, $0 < s < 1$. Note that, in the left side of (4.4), the contribution away from the boundary is easily estimated. Furthermore, via a partition of unity, there is no loss of generality in assuming that f has support contained in a coordinate patch where $\partial\Omega$ is given by the graph of a Lipschitz function $\varphi\colon \mathbb{R}^{n-1} \to \mathbb{R}$. Finally, since $\nabla\mathscr{D}$ annihilates constants, matters can be reduced, by a change of variables, to estimating

$$\int_0^1 \frac{1}{t^s}\int_{\mathbb{R}^{n-1}} D_j\left(\int_{\mathbb{R}^{n-1}} \frac{(t+\varphi(x)-\varphi(y)-(x-y)\,\nabla\varphi(y))}{(|x-y|^2+(t+\varphi(x)-\varphi(y))^2)^{n/2}}\right.$$
$$\left.\times(\tilde{f}(x)-\tilde{f}(y))\,dy\right)dx\,dt \tag{4.5}$$

for $j=0, 1, 2, ..., n-1$. Here $D_j := \partial/\partial t$ for $j=0$ and $D_j := \partial/\partial x_j$ for $j=1, ..., n-1$. Also, $\tilde{f}(x) := f(x, \varphi(x))$, extended by zero outside of the support; note that $\tilde{f} \in B_s^1(\mathbb{R}^{n-1})$.

We shall only indicate how the derivative with respect to t (i.e., $j=0$) can be bounded since the case $1 \leqslant j \leqslant n-1$ is quite similar. To this effect, for some large, fixed $c>0$, it suffices to control

$$
\begin{aligned}
&\int_0^1 \frac{1}{t^s} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \frac{(\tilde{f}(x)-\tilde{f}(y))}{(|x-y|^2+(t+\varphi(x)-\varphi(y))^2)^{n/2}}\, dy\, dx\, dt \\
&\quad = \int_0^1 \frac{1}{t^s} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \frac{(\tilde{f}(x)-\tilde{f}(x-z))}{(|z|^2+(t+\varphi(x)-\varphi(x-z))^2)^{n/2}}\, dz\, dx\, dt \\
&\quad \leqslant C \int_0^1 \frac{1}{t^s} \int_{\mathbb{R}^{n-1}} \int_{|z|>ct} \frac{(\tilde{f}(x)-\tilde{f}(x-z))}{(|z|^2+(t+\varphi(x)-\varphi(x-z))^2)^{n/2}}\, dz\, dx\, dt \\
&\qquad + C \int_0^1 \frac{1}{t^s} \int_{\mathbb{R}^{n-1}} \int_{|z|<ct} \frac{(\tilde{f}(x)-\tilde{f}(x-z))}{(|z|^2+(t+\varphi(x)-\varphi(x-z))^2)^{n/2}}\, dz\, dx\, dt \\
&\quad =: I + II.
\end{aligned} \tag{4.6}
$$

To continue, denote by $(\omega_1 g)(x) := \|g(x+\cdot)-g(\cdot)\|_{L^1(\mathbb{R}^{n-1})}$ the L^1-modulus of continuity of an arbitrary function $g\colon \mathbb{R}^{n-1} \to \mathbb{R}$. Then, changing the order of integration and integrating first with respect to t, we obtain for the first term above

$$
|I| \leqslant C \int_0^1 \frac{1}{t^s} \int_{|z|>ct} \frac{(\omega_1 \tilde{f})(z)}{|z|^n}\, dz\, dt \leqslant C \int_{\mathbb{R}^{n-1}} \frac{(\omega_1 \tilde{f})(z)}{|z|^{n-1+s}}\, dz \leqslant C\, \|\tilde{f}\|_{B_s^1(\mathbb{R}^{n-1})}, \tag{4.7}
$$

which has the right order. In a similar manner, for the second term in (4.6) we have

$$
\begin{aligned}
|II| &\leqslant C \int_0^\infty \frac{1}{t^{1+s}} \int_{|z|<ct} \frac{(\omega_1 \tilde{f})(z)}{|z|^{n-1}}\, dz\, dt \\
&\leqslant C \int_{\mathbb{R}^{n-1}} \frac{(\omega_1 \tilde{f})(z)}{|z|^{n-1+s}}\, dz \leqslant C\, \|\tilde{f}\|_{B_s^1(\mathbb{R}^{n-1})}.
\end{aligned} \tag{4.8}
$$

This concludes the proof of Lemma 4.3. ∎

We are finally ready for the

Proof of Theorem 4.1. If $s+1/p \neq 1$ then Lemmas 4.2, 4.3, real or complex interpolation and Theorem 1.4 may be used to conclude in this case. Now, since $\mathscr{D}\colon B^p_{1-1/p}(\partial\Omega) \to L^p_1(\Omega)$, for $1<p<\infty$, is known (cf., e.g., Lemma 3.1 in [MiMiPi]), the full statement follows (alternatively, one may use the first part and a reiteration theorem).

Given what we have proved so far, the last point in the statement is a consequence of the well-known trace formula for Lipschitz continuous functions and a density argument. ▮

5. THE BANACH ENVELOPE ANATOMIC HARDY SPACES

Fix an arbitrary bounded Lipschitz domain Ω in $\mathbb{R}^n$ and for each $(n-1)/n < p \leqslant 1$ set $s := (n-1)(1/p-1)$. Note that $0 \leqslant s < 1$. A function $a \in L^\infty(\partial\Omega)$ is called an $H^p(\partial\Omega)$-atom if $\int_{\partial\Omega} a \, d\sigma = 0$ and, for some boundary point $P \in \partial\Omega$ and some $0 < r < \operatorname{diam} \Omega$,

$$\operatorname{supp} a \subset \partial\Omega \cap B(P, r), \qquad \|a\|_{L^\infty(\partial\Omega)} \leqslant r^{-(n-1)/p}. \tag{5.1}$$

For $(n-1)/n < p < 1$ (and $p = 1$, respectively) we recall that the atomic Hardy space $H^p(\partial\Omega)$ is defined as the vector subspace of $(B_s^\infty(\partial\Omega)/\langle 1 \rangle)^*$ (and $L^1(\partial\Omega)$, respectively) consisting of all linear functionals f which can be represented as

$$f = \sum_{i=0}^{\infty} \lambda_i a_i, \qquad (\lambda_i)_i \in \ell^p, \quad a_i\text{'s are } H^p(\partial\Omega)\text{-atoms}, \tag{5.2}$$

in the sense of convergence in $(B_s^\infty(\partial\Omega)/\langle 1 \rangle)^*$ (and $L^1(\partial\Omega)$, respectively). We equip $H^p(\partial\Omega)$ with the (quasi-)norm

$$\|f\|_{H^p(\partial\Omega)} := \inf \left\{ \left(\sum_{i=0}^{\infty} |\lambda_i|^p \right)^{1/p} ; f = \sum_{i=0}^{\infty} \lambda_i a_i \text{ as in (5.2)} \right\}. \tag{5.3}$$

Note that $L_0^2(\partial\Omega) \subset H^p(\partial\Omega)$ and, as is well known,

$$(H^p(\partial\Omega))^* = \begin{cases} B_s^\infty(\partial\Omega)/\langle 1 \rangle, & \text{if } \ p < 1, \\ BMO(\partial\Omega), & \text{if } \ p = 1. \end{cases} \tag{5.4}$$

See, e.g., [CoWe] for a more detailed account.

Occasionally, we shall also find it convenient to consider the constant function as an atom, in which case we shall denote the corresponding

Hardy space by $H^p_{ct}(\partial\Omega)$. In this case, $H^p_{ct}(\partial\Omega) \subseteq (B^\infty_s(\partial\Omega))^*$ and $(H^p_{ct}(\partial\Omega))^* = B^\infty_s(\partial\Omega)$ with $0 < s = (n-1)(1/p - 1) < 1$. If we also set $H^p(\partial\Omega) := L^p(\partial\Omega)$ for $1 < p < \infty$ then it is well known that

$$\{H^p(\partial\Omega)\}_{(n-1)/n \leqslant p < \infty} \text{ is an interpolation scale for the complex method.} \tag{5.5}$$

See the discussion in [KaMi].

In the sequel, it will be necessary to consider the space of functions in $H^p(\partial\Omega)$ with (tangential) derivatives also in $H^p(\partial\Omega)$. First, from the atomic characterization of Besov spaces in Section 1 it follows that $\nabla_{\tan}: B^1_{1-s}(\partial\Omega) \to (B^\infty_s(\partial\Omega))^*$ is a well-defined and bounded operator. Hence, for $0 < s = (n-1)(1/p - 1) < 1$, we may consider

$$H^p_1(\partial\Omega) := \{f \in B^1_{1-s}(\partial\Omega); \nabla_{\tan} f \in H^p(\partial\Omega)\} \tag{5.6}$$

endowed with the quasi-norm

$$\|f\|_{H^p_1(\partial\Omega)} := \|\nabla_{\tan} f\|_{H^p(\partial\Omega)} + \|f\|_{B^1_{1-s}(\partial\Omega)}. \tag{5.7}$$

In particular, for p as above, $H^p_1(\partial\Omega)$ becomes a quasi-Banach space continuously embedded into $B^1_{1-s}(\partial\Omega)$. Let us point out that another variant of the "regular" Hardy space (5.6) is to consider the ℓ^p-span of $B^1_{1-s}(\partial\Omega)$-atoms; cf. [Br]. However, in this latter case, the boundedness of certain natural boundary integral operators of Calderón–Zygmund type becomes rather delicate and we prefer to avoid this issue by adopting the present definition.

Our main results in this section involve the "minimal enlargement" of $H^p(\partial\Omega)$ to a Banach space, its so-called Banach envelope. To define this properly, we digress momentarily for the purpose of explaining a somewhat more general functional analytic setting. A good reference is [KaPeRo].

Let $\mathscr{V}$ be a locally bounded topological vector space, whose dual separates points and fix U a bounded neighborhood of the origin. Then, with co A standing for the convex hull of a set $A \subseteq \mathscr{V}$, the functional given by

$$\|x\|_{\hat{\mathscr{V}}} := \inf\{\gamma > 0; \gamma^{-1} x \in \mathrm{co}(U)\} \tag{5.8}$$

defines a (continuous) norm on $\mathscr{V}$ (any other such norm corresponding to a different choice of U is in fact equivalent to (5.8)). Then, $\hat{\mathscr{V}}$, the Banach envelope of $\mathscr{V}$, is the completion of $\mathscr{V}$ in the norm (5.8). Thus, $\hat{\mathscr{V}}$ is a well defined Banach space, uniquely defined up to an isomorphism. Also, the inclusion $\mathscr{V} \hookrightarrow \hat{\mathscr{V}}$ is continuous and has a dense image.

PROPOSITION 5.1. *Let $\mathscr{V}_1$, $\mathscr{V}_2$ be two topological vector spaces as above. Then any bounded linear operator $T\colon \mathscr{V}_1 \to \mathscr{V}_2$ extends to a bounded linear operator*

$$\hat{T}\colon \hat{\mathscr{V}}_1 \to \hat{\mathscr{V}}_2. \tag{5.9}$$

In fact,

$$\sup_{x \neq 0} \frac{\|\hat{T}(x)\|_{\hat{\mathscr{V}}_2}}{\|x\|_{\hat{\mathscr{V}}_1}} \leqslant \inf\{\varepsilon > 0;\ T(U_1) \subset \varepsilon U_2\}, \tag{5.10}$$

where $U_i \subset \mathscr{V}_i$ is the bounded neighborhood with respect to which the norm $\|\cdot\|_{\hat{\mathscr{V}}_i}$ has been defined, $i = 1, 2$.

Proof. The verification is straightforward and, hence, omitted. ∎

As a consequence of this result, we note the following.

COROLLARY 5.2. *If $\mathscr{V}$ is as before then $\mathscr{V}$ and $\hat{\mathscr{V}}$ have the same dual.*

Another more or less direct corollary of Proposition 5.1 is the result recorded below.

COROLLARY 5.3. *Let $\mathscr{V}_1$, $\mathscr{V}_2$ and T be as above, and in addition let T be an isomorphism. Then $\hat{T}$ in (5.9) is an isomorphism of Banach spaces.*

Proof. Apply Proposition 5.1 both to T and to T^{-1}. ∎

Next we come to the main result concerning the Banach envelope of atomic Hardy spaces.

THEOREM 5.4. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and fix some index $(n-1)/n < p < 1$. Then $\hat{H}^p(\partial\Omega)$, the Banach envelope of $H^p(\partial\Omega)$ is the subspace of $(B_s^\infty(\partial\Omega))^*$, with $s = (n-1)(1/p - 1)$, given by*

$$\left\{ f = \sum \lambda_i a_i \text{ in } (B_s^\infty(\partial\Omega))^*;\ (\lambda_i)_i \in \ell^1,\ a_i\text{'s are } H^p(\partial\Omega) - \text{atoms} \right\} \tag{5.11}$$

endowed with the norm

$$f \mapsto \inf\left\{ \sum |\lambda_i|;\ f = \sum \lambda_i a_i, \text{ where } \lambda_i\text{'s}, a_i\text{'s are as in (5.11)} \right\}.$$

Also, $\hat{H}^p_{\mathrm{ct}}(\partial\Omega)$, the Banach envelope of $H^p_{\mathrm{ct}}(\partial\Omega)$, has a similar description except that $H^p(\partial\Omega)$-atoms are replaced by $H^p_{\mathrm{ct}}(\partial\Omega)$-atoms. In particular, $(\hat{H}^p(\partial\Omega))^* = B^\infty_s(\partial\Omega)/\langle 1\rangle$ and $(\hat{H}^p_{\mathrm{ct}}(\partial\Omega))^* = B^\infty_s(\partial\Omega)$.

Proof. Let us temporarily denote by $\mathscr{X}$ the space (5.11) equipped with the norm (5.12). A routine argument gives that $\mathscr{X}$ is complete. Observe now that the inclusion $\imath\colon H^p(\partial\Omega) \to \mathscr{X}$ is continuous so that, by Proposition 5.1, it extends to a bounded mapping

$$\hat{\imath}\colon \hat{H}^p(\partial\Omega) \to \mathscr{X}. \tag{5.13}$$

The idea is to show that $\hat{\imath}$ above is an isomorphism. The first order of business is to check that $\hat{\imath}$ is injective. Concretely, from the various definitions introduced before, this comes down to proving the following claim:

if $\{f_k\}_k$ is a sequence in $H^p(\partial\Omega)$ which converges to $f \in \hat{H}^p(\partial\Omega)$ in $\hat{H}^p(\partial\Omega)$ and to zero in $\mathscr{X}$, then $f = 0$. (5.14)

To see this, fix $\varphi \in (\hat{H}^p(\partial\Omega))^* = B^\infty_s(\partial\Omega)/\langle 1\rangle$ with $\|\varphi\|_{B^\infty_s(\partial\Omega)/\langle 1\rangle} \leqslant 1$. Note that for any $g \in H^p(\partial\Omega) \hookrightarrow \mathscr{X}$ we have

$$|\langle \varphi, g\rangle| \leqslant C\,\|g\|_{\mathscr{X}}. \tag{5.15}$$

Indeed, corresponding to any expansion $g = \sum \lambda_i a_i$ in $(B^\infty_s(\partial\Omega))^*$, where $(\lambda_i)_i \in \ell^1$ and a_i's are $H^p(\partial\Omega)$-atoms, we have $|\langle \varphi, g\rangle| \leqslant \sum |\lambda_i|$. Now (5.15) follows by taking the infimum over all such decompositions of g. Next we write, on account of the hypotheses in (5.14) and (5.15), that

$$|\langle \varphi, f\rangle| = \lim_k |\langle \varphi, f_k\rangle| \leqslant C \lim_k \|f_k\|_{\mathscr{X}} = 0. \tag{5.16}$$

From the Hahn–Banach theorem it now follows that $f = 0$, as desired.

To finish the proof, via Banach's open mapping theorem, it suffices to check that

$$\tfrac{1}{2} B_{\mathscr{X}} \subseteq \hat{\imath}(B_{\hat{H}^p}) \tag{5.17}$$

where $B_{\mathscr{X}}$, $B_{\hat{H}^p}$ are the unit balls in $\mathscr{X}$ and $\hat{H}^p(\partial\Omega)$, respectively, and $\hat{\imath}$ is as in (5.13). To this end, let $f = \sum \lambda_i a_i$ be some atomic decomposition for an arbitrary, fixed $f \in B_{\mathscr{X}}$. We can assume $\sum |\lambda_i| \leqslant 2$ which implies that

$$f_m := \sum_{i=0}^{m} \lambda_i a_i \in 2\mathrm{co}(B_{H^p}) \subseteq 2B_{\hat{H}^p}, \tag{5.18}$$

with B_{H^p} standing for the unit ball in $H^p(\partial\Omega)$. Since $\|f_m - f_{m+l}\|_{\hat{H}^p(\partial\Omega)} \leqslant \sum_{i=m}^{m+l} |\lambda_i|$, it follows that $(f_m)_m$ converges to some $g \in \hat{H}^p(\partial\Omega)$ in the

$\hat{H}^p(\partial\Omega)$-norm. In particular, $g \in B_{\hat{H}^p}$ so that $\hat{\imath}(g) \in 2\hat{\imath}(B_{\hat{H}^p})$. On the other hand, $\hat{\imath}(f_m) = f_m$ and, passing to the limit in $\mathscr{X}$, we arrive at $f = \hat{\imath}(g) \in 2\hat{\imath}(B_{\hat{H}^p})$, as claimed. ∎

An important conjecture related to the discussion in this section is that $\hat{H}^p_1(\partial\Omega) = B^1_{1-s}(\partial\Omega)$ for any $(n-1)/n < p < 1$ and $s = (n-1)(1/p - 1)$.

6. AN ENDPOINT NEUMANN PROBLEM

Here and in the next section we initiate the study of the Neumann and Dirichlet problems with boundary data in Besov spaces. Specifically, we shall be concerned with the case when the boundary data belong, respectively, to $\hat{H}^p(\partial\Omega)$ and $B^1_s(\partial\Omega)$, for $1-\varepsilon < s$, $p < 1$, where $1 \geqslant \varepsilon > 0$ depends on the domain. Not only are these problems important in themselves but, as we shall see shortly, such results are also useful for establishing invertibility properties of boundary layers on Besov spaces. A complete analysis of these problems with boundary data in more general Besov spaces will be carried out in Sections 9 and 10. In this section we start by formulating and solving the corresponding Neumann problem. Hereafter, Ω will stand for a fixed, bounded Lipschitz domain in $\mathbb{R}^n$. Also, recall that δ denotes the distance function to $\partial\Omega$. First we need an extension result.

LEMMA 6.1. *Let $\psi \in B^\infty_s(\partial\Omega)$ for some $0 < s < 1$. Then there exists $\tilde{\psi} \in C^0(\bar{\Omega})$, locally Lipschitz in Ω, which extends ψ and such that $\delta^{1-s}\,\nabla\tilde{\psi} \in L^\infty(\Omega)$.*

Proof. The problem localizes and, via a bi-Lipschitz change of variables can be transported to the upper-half space. Now, if $\psi \in B^\infty_s(\mathbb{R}^{n-1})$ with compact support and if u is its Poisson extension to $\mathbb{R}^n_+$ then it is known that

$$\|\delta^{1-s}\,\nabla u\|_{L^\infty(\mathbb{R}^n_+)} \leqslant C\,\|\psi\|_{B^\infty_s(\mathbb{R}^{n-1})}. \tag{6.1}$$

See, e.g., [St]. The proof is finished. ∎

For a fixed, arbitrary $(n-1)/n < p < 1$ set $s := (n-1)(1/p - 1)$. We will be concerned with the Neumann boundary problem with boundary data in $\hat{H}^p(\partial\Omega)$, i.e.

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \partial u/\partial N = f \in \hat{H}^p(\partial\Omega), \\ \delta^{s-1}\,\nabla u \in L^1(\Omega). \end{cases} \tag{6.2}$$

The boundary condition in (6.2) is interpreted as the equivalent of

$$\int_{\Omega} \langle \nabla u(X), \nabla\tilde{\psi}(X)\langle\, dX = \langle f, \psi\rangle, \qquad \forall \psi \in B_s^{\infty}(\partial\Omega), \tag{6.3}$$

where tilde is the extension operator introduced in Lemma 6.1, and $\langle\cdot,\cdot\rangle$ in the right side stands for the natural pairing, between $\hat{H}^p(\partial\Omega) \subseteq (B_s^{\infty}(\partial\Omega))^*$ and $B_s^{\infty}(\partial\Omega)$. It is to be noted that, by Lemma 6.1, the integral in the left-side of (6.3) is absolutely convergent.

A similar definition works for the case of the exterior Neumann problem with natural modifications. Most notably, this time u is assumed to decay at infinity, $\delta^{s-1}\nabla u$ is locally integrable in $\mathbb{R}^n\backslash\Omega$ and the boundary data are from $\hat{H}^p_{\mathrm{ct}}(\partial\Omega)$.

Before we come to the main result of this section we would like to include a comment to the effect that the space of natural boundary data in (6.2) is indeed $\hat{H}^p(\partial\Omega)$ and not the larger space $(B_s^{\infty}(\partial\Omega)/\langle 1\rangle)^*$. This is supported by the observation that even though $-\frac{1}{2}I + K^*$ is, as we shall see momentarily, an isomorphism of the larger space $(B_s^{\infty}(\partial\Omega)/\langle 1\rangle)^*$ for small s, and even though $\mathscr{S}$ maps the latter space boundedly into $B_{2-s}^{\infty}(\Omega)$, the natural jump formula

$$\frac{\partial \mathscr{S} f}{\partial N} = (-\tfrac{1}{2}I + K^*)\, f$$

necessarily fails for general $f \in (B_s^{\infty}(\partial\Omega)/\langle 1\rangle)^*$. This is because, as it will be shown in Theorem 7.1, the normal derivative of $\mathscr{S} f \in B_{2-s}^1(\Omega)$ always belongs to a smaller subspace of $(B_s^{\infty}(\partial\Omega))^*$, namely $\hat{H}^p(\partial\Omega)$.

THEOREM 6.2. *There exists $\varepsilon > 0$ depending only on Ω with the following relevance. If $1 - \varepsilon < p < 1$ and $s = (n-1)(1/p - 1)$ then for each $f \in \hat{H}^p(\partial\Omega)$ there exists a unique (modulo constants) solution u to the Neumann problem* (6.2). *Moreover, u belongs to $B_{2-s}^1(\Omega)$ and satisfies the estimate*

$$\|u\|_{B_{2-s}^1(\Omega)/\langle 1\rangle} + \|\delta^{s-1}\nabla u\|_{L^1(\Omega)} \leqslant C(\Omega)\, \|f\|_{\hat{H}^p(\partial\Omega)}. \tag{6.4}$$

In particular,

$$\|\mathrm{Tr}\, u\|_{B_{1-s}^1(\partial\Omega)/\langle 1\rangle} \leqslant C(\Omega)\, \|f\|_{\hat{H}^p(\partial\Omega)}. \tag{6.5}$$

Finally, a similar statement is valid for the exterior Neumann problem.

Before we present the proof of this theorem, we isolate a result which will also be of importance latter on. To state it, recall the double layer potential operator K in (4.2) and denote by K^* its formal adjoint.

LEMMA 6.3. *There exists* $\varepsilon > 0$ *such that*

$$-\tfrac{1}{2}I + K^* : \hat{H}^p(\partial\Omega) \to \hat{H}^p(\partial\Omega) \qquad \textit{and} \qquad \tfrac{1}{2}I + K^* : \hat{H}^p_{\mathrm{ct}}(\partial\Omega) \to \hat{H}^p_{\mathrm{ct}}(\partial\Omega) \tag{6.6}$$

are isomorphisms for each $1 - \varepsilon < p \leqslant 1$.

Proof. For $p = 1$ this is contained in [DaKe]. Our result then follows from (5.5), Corollary 5.3, and the stability results on complex interpolation scales of quasi-Banach spaces in [KaMi]. ∎

Proof of Theorem 6.2. Let $\varepsilon > 0$ be as in Lemma 6.3, and s, p as in the statement of Theorem 6.2. Then Lemma 6.3 gives that $g: (-\frac{1}{2}I + K^*)^{-1} f$ exists in $\hat{H}^p(\partial\Omega)$ and $\|g\|_{\hat{H}^p(\partial\Omega)} \leqslant C \|f\|_{\hat{H}^p(\partial\Omega)}$. If we now set $u := \mathscr{S}g$, it follows that u is harmonic in Ω and, by invoking Lemma 3.3 and Theorem 1.4, $u \in B^{\frac{1}{2}}_{2-s}(\partial\Omega)$ and (6.4) holds. To see that u is actually a solution to (6.2) we shall prove that u also satisfies (6.3). Indeed, this is a consequence of the general identity

$$\int_\Omega \langle \nabla \mathscr{S}h(X), \nabla\tilde{\psi}(X)\rangle \, dX = \langle (-\tfrac{1}{2}I + K^*)\, h, \psi\rangle, \qquad \forall \psi \in B^\infty_s(\partial\Omega), \tag{6.7}$$

which we claim is valid for arbitrary $h \in \hat{H}^p(\partial\Omega)$. In turn, this is easily checked on $H^p(\partial\Omega)$-atoms and, hence, extends by linearity to the whole $\hat{H}^p(\partial\Omega)$ by invoking Lemma 3.3 and Theorem 1.4. This completes the proof of the existence part.

Turning to uniqueness, assume that $u \in C^2(\Omega)$ solves the homogeneous version of (6.2) so that, in particular,

$$\int_\Omega \langle \nabla u(X), \nabla\tilde{\psi}(X)\rangle \, dX = 0 \qquad \forall \psi \in B^\infty_s(\partial\Omega). \tag{6.8}$$

Let $(\Omega_j)_j$ be a sequence of C^∞ sub-domains approximating Ω as in [Ve] and, for some fixed point $X_0 \in \Omega$, consider the Neumann function for Ω_j with pole at X_0, i.e.

$$\mathscr{N}_j(X) := \Gamma(X_0 - X) - \mathscr{S}_j((-\tfrac{1}{2}I + K_j^*)^{-1}\,(\langle \nabla\Gamma(X_0 - \cdot), N_j\rangle - \omega_n)(X) \tag{6.9}$$

for $X \in \Omega_j$, where ω_n stands for the area of the unit sphere in $\mathbb{R}^n$. Also, hereafter, $\mathscr{S}_j$, K_j^*, etc. will denote operators similar to $\mathscr{S}$, K^*, etc, but constructed in connection with $\partial\Omega_j$ rather than $\partial\Omega$. Take a smooth function $0 \leqslant \theta \leqslant 1$ which vanishes identically near X_0 and is identically 1 near $\partial\Omega$. Green's formula and an integration by parts then give

$$
\begin{aligned}
u(X_0) &= \int_{\partial\Omega_j} \mathcal{N}_j \frac{\partial u}{\partial N_j}\, d\sigma_j + \int_{\partial\Omega_j} \frac{\partial \mathcal{N}_j}{\partial N_j}\, u\, d\sigma_j \\
&= \int_{\partial\Omega_j} \theta \mathcal{N}_j \frac{\partial u}{\partial N_j}\, d\sigma_j + \omega_n \int_{\partial\Omega_j} u\, d\sigma_j \\
&= \int_{\Omega_j} \langle \nabla(\theta \mathcal{N}_j)(X), \nabla u(X) \rangle\, dX + \omega_n \int_{\partial\Omega_j} u\, d\sigma_j. \qquad (6.10)
\end{aligned}
$$

Note that there is no loss of generality in assuming that $\omega_n \int_{\partial\Omega_j} u\, d\sigma_j$ converges to some constant c as $j \to \infty$. The key step is to prove that

$$
\int_{\Omega_j} \langle \nabla(\theta \mathcal{N}_j)(X), \nabla u(X) \rangle\, dX \to \int_{\Omega} \langle \nabla(\theta \mathcal{N})(X), \nabla u(X) \rangle\, dX = 0 \qquad (6.11)
$$

as $j \to \infty$. Then, passing to the limit in (6.10) will give that $u \equiv c$ in Ω as desired. First, we shall prove that the second integral in (6.11) vanishes. To this end, recall that any harmonic function in Ω whose normal derivative vanishes on a portion of the boundary, say on $\partial\Omega \cap B(P, r)$, with $P \in \partial\Omega$, $r > 0$, actually belongs to $C^{\beta}(\bar{\Omega} \cap B(P, r))$, for some small, positive $\beta = \beta(\Omega, r)$; see [Ke]. This, in concert with the classical estimate of Zaremba, gives that there exists $\beta > 0$ such that

$$
\delta_j^{1-\beta}\, |\nabla \mathcal{N}_j|, \qquad \delta^{1-\beta}\, |\nabla \mathcal{N}| \leqslant C \quad \text{away from} \quad X_0, \quad \text{uniformly in } j, \qquad (6.12)
$$

so that

$$
\delta_j^{1-\beta}\, |\nabla(\theta \mathcal{N}_j)|, \qquad \delta^{1-\beta}\, |(\theta \nabla \mathcal{N})| \leqslant C \quad \text{in } \Omega, \quad \text{uniformly in } j, \qquad (6.13)
$$

With this at hand, the second integral in (6.11) vanishes on account of (6.8) if $0 < s < \beta$ which we can, and will, assume for the remaining part of the proof.

At this point we are left with proving the convergence in (6.11) which we tackle next. For k large we write

$$
\begin{aligned}
&\int_{\Omega_j} \langle \nabla(\theta \mathcal{N}_j), \nabla u \rangle\, dX \\
&\quad = \int_{\Omega_j \setminus \Omega_k} \langle \nabla(\theta \mathcal{N}_j), \nabla u \rangle\, dX + \int_{\Omega_k} \langle \nabla(\theta \mathcal{N}_j) - \nabla(\theta \mathcal{N}), \nabla u \rangle\, dX \\
&\qquad + \int_{\Omega_k} \langle \nabla(\theta \mathcal{N}), \nabla u \rangle\, dX \\
&\quad =: I_{j,k} + II_{j,k} + III_k.
\end{aligned}
$$

Now, by (6.13) and Lebesgue's dominated convergence theorem, $\lim_{k\to\infty} III_k = 0$. Further,

$$|I_{j,k}| \leqslant \left(\sup_{X\in\Omega_j\setminus\Omega_k} |\delta_j(X)^{1-s}\, \nabla(\theta\mathcal{N}_j)(X)| \right) \int_{\Omega_j\setminus\Omega_k} \delta_j^{s-1}\, |\nabla u|\, dX. \tag{6.15}$$

Since the first factor in the right side of (6.15) is bounded uniformly in j, k, by (6.13) and our assumption on s, and since the second one is $\leqslant C\, \|u\|_{B^1_{2-s}(\Omega_j\setminus\Omega_k)}$, it follows that $|I_{j,k}|$ is small if j, k are large enough.

To conclude the proof, we only need to show that, for a fixed k, $|II_{j,k}|$ is small if j is large enough. Thus, if set $f_j := (-\frac{1}{2}I + K_j^*)^{-1}(\langle \nabla\Gamma(X_0 - \cdot), N_j\rangle)$ and $f := (-\frac{1}{2}I + K^*)^{-1}(\langle \nabla\Gamma(X_0 - \cdot), N\rangle)$, it suffices to prove that

$$\nabla(\theta\mathcal{S}_j f_j)|_{\Omega_k} \to \nabla(\theta\mathcal{S} f)|_{\Omega_k} \quad \text{in} \quad L^2(\Omega_k) \quad \text{as} \quad j\to\infty. \tag{6.16}$$

This, in turn, follows from Lebesgue's dominated convergence theorem. First, it can be proved that $f_j \circ \Lambda_j \to f$ in $L^2(\partial\Omega)$ (cf. Section 9 in [MiMiPi] for details) which gives pointwise convergence; domination is trivially given by $|\nabla(\theta\mathcal{S}_j f_j)| \leqslant C$ on Ω_k uniformly in j. Note that (6.4) and (6.5) follow from Lemmas 3.3, 6.3 and the integral representation of the solution. Since similar arguments apply to the exterior case, the proof of Theorem 6.2 is now completed. ∎

7. AN ENDPOINT DIRICHLET PROBLEM

In this section we study the Dirichlet problem with boundary data in $B^1_{1-s}(\partial\Omega)$. Our main result, which is a strengthened form of Theorem 5.8 in [JeKe], establishes the well posedness of this problem for small s. The basic novel feature in our approach is that we are able to show that the solution has a normal derivative in $\hat{H}^p(\partial\Omega)$, where p is so that $s = (n-1)(1/p - 1)$. Moreover, this is accompanied by a natural estimate. Recall that Tr stands for the usual trace operator.

THEOREM 7.1. *Let Ω be an arbitrary bounded Lipschitz domain in $\mathbb{R}^n$. Then there exists $\varepsilon = \varepsilon(\Omega) > 0$ with the following significance. For any $0 < s < \varepsilon$ and each $f \in B^1_{1-s}(\partial\Omega)$ the Dirichlet problem*

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \operatorname{Tr} u = f & \text{on } \partial\Omega, \\ \delta^{s-1}\, |\nabla u| + |u| \in L^1(\Omega), \end{cases} \tag{7.1}$$

has a unique solution. The solution satisfies

$$\|u\|_{B^1_{2-s}(\Omega)} \leqslant C(\Omega, s) \, \|f\|_{B^1_{1-s}(\partial\Omega)} \tag{7.2}$$

and has $\partial u/\partial N \in \hat{H}^p(\partial\Omega)$, *where* $s=(n-1)(1/p-1)$. *More specifically, there exists* $g \in \hat{H}^p(\partial\Omega)$ *such that* u *is also a solution of the Neumann problem* (6.2) *with boundary datum* g. *In addition,*

$$\left\| \frac{\partial u}{\partial N} \right\|_{\hat{H}^p(\partial\Omega)} \leqslant C(\Omega, s) \, \|f\|_{B^1_{1-s}(\partial\Omega)}. \tag{7.3}$$

Finally, similar results are valid for the exterior Dirichlet problem (*note that, in this case,* $\partial u/\partial N \in \hat{H}^p_{\mathrm{ct}}(\partial\Omega)$).

Proof. We start with the existence part. As mentioned before, this has been already proved in [JeKe]. Here we develop a different approach which will eventually give us information about the normal derivative of the solution.

Choose ε as in Theorem 6.2 and let s, p be as in the statement of the theorem. First, we shall prove a technical result to the effect that

$$\exists C>0 \qquad \text{such that} \quad \|S^{-1}a\|_{H^p_{\mathrm{ct}}(\partial\Omega)} \leqslant C, \qquad \forall a \; B^1_{1-s}(\partial\Omega)\text{-atom}. \tag{7.4}$$

In the above, $B^1_{1-s}(\partial\Omega)$-atoms are regarded as elements in $L^2_1(\partial\Omega)$ so that $S^{-1}a$ belongs to $L^2(\partial\Omega) \subseteq H^p_{\mathrm{ct}}(\partial\Omega)$. To see this, we first note that for each $1-\varepsilon < p \leqslant 1$,

$$\|h\|_{H^p(\partial\Omega)} \leqslant C \, \|\nabla_{\tan} Sh\|_{H^p(\partial\Omega)} \tag{7.5}$$

uniformly for $h \in H^p(\partial\Omega)$. For $p=1$ this has been proved in [DaKe] and our inequality follows from it via the perturbation techniques devised in [KaMi]. This implies that the operator $S\colon H^p(\partial\Omega) \to H^p_1(\partial\Omega)/\langle 1 \rangle$ is injective with closed range which, in turn, yields a similar conclusion for $S\colon H^p_{\mathrm{ct}}(\partial\Omega) \to H^p_1(\partial\Omega)$. Note that, by the L^2-theory, $B^1_{1-s}(\partial\Omega)$-atoms belong to the range of S so that (7.4) is a consequence of this and the open mapping theorem.

Returning to the main line of reasoning, fix an arbitrary $f \in B^1_{1-s}(\partial\Omega)$. By Theorem 1.3, there exist a sequence of scalars $(\lambda_i)_i \in \ell^1$ and a sequence $(a_i)_i$ of $B^1_{1-s}(\partial\Omega)$-atoms such that

$$f = \sum_{i=0}^{\infty} \lambda_i a_i, \qquad \sum_{i=0}^{\infty} |\lambda_i| \leqslant 2 \, \|f\|_{B^1_{1-s}(\partial\Omega)}. \tag{7.6}$$

If we now set $u_i := \mathscr{S}(S^{-1}a_i)$ in Ω then, so we claim,

$$u(X) := \sum_{i=0}^{\infty} \lambda_i u_i(X), \qquad X \in \Omega, \tag{7.7}$$

solves (7.1). Indeed, clearly u is harmonic and

$$\|u\|_{B^1_{2-s}(\Omega)} \leqslant C \sum_{i=0}^{\infty} |\lambda_i| \leqslant C \|f\|_{B^1_{1-s}(\partial\Omega)} \tag{7.8}$$

by Theorem 3.1 and (7.4). It is also implicit in the above that $\sum_{i=0}^m \lambda_i u_i \to u$ in $B^1_{2-s}(\Omega)$ as $m \to \infty$ so that, by the continuity of the trace operator,

$$\operatorname{Tr} u = \sum_{i=0}^{\infty} \lambda_i \operatorname{Tr} u_i = \sum_{i=0}^{\infty} \lambda_i a_i = f, \tag{7.9}$$

which concludes the proof of the existence part.

Uniqueness can be proved along the lines of [JeKe]. Since the argument is short, we include it here for the sake of completeness. Specifically, let us assume that u solves the homogeneous version of (7.1) and consider $(\Omega_j)_j$ an approximating sequence of smooth subdomains of Ω as in [Ve]. First, owing to the existence of a "good" Green function in each Ω_j, it is not difficult to show that, in each Ω_j, the function u must coincide with the solution to the Dirichlet problem with boundary data $\operatorname{Tr}_j u$ constructed as above (here Tr_j stands for the boundary trace operator corresponding to Ω_j). Most importantly, by what we have proved so far, the estimate

$$\|u\|_{B^1_{2-s}(\Omega_j)} \leqslant C \|\operatorname{Tr}_j u\|_{B^1_{1-s}(\partial\Omega_j)} \tag{7.10}$$

holds uniformly in j. If we now recall that $u \in B^1_{2-s,0}(\Omega)$ and take $u_j \in C^\infty_{\text{comp}}(\Omega_j)$ approximating u in the norm of $B^1_{2-s}(\Omega)$, we have

$$\|\operatorname{Tr}_j u\|_{B^1_{1-s}(\partial\Omega_j)} = \|\operatorname{Tr}_j(u - u_j)\|_{B^1_{1-s}(\partial\Omega_j)} \leqslant C \|u - u_j\|_{B^1_{2-s}(\Omega)} \to 0. \tag{7.11}$$

Now the desired conclusion follows from (7.10) and (7.11).

In order to show that u constructed above solves a Neumann problem (in the sense of Section 6) with an appropriate boundary datum in $\hat{H}^p(\partial\Omega)$ it suffices to check that

$$\frac{\partial u_i}{\partial N} \in H^p(\partial\Omega) \qquad \text{and} \qquad \left\|\frac{\partial u_i}{\partial N}\right\|_{H^p(\partial\Omega)} \leqslant C, \tag{7.12}$$

uniformly in i. However, this easily follows from the integral representation of each u_i, the results in Section 3 and the boundedness of K^* on $\hat{H}^p_{\mathrm{ct}}(\partial\Omega)$. Moreover, (7.3) also follows in light of (7.12) and (7.6).

Finally, for the exterior domain the reasoning is similar (alternatively, one may use a Kelvin transform to reduce matters to the case already proved). ∎

8. INVERTIBILITY RESULTS

For each $0 \leqslant \varepsilon \leqslant 1$ consider the region $\mathscr{R}_\varepsilon$ depicted in Fig. 8.1. The points labeled in this picture are $O=(0,0)$, $A=(\varepsilon, 0)$, $B=(1, (1-\varepsilon)/2))$, $C(1,1)$, $D(1-\varepsilon, 1)$, $E=(0, (1+\varepsilon)/2)$, and $\mathscr{R}_\varepsilon$ is taken to be the interior of the hexagon OABCDE.

To state the main result of this section, define for $\Omega \subseteq \mathbb{R}^n$ bounded Lipschitz domain and $1 \leqslant p \leqslant \infty$, $0 < s < 1$,

$$\tilde{B}^p_{-s}(\partial\Omega) := \{ f \in B^p_{-s}(\partial\Omega); \langle f, 1 \rangle = 0 \}. \tag{8.1}$$

THEOREM 8.1. *For each bounded Lipschitz domain Ω in $\mathbb{R}^n$ there exists $\varepsilon > 0$ with the following significance. If $1 < p < \infty$ and $0 < s < 1$ are such that*

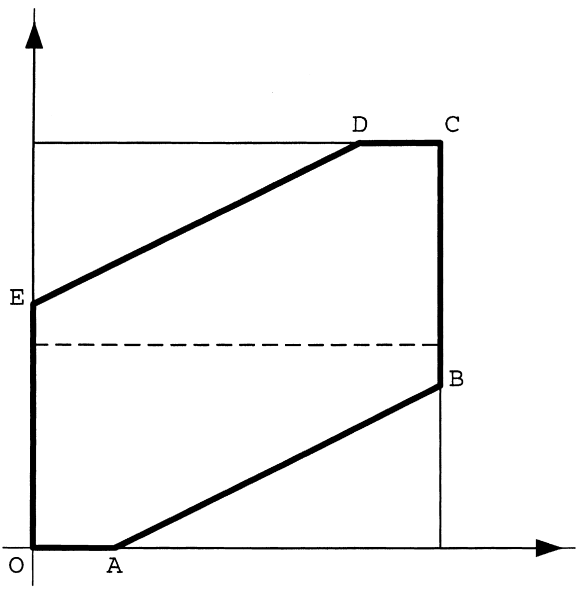

FIGURE 8.1

the point with coordinates $(s, 1/p)$ belongs to $\mathscr{R}_\varepsilon$ then the operators listed below are invertible:

(1) $\frac{1}{2}I+K\colon B^p_s(\partial\Omega)\to B^p_s(\partial\Omega)$;

(2) $\pm\frac{1}{2}I+K\colon B^p_s(\partial\Omega)/\langle 1\rangle\to B^p_s(\partial\Omega)/\langle 1\rangle$;

(3) $\frac{1}{2}I+K^*\colon B^q_{-s}(\partial\Omega)\to B^q_{-s}(\partial\Omega)$, *where* $1/p+1/q=1$;

(4) $\pm\frac{1}{2}I+K^*\colon \tilde{B}^q_{-s}(\partial\Omega)\to \tilde{B}^q_{-s}(\partial\Omega)$;

(5) $S\colon B^q_{-s}(\partial\Omega)\to B^q_{1-s}(\partial\Omega)$;

(6) $S\colon \tilde{B}^q_{-s}(\partial\Omega)\to B^q_{1-s}(\partial\Omega)/\langle 1\rangle$.

The above results are sharp in the class of Lipschitz domains. However, if $\partial\Omega\in C^1$ then we may always take $\varepsilon=1$.

The sense in which (1)–(6) are optimal is that for each $\varepsilon>0$ and for each point $(s, 1/p)\notin\mathscr{R}_\varepsilon$ there exists a Lipschitz domain $\Omega\subset\mathbb{R}^n$ such that (1)–(6) in Theorem 8.1 fail. Note that the region encompassed by the parallelogram with vertices at $(0,0)$, $(1,\frac{1}{2})$, $(1,1)$, and $(0,\frac{1}{2})$ is common for all Lipschitz domains, and that $\mathscr{R}_\varepsilon$ can be thought as an enhancement of it. Also, for $\varepsilon=1$, $\mathscr{R}_\varepsilon$ simply becomes the standard unit square in the plane.

The proof of Theorem 8.1 uses interpolation, and several special cases of interest are singled out below.

PROPOSITION 8.2. *For each bounded Lipschitz domain Ω in $\mathbb{R}^n$ there exists $\varepsilon=\varepsilon(\Omega)>0$ such that for $0<s<\varepsilon$ the operators*

$$\tfrac{1}{2}I+K\colon B^1_{1-s}(\partial\Omega)\to B^1_{1-s}(\partial\Omega) \tag{8.2}$$

and

$$\pm\tfrac{1}{2}I+K\colon B^1_{1-s}(\partial\Omega)/\langle 1\rangle\to B^1_{1-s}(\partial\Omega)/\langle 1\rangle \tag{8.3}$$

are isomorphisms.

Proof. One way of proving this is as follows. Let $\Omega_\pm$ stand, respectively, for Ω and $\mathbb{R}^n\setminus\bar{\Omega}$ and let ε be as in Theorem 7.1, $0<s<\varepsilon$. Also, denote by $\mathrm{Tr}_\pm$ the boundary trace operators corresponding to $\Omega_\pm$. From Section 7, we know that functions u harmonic in $\Omega_\pm$ with $u\in B^1_{2-s}(\bar{\Omega}_\pm, \mathrm{loc})$ satisfy

$$\left\| \frac{\partial u}{\partial N}\bigg|_{\partial\Omega_\pm} \right\|_{\hat{H}^p(\partial\Omega)} \approx \|\mathrm{Tr}_\pm u\|_{B^1_{1-s}(\partial\Omega)/\langle 1\rangle} \tag{8.4}$$

where the normal derivatives on $\partial\Omega_\pm$ are taken in the sense discussed in Section 6. We shall utilize this when $u:=\mathscr{D}f$ in $\Omega_\pm$, for some arbitrary, fixed $f \in B^1_{1-s}(\partial\Omega)$. Since, in this case, $(\partial u/\partial N)|_{\partial\Omega_+} = (\partial u/\partial N)|_{\partial\Omega_-}$ while $\mathrm{Tr}_{\pm} u = (\pm\frac{1}{2}I + K) f$, the triangle inequality readily gives

$$\|f\|_{B^1_{1-s}(\partial\Omega)/\langle 1\rangle} \leqslant C \,\|(\pm\tfrac{1}{2}I + K) f\|_{B^1_{1-s}(\partial\Omega)/\langle 1\rangle}. \tag{8.5}$$

In particular, the operators in (8.3) are injective with closed ranges. Now, the $L^2_1(\partial\Omega)$-theory in [Ve] together with the atomic decomposition of $B^1_{1-s}(\partial\Omega)$ give that they also have dense ranges and, hence, are isomorphisms.

Finally, since $K1 = \frac{1}{2}$ the fact that the operator in (8.2) is an isomorphism follows easily from what we have proved already. ∎

PROPOSITION 8.3. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Then there exists $\varepsilon = \varepsilon(\Omega) > 0$ such that for $0 < s < \varepsilon$ the operators*

$$S\colon B^\infty_{s-1}(\partial\Omega) \to B^\infty_s(\partial\Omega) \tag{8.6.}$$

and

$$S\colon \tilde{B}^\infty_{s-1}(\partial\Omega) \to B^\infty_s(\partial\Omega)/\langle 1\rangle \tag{8.7}$$

are isomorphisms.

Proof. This will follow by duality from the fact that for $0 < s < \varepsilon$, ε as in Theorem 7.1, and $s = (n-1)(1/p - 1)$, the operators

$$S\colon \hat{H}^p_{\mathrm{ct}}(\partial\Omega) \to B^1_{1-s}(\partial\Omega), \qquad S\colon \hat{H}^p(\partial\Omega) \to B^1_{1-s}(\partial\Omega)/\langle 1\rangle \tag{8.8}$$

are isomorphisms. Note that, by the results in Sections 3 and 5, they are well defined and bounded. We shall only indicate a proof for the first operator since the remaining case follows more or less directly from this. To this end, consider some $f \in B^1_{1-s}(\partial\Omega)$ which has an atomic decomposition of the form $f = \sum \lambda_i a_i$, where $(\lambda_i)_i \in \ell^1$ and the a_i's are $B^1_{1-s}(\partial\Omega)$-atoms. By (7.4), we can find $h_i \in H^p_{\mathrm{ct}}(\partial\Omega)$ so that $\|h_i\|_{H^p_{\mathrm{ct}}(\partial\Omega)} \leqslant C$ and $Sh_i = a_i$. Then $\sum \lambda_i h_i$ converges in $\hat{H}^p(\partial\Omega)$ to an element g which S should send into f. Thus, S is onto.

Finally, to see that S is also one-to-one, take some $f \in \hat{H}^p_{\mathrm{ct}}(\partial\Omega)$ so that $Sf = 0$. It follows from the uniqueness in the interior and the exterior Dirichlet problem with data in $B^1_{1-s}(\partial\Omega)$ that $\mathcal{S}f$ must vanish identically both in Ω and in $\mathbb{R}^n \setminus \bar{\Omega}$. Since f is the jump of $\partial\mathcal{S}f/\partial N$ across $\partial\Omega$, we infer that $f = 0$. ∎

As we shall see momentarily, our proof of Theorem 8.1 requires one extra technical ingredient which we now discuss. The idea is that, contrary to what goes on for boundedness, invertibility is not preserved under interpolation (generally speaking); for counterexamples, see [FaJoLe]. Below we explain a specific setting in which this phenomenon is true nonetheless. To set the stage, consider (X_0, X_1) a couple of compatible Banach spaces and, as usual, equip $X_0 \cap X_1$ and $X_0 + X_1$, respectively, with the norms

$$\|x\|_{X_0 \cap X_1} := \max\{\|x\|_{X_0}, \|x\|_{X_1}\} \tag{8.9}$$

and

$$\|z\|_{X_0+X_1} = \inf\{\|x_0\|_{X_0} + \|x_1\|_{X_1};\ z = x_0 + x_1,\ x_i \in X_i,\ i = 0, 1\}. \tag{8.10}$$

LEMMA 8.4. *With the above notation, let $T\colon X_0 + X_1 \to X_0 + X_1$ be a linear operator such that $T\colon X_i \to X_i$ is an isomorphism, $i = 0, 1$. Assume that there exists a Banach space Y such that the inclusion $Y \hookrightarrow X_0 \cap X_1$ is continuous with dense range, and that $T\colon Y \to Y$ is an isomorphism.*

Then, for any $0 \leqslant \theta \leqslant 1$, the operator $T\colon [X_0, X_1]_\theta \to [X_0, X_1]_\theta$ is an isomorphism. Moreover, a similar statement is valid for the real method of interpolation.

Proof. Denote by $R_i\colon X_i \to X_i$ the inverse of T on X_i, $i = 0, 1$. First, R_0 and R_1 coincide on Y and, by density, they also agree on $X_0 \cap X_1$. It is therefore meaningful to define

$$R\colon X_0 + X_1 \to X_0 + X_1, \qquad \text{by} \quad R(x_0 + x_1) := R_0(x_0) + R_1(x_1),\ x_i \in X_i. \tag{8.11}$$

Hence, R is a bounded, linear operator which leaves both X_0 and X_1 invariant. In particular, R maps $[X_0, X_1]_\theta$ boundedly into itself and, as such, it provides an inverse for $T\colon [X_0, X_1]_\theta \to [X_0, X_1]_\theta$ (since $TR = RT = I$ on $X_0 \cap X_1$, etc). ∎

With all major ingredients in place, we are now ready to present the

Proof of Theorem 8.1. Let us deal first with the operators $\pm\frac{1}{2}I + K$. The segments (E, O) and (B, C) corresponding to invertibility on $L^p(\partial\Omega)$ or $L^p(\partial\Omega)/\langle 1 \rangle$, for $0 < 1/p < (1+\varepsilon)/2$, and on $L^p_1(\partial\Omega)$ or $L^p_1(\partial\Omega)/\langle 1 \rangle$, for $(1-\varepsilon)/2 < 1/p < 1$, respectively, have been treated in [DaKe]. Also, the segment (O, A), corresponding to invertibility on $B^\infty_s(\partial\Omega)$ or $B^\infty_s(\partial\Omega))/\langle 1 \rangle$, for s small, has been taken care of in [Br], [KaMi], while invertibility on the segment (C, D) is covered by Proposition 8.2. Notice that the interior of the convex hull of these segments is precisely the region $\mathscr{R}_\varepsilon$ (cf. Fig. 8.1). Now, the desired result follows by repeated applications of the real and complex methods of interpolation together with Lemma 8.4. We leave the details to the reader.

By duality, we obtain results for $\pm\frac{1}{2}I + K^*$ on the corresponding dual scales. Furthermore, a similar reasoning applies to the operator S, given the results in [Ve], [DaKe] for $L_1^p(\partial\Omega)$ and Proposition 8.3. Once again, we omit the straightforward details.

The fact that these results are sharp in the class of Lipschitz domains is guaranteed (in the light of our results in Sections 3 and 4) by the counterexamples in [JeKe] to the solvability of the Dirichlet problem. Finally, that $\partial\Omega \in C^1$ allows us to take $\varepsilon = 1$ follows from the results in [FaJoRi] via the same interpolation patterns. ∎

Remark. Let us note that it is possible to extend the results of this section to Lipschitz domains with *arbitrary topology* by appropriately adapting the techniques of [MiD]. In order to explain how the statement of Theorem 8.1 should be altered, we need more notation. For an arbitrary bounded Lipschitz domain Ω in $\mathbb{R}^n$ set $\Omega_+ := \Omega$, $\Omega_- := \mathbb{R}^n \setminus \bar{\Omega}$ and

$$\begin{aligned} \mathbb{R}_{\Omega_\pm} &:= \operatorname{span}_{\mathbb{R}} \{\chi_{\Omega'};\ \Omega' \text{ bounded connected component of } \Omega_\pm\} \\ \mathbb{R}_{\partial\Omega_\pm} &:= \operatorname{span}_{\mathbb{R}} \{\chi_{\partial\Omega'};\ \Omega' \text{ bounded connected component of } \Omega_\pm\} \\ \mathbb{R}_{\partial\Omega} &:= \operatorname{span}_{\mathbb{R}} \{\chi_{\omega};\ \omega \text{ connected component of } \partial\Omega\} \end{aligned} \tag{8.12}$$

Then, for the same range of indices as in Theorem 8.1, the following operators are isomorphisms:

(1) $\pm\frac{1}{2}I + K\colon B_s^p(\partial\Omega)/\mathbb{R}_{\partial\Omega_\mp} \to B_s^p(\partial\Omega)/\mathbb{R}_{\partial\Omega_\mp}$;

(2) $\pm\frac{1}{2}I + K\colon B_s^p(\partial\Omega)/\mathbb{R}_{\partial\Omega} \to B_s^p(\partial\Omega)/\mathbb{R}_{\partial\Omega}$;

(3) $\pm\frac{1}{2}I + K^*$ acting from $\{f \in B_{-s}^q(\partial\Omega);\ \langle f, \chi\rangle = 0,\ \forall \chi \in \mathbb{R}_{\partial\Omega_\mp}\}$ onto itself;

(4) $\pm\frac{1}{2}I + K^*$ acting from $\{f \in B_{-s}^q(\partial\Omega);\ \langle f, \chi\rangle = 0,\ \forall \chi \in \mathbb{R}_{\partial\Omega}\}$ onto itself;

(5) $S\colon B_{-s}^q(\partial\Omega) \to B_{1-s}^q(\partial\Omega)$;

(6) $S\colon \{f \in B_{-s}^q(\partial\Omega);\ \langle f, \chi\rangle = 0,\ \forall \chi \in \mathbb{R}_{\partial\Omega_\pm}\} \to B_{1-s}^q(\partial\Omega)/\mathbb{R}_{\partial\Omega_\pm}$;

(7) $S\colon \{f \in B_{-s}^q(\partial\Omega);\ \langle f, \chi\rangle = 0,\ \forall \chi \in \mathbb{R}_{\partial\Omega}\} \to B_{1-s}^q(\partial\Omega)/\mathbb{R}_{\partial\Omega}$.

9. THE GENERAL POISSON PROBLEM WITH NEUMANN BOUNDARY CONDITIONS

The first order of business is to properly formulate the Poisson problem for the Laplace operator with Neumann boundary conditions in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$. We commence by noting that the pairing between $L_{-s+1/q}^q(\Omega)$ and $L_{s-1+1/p}^p(\Omega)$ is well defined for any $0 < s < 1$, $1 < p,\ q < \infty$ with $1/p + 1/q = 1$. In fact, since $C_{\mathrm{comp}}^\infty(\Omega)$ is dense in $L_\alpha^q(\Omega)$ for $0 \leqslant \alpha < 1/q$, it is not difficult to see that $L_{\alpha,0}^q(\Omega) = L_\alpha^q(\Omega)$ for $0 \leqslant \alpha < 1/q$,

$1 < p < \infty$ (eventually by invoking Proposition 3.5 in [JeKe]; cf. also Theorem 11.4 in [LiMa]). In particular,

$$L^q_{-s+1/q}(\Omega) = (L^p_{s-1+1/p}(\Omega))^*, \qquad \text{if} \quad -s + \frac{1}{q} \leqslant 0, \tag{9.1}$$

and

$$L^p_{s-1+1/p}(\Omega) = (L^q_{-s+1/q}(\Omega))^*, \qquad \text{if} \quad 0 \leqslant -s + \frac{1}{q}. \tag{9.2}$$

This observation allows us to define the normal component of any vector field F with components in $L^q_{-s+1/q}(\Omega)$ for $0 < s < 1$ and $1 < p,\ q < \infty$, $1/p + 1/q = 1$. Namely, for any extension $f \in (L^p_{s+1/p}(\Omega))^* = L^q_{-s-1/p,\,0}(\Omega)$ of the distribution $\operatorname{div} F \in (C^\infty_{\text{comp}}(\Omega))'$ (as usual, $\langle \operatorname{div} F, \phi \rangle = -\langle F, \nabla\phi \rangle$, for all $\phi \in C^\infty_{\text{comp}}(\Omega)$), we denote by $F \cdot N_f$ the normal component of F (with respect to the extension f) and define it as the linear functional in $B^q_{-s}(\partial\Omega) = (B^p_s(\partial\Omega))^*$ given by

$$\langle F \cdot N_f, \phi \rangle := \langle f, \tilde{\phi} \rangle + \langle F, \nabla\tilde{\phi} \rangle, \qquad \forall \phi \in B^p_s(\partial\Omega), \tag{9.3}$$

where $\tilde{\phi} \in L^p_{s+1/p}(\Omega)$ is an extension (in the trace sense) of ϕ. The second pairing in the right side of (9.3) is understood in the sense of (9.1) and (9.2) and is well defined since $\nabla\tilde{\phi} \in L^p_{s+(1/p)-1}(\Omega)$. In turn, this membership is a consequence of our assumptions and the lemma below.

Lemma 9.1. *For any* $1 < p < \infty$ *and* $s > 0$, *the operator* $\nabla \colon L^p_s(\Omega) \to L^p_{s-1}(\Omega)$ *is well defined and bounded.*

Proof. The case when $s \geqslant 1$ is treated in [JeKe; Proposition 2.18] so we restrict attention to $0 < s < 1$. To this end, with Ω replaced by $\mathbb{R}^n$ the above lemma is seen by considering the trivial case $s = 1$, duality (cf. Corollary 6.2.8 in [BeLö]) and interpolation. Now, if $u \in L^p_s(\Omega)$ and $U \in L^p_s(\mathbb{R}^n)$ is such that $U|_\Omega = u$, $\|U\|_{L^p_s(\mathbb{R}^n)} \leqslant 2\,\|u\|_{L^p_s(\Omega)}$ then, for any $\psi \in C^\infty_{\text{comp}}(\mathbb{R}^n)$ with $\operatorname{supp} \psi \subset \Omega$,

$$\begin{aligned} |\langle \nabla u, \psi \rangle| = |\langle \nabla U, \psi \rangle| &\leqslant \|\nabla U\|_{L^p_{s-1}(\mathbb{R}^n)}\, \|\psi\|_{L^q_{1-s}(\mathbb{R}^n)} \\ &\leqslant C\,\|U\|_{L^p_s(\mathbb{R}^n)}\, \|\psi\|_{L^q_{1-s}(\mathbb{R}^n)} \leqslant C\,\|u\|_{L^p_s(\Omega)}\, \|\psi\|_{L^q_{1-s}(\mathbb{R}^n)}. \end{aligned}$$

Thus, $\nabla u \in (L^q_{1-s,\,0}(\Omega))^* = L^p_{s-1}(\Omega)$ and $\|\nabla u\|_{L^p_{s-1}(\Omega)} \leqslant C\,\|u\|_{L^p_s(\Omega)}$, by (1.8) and (1.9). ∎

Returning to (9.3), it is not difficult to check that the definition is correct and that

$$\|F \cdot N_f\|_{B^q_{-s}(\partial\Omega)} \leqslant C\,\|F\|_{L^q_{-s+1/q}(\Omega)} + C\,\|f\|_{(L^p_{s+1/p}(\Omega))^*}. \tag{9.5}$$

In particular, if the divergence of a field $F \in L^q(\Omega)$ (taken in the sense of distributions) belongs to $L^q(\Omega)$, $1<q<\infty$, then $F\cdot N_f$, the normal component of F, defined as in (9.3) for $f := \operatorname{div} F$, belongs to $B^q_{-1/q}(\partial\Omega)$ and satisfies the estimates (9.5) with $s=1/q$. This particular instance will be relevant in the study of Helmholtz type decompositions in Section 11.

We notice, moreover, that for any function $u \in L^q_{1-s+1/q}(\Omega)$ and any extension f of $\Delta u = \operatorname{div}(\nabla u)$, considered first as a distribution in Ω to an element in $L^q_{-1-s+1/q,\,0}(\Omega)$, the normal derivative $(\partial u/\partial N)_f$ (with respect to the extension f) can be defined, in the sense of (9.3), as $\nabla u \cdot N_f$. In what follows, when no confusion is likely to occur, we will drop the subindex f and simply write $\partial u/\partial N$, the dependence on the particular extension of Δu being implicitly understood.

For further reference we also note the integral formulas

$$u = \Pi_\Omega(f) + \mathscr{D}(\operatorname{Tr} u) - \mathscr{S}\left(\frac{\partial u}{\partial N}\right), \tag{9.6}$$

valid for arbitrary $u \in L^q_{1-s+1/q}(\Omega)$ with Δu extendible to $f \in (L^p_{s+1/p}(\Omega))^*$, and

$$\left\langle \frac{\partial \Pi_\Omega(f)}{\partial N}, \phi \right\rangle = \langle f, \mathscr{D}\phi\rangle, \qquad \forall \phi \in B^p_s(\partial\Omega), \tag{9.7}$$

valid for any $f \in (L^p_{s+1/p}(\Omega))^*$. They can be easily justified starting from (9.3), using a limiting argument and invoking the mapping properties of Π_Ω, $\mathscr{S}$, $\mathscr{D}$ established in Sections 2–4.

The main focus of this section is the boundary problem

$$\begin{cases} \Delta u = f \in L^q_{(1/q)-s-1,\,0}(\Omega), \\ \dfrac{\partial u}{\partial N} = g \in B^q_{-s}(\partial\Omega), \\ u \in L^q_{1-s+1/q}(\Omega), \end{cases} \tag{9.8}$$

subject to the (necessary) compatibility condition

$$\langle f, 1\rangle = \langle g, 1\rangle. \tag{9.9}$$

In this regard, our main result is the following.

THEOREM 9.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, Then there exists $\varepsilon = \varepsilon(\Omega) > 0$ having the following relevance. If $1 < p,\ q < \infty$, $1/p + 1/q = 1$ and $0 < s < 1$ are such that $(s, 1/p)$ belongs to the region $\mathcal{R}_\varepsilon$ (in Fig. 8.1) then the Poisson problem with Neumann boundary condition (9.8) has a unique (modulo additive constants) solution u for any f, g satisfying the compatibility condition (9.9). Moreover,*

$$u = \Pi_\Omega(f) + \mathcal{S}\left(-\frac{1}{2}I + K^*\right)^{-1}\left(g - \frac{\partial \Pi_\Omega(f)}{\partial N}\right) \tag{9.10}$$

and there exists a positive constant C which depends only on Ω, p, s, such that

$$\|u\|_{L^q_{1-s+1/q}(\Omega)/\langle 1\rangle} \leqslant C\,\|f\|_{L^q_{(1/q)-s-1,\,0}(\Omega)} + C\,\|g\|_{B^q_{-s}(\partial\Omega)}. \tag{9.11}$$

Also, a similar result is valid for the exterior domain $\mathbb{R}^n \backslash \bar{\Omega}$ too. Finally, if $\partial\Omega \in C^1$ then we may take $\varepsilon = 1$.

Remark. It should be noted that similar results are valid for the scales of Besov spaces, i.e. when $f \in (B^p_{s+1/p}(\Omega))^*$. Of course, in this case the solution u belongs to $B^q_{1-s+1/q}(\Omega)$ and the second pairing in (9.3) remains meaningful because of Theorem 1.4.4.6 and Corollary 1.4.4.5 in [Gr].

Proof of Theorem 9.2. In view of Proposition 2.1, (9.7) and (9.9), subtracting $\Pi_\Omega(f)$ reduces the problem to solving (9.8) with $f = 0$ and $\tilde{g} := g - \partial \Pi_\Omega(f)/\partial N \in \tilde{B}^q_{-s}(\partial\Omega)$. For this latter problem, a solution is given by $\mathcal{S}(-\frac{1}{2}I + K^*)^{-1}\,\tilde{g}$; cf. Theorems 3.1 and 8.1. This finishes the proof of the existence part. Note that (9.11) follows from the integral representation formula (9.10) of the solution and mapping properties of layer potentials.

There remains to establish uniqueness. To this end, if $u \in L^q_{1-s+1/q}(\Omega)$ solve the homogeneous version of (9.8), then taking the boundary trace in (9.6) gives that $(-\frac{1}{2}I + K)(\mathrm{Tr}\,u) = 0$. The important thing is that the region $\mathcal{R}_\varepsilon$ is invariant to the transformation $(s, 1/p) \mapsto (1-s, 1-1/p)$ and that $\mathrm{Tr}\,u \in B^q_{1-s}(\partial\Omega)$. Thus, on account of Theorem 8.1, $\mathrm{Tr}\,u \equiv$ constant. Utilizing this back in (9.6) yields $u \equiv$ constant in Ω, as desired.

The argument for the exterior problem (and C^1 domains) is similar and, hence, omitted. ∎

An important particular case, corresponding to $s = -1/p$, is singled out below for further reference.

COROLLARY 9.3. *For Ω bounded Lipschitz domain in $\mathbb{R}^n$, there exists a positive number $\varepsilon = \varepsilon(\Omega)$ with the following significance. If $\frac{3}{2} - \varepsilon < p < 3 + \varepsilon$,* *then for any* $f \in L^p_{-1,0}(\Omega)$ *and any* $g \in B^p_{-1/p}(\partial\Omega)$ *satisfying the compatibility condition* (9.9) *the Neumann problem*

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ \dfrac{\partial u}{\partial N} = g & \text{on } \partial\Omega, \\ u \in L^p_1(\Omega), \end{cases} \tag{9.12}$$

has a unique (*modulo additive constants*) *solution u. Recall that the Neumann boundary condition in* (9.12) *is interpreted in the sense that*

$$\int_\Omega \langle \nabla u(X), \nabla \phi(X) \rangle \, dX = -\langle f, \phi \rangle + \langle g, \operatorname{Tr} \phi \rangle \qquad \text{for any} \quad \phi \in L^q_1(\Omega). \tag{9.13}$$

Moreover, ∇u *satisfies the estimate*

$$\|\nabla u\|_{L^p(\Omega)} \leqslant C(\Omega, p)(\|f\|_{L^p_{-1,0}(\Omega)} + \|g\|_{B^p_{-1/p}(\partial\Omega)}). \tag{9.14}$$

Proof. One only needs to observe that $(1/p, 1+1/p) \in \mathcal{R}_\varepsilon$ for p in a neighborhood of the interval $[\frac{3}{2}, 3]$. ∎

As we shall see in Section 2, the results in this section are sharp in the class of Lipschitz domains.

Remark. Once again, the results of this section can be extended to bounded Lipschitz domains with arbitrary topology. In particular, the compatibility condition (9.9) for $\Omega_\pm$ should read in this case

$$\langle f, \chi_{\Omega'} \rangle = \langle g, \chi_{\partial\Omega'} \rangle, \qquad \forall \Omega' \text{ bounded connected component of } \Omega_\pm \tag{9.15}$$

and the spaces of null-solutions for (9.12) corresponding to $\Omega_\pm$ are $\mathbb{R}_{\Omega_\pm}$. Cf. also [MiD].

10. THE GENERAL POISSON PROBLEM WITH DIRICHLET BOUNDARY CONDITIONS

In this section we shall prove the following companion to Theorem 9.2, refining previous work in [JeKe].

THEOREM 10.1. *For each bounded Lipschitz domain* Ω *in* $\mathbb{R}^n$ *there exists* $\varepsilon = \varepsilon(\Omega) > 0$ *with the following interpretation. If* $1 < p < \infty$ *and* $0 \leqslant s \leqslant 1$ *are* *such that* $(s, 1/p) \in \mathcal{R}_\varepsilon$ (*cf.* Fig. 8.1) *then for any* $f \in L^p_{s+1/p-2}(\Omega)$ *and any* $g \in B^p_s(\partial\Omega)$ *the Dirichlet problem*

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ \operatorname{Tr} u = g & \text{on } \partial\Omega, \\ u \in L^p_{s+1/p}(\Omega), \end{cases} \tag{10.1}$$

has a unique solution. Also, there exists $C > 0$ *depending only on* Ω, p, s, *such that the solution satisfies the estimate*

$$\|u\|_{L^p_{s+1/p}(\Omega)} \leqslant C\, \|f\|_{L^p_{(1/p)-s-2}(\Omega)} + C\, \|g\|_{B^p_s(\partial\Omega)}. \tag{10.2}$$

In fact,

$$u = \Pi(\tilde{f})|_\Omega + \mathcal{D}((\tfrac{1}{2}I + K)^{-1}\,(g - \operatorname{Tr}\Pi(\tilde{f}))), \qquad \text{in } \Omega \tag{10.3}$$

where $\tilde{f}$ *is an extension of* f *to an element in* $L^p_{s+(1/p)-2}(\mathbb{R}^n)$.

In particular, if $f \in L^p_{(1/p)-s-2,\,0}(\Omega)$, *then the solution has a normal derivative in* $B^p_{s-1}(\partial\Omega)$ (*in the sense discussed in Section* 9) *and*

$$\left\|\frac{\partial u}{\partial N}\right\|_{B^p_{s-1}(\partial\Omega)} \leqslant C(\|f\|_{L^p_{(1/p)-s-2,\,0}(\Omega)} + \|g\|_{B^p_s(\partial\Omega)}). \tag{10.4}$$

A similar statement is valid for the exterior Dirichlet problem. Finally, if $\partial\Omega \in C^1$ *then we can actually take* $\varepsilon = 1$.

Remark. Once again, as it will be apparent from the proof, similar results are valid on the scale of Besov spaces, i.e. when $f \in B^p_{(1/p)-s-2}(\Omega)$. In this case, the solution u belongs to $B^p_{s+1/p}(\Omega)$; we omit the details.

Proof of Theorem 10.1. Fix an arbitrary $f \in L^p_{(1/p)-s-2}(\Omega) = (L^q_{1+(1/q)+s,0}(\Omega))^*$. Since $L^q_{1+(1/q)+s,\,0}(\Omega)$ can be identified (via extension by zero outside the support and restriction to Ω) with $\{\psi \in L^q_{1+(1/q)+s}(\mathbb{R}^n);\ \operatorname{supp}\psi \subseteq \bar{\Omega}\}$, we may invoke Hahn–Banach's extension theorem to produce $\tilde{f} \in L^p_{(1/p)-s-2}(\mathbb{R}^n)$ so that $\tilde{f}|_\Omega = f$ and the norm of $\tilde{f}$ is controlled by that of f. Of course, there is no loss of generality in assuming that $\tilde{f}$ is compactly supported.

Then, existence is obtained from (10.3) and the mapping properties of the operators involved. Notice that this also gives the estimate (10.2). Uniqueness can be established by mimicking the argument already utilized in the proof of Theorem 9.2.

Since we may express u in the alternative form

$$u = \Pi(\tilde{f})|_\Omega + \mathcal{S}(S^{-1}(g - \operatorname{Tr}\Pi(\tilde{f}))) \qquad \text{in } \Omega, \tag{10.5}$$

the estimate (10.4) is implied by the discussion in the previous section.

The corresponding statement for the exterior Dirichlet problem follows along similar lines (or by using a Kelvin transform). Finally, invoking [FaJoRi] and proceeding as before, it is clear that we may take $\varepsilon = 1$ if $\partial\Omega \in C^1$. ∎

In closing, let us point out that the results of this section also extend to the case of Lipschitz domains with arbitrary topology.

11. HELMHOLTZ TYPE DECOMPOSITIONS IN LIPSCHITZ DOMAINS

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ with arbitrary topology. The goal is to establish L^p based Helmholtz type decompositions for vector fields in Ω. In what follows, we shall make no notational distinction between scalar-valued and vector-valued function with components in $L^p(\Omega)$; both are going to be denoted by $L^p(\Omega)$. Recall from Section 9 that the mapping

$$\{u \in L^p(\Omega);\, \operatorname{div} u \in L^p(\Omega)\} \ni u \mapsto u \cdot N \in B^p_{-1/p}(\partial\Omega) \tag{11.1}$$

for $1 < p,\ q < \infty$, $1/p + 1/q = 1$, is well defined and bounded in the sense that

$$\|u \cdot N\|_{B^p_{-1/p}(\partial\Omega)} \leqslant C(\Omega, p)(\|u\|_{L^p(\Omega)} + \|\operatorname{div} u\|_{L^p(\Omega)}). \tag{11.2}$$

Notice that if $u \in L^p(\Omega)$ has $\operatorname{div} u = 0$ then $u \cdot N$, as a functional in $(B^q_{1-1/q}(\partial\Omega))^*$, annihilates all functions of the form $\chi_{\partial\Omega'}$, with Ω' connected component of Ω. We denote the collection of all such functionals by $\widetilde{B}^p_{-1/p}(\partial\Omega)$.

Next, we introduce

$$L^p_{\operatorname{div},0}(\Omega) := \{u \in L^p(\Omega);\, \operatorname{div} u = 0 \text{ and } u \cdot N = 0\} \tag{11.3}$$

and

$$\operatorname{grad} L^p_1(\Omega) := \{\nabla u;\, u \in L^p_1(\Omega)\}. \tag{11.4}$$

They are easily seen to be closed subspaces of $L^p(\Omega)$ and, for $p = 2$, we denote by $\mathscr{P}$, $\mathscr{Q}$ the corresponding orthogonal projections from $L^2(\Omega)$ onto $L^2_{\operatorname{div},0}(\Omega)$ and $\operatorname{grad} L^2_1(\Omega)$, respectively.

THEOREM 11.1. *For each Lipschitz domain Ω in $\mathbb{R}^n$, with arbitrary topology, there exists a positive number ε depending on Ω such that $\mathscr{P}$ and*

$\mathcal{Q}$ extend to bounded operators from $L^p(\Omega)$ onto $L^p_{\mathrm{div},0}(\Omega)$ and onto grad $L^p_1(\Omega)$, *respectively, for each $\frac{3}{2}-\varepsilon<p<3+\varepsilon$. Hence, in this range,*

$$L^p(\Omega)=\mathrm{grad}\, L^p_1(\Omega)\oplus L^p_{\mathrm{div},0}(\Omega) \tag{11.5}$$

where the direct sum is topological.

In the class of Lipschitz domains, this result is sharp. If, however, $\partial\Omega\in C^1$ then we may take $1<p<\infty$.

Proof. Let $1<p<\infty$ be such that the boundary problem (9.12), subject to the compatibility condition (9.15), is solvable (uniquely modulo $\mathbb{R}_\Omega$, with the gradient of the solution satisfying natural estimates) for arbitrary data. Recall that Π_Ω stands for the Newtonian potential which acts component-wise on vector fields. We define $\tilde{\mathcal{P}}: L^p(\Omega)\to L^p_{\mathrm{div},0}(\Omega)\hookrightarrow L^p(\Omega)$ by setting

$$\tilde{\mathcal{P}}u:=u-\nabla\,\mathrm{div}\,\Pi_\Omega(u)-\nabla\psi,\qquad \forall u\in L^p(\Omega), \tag{11.6}$$

where ψ is the unique solution to the Neumann boundary problem

$$\begin{cases}\Delta\psi=0 & \text{in } \Omega,\\ \dfrac{\partial\psi}{\partial N}=(u-\nabla\,\mathrm{div}\,\Pi_\Omega(u))\cdot N\in\tilde{B}^p_{-1/p}(\partial\Omega),\\ \psi\in L^p_1(\Omega).\end{cases} \tag{11.7}$$

By assumption, $\tilde{\mathcal{P}}$ is well-defined, linear, and bounded and, moreover, $I-\tilde{\mathcal{P}}$ maps $L^p(\Omega)$ boundedly into grad $L^p_1(\Omega)$.

Next, we aim to prove that $\tilde{\mathcal{P}}$ is onto $L^p_{\mathrm{div},0}(\Omega)$. Indeed, so we claim,

$$\tilde{\mathcal{P}}|_{L^p_{\mathrm{div},0}(\Omega)}=I,\qquad \text{the identity operator on } L^p_{\mathrm{div},0}(\Omega). \tag{11.8}$$

To see this, note that if $u\in L^p_{\mathrm{div},0}(\Omega)$ and if ψ solves (11.7), then the function $\psi+\mathrm{div}\,\Pi_\Omega(u)$ is harmonic, belongs to $L^p_1(\Omega)$ and has vanishing normal derivative. Invoking uniqueness for the Neumann problem, it follows that $\tilde{\mathcal{P}}(u)=u-\nabla\,\mathrm{div}\,\Pi_\Omega(u)-\nabla\psi=u$ as claimed.

The fact that on $L^2(\Omega)\cap L^p(\Omega)$ the operator $\tilde{\mathcal{P}}$ acts as the orthogonal projection onto $L^2_{\mathrm{div},0}(\Omega)$ is easily seen from (11.6). Thus, $\mathcal{P}$ extends to a bounded mapping of $L^p(\Omega)$ onto $L^p_{\mathrm{div},0}(\Omega)$, as desired. From this, the statement about $\mathcal{Q}=I-\mathcal{P}$ follows as well.

Finally, the range $p\in(\frac{3}{2}-\varepsilon,3+\varepsilon)$, with ε as in Corollary 9.3, ensures the solvability of the boundary problem (9.12), (9.15). The optimality of this range is proved in Section 12. ∎

Remark. It is clear that, for any $1<p<\infty$, the L^p-Helmholtz decomposition (11.5) holds if and only if the projection $\mathcal{P}$ extends to a bounded operator on $L^p(\Omega)$. Since $\mathscr{P}$ is (L^2) self-adjoint, the latter condition is also equivalent to $\mathscr{P}$ being extendible to a bounded operator on $L^q(\Omega)$, where $1/p+1/q=1$. In particular, the L^p-Helmholtz decomposition is valid if and only if the L^q-Helmholtz decomposition is valid.

In a similar manner, we may also consider

$$L^p_{\mathrm{div}}(\Omega) := \{u \in L^p(\Omega);\ \mathrm{div}\, u = 0\} \tag{11.9}$$

and state the following.

THEOREM 11.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, with arbitrary topology. Then there exists $\varepsilon=\varepsilon(\Omega)>0$ such that*

$$L^p(\Omega) = \mathrm{grad}\, L^p_{1,0}(\Omega) \oplus L^p_{\mathrm{div}}(\Omega), \qquad \textit{for each}\ \ \tfrac{3}{2}-\varepsilon<p<3+\varepsilon, \tag{11.10}$$

where the direct sum is topological.

Once again, in the class of Lipschitz domains, this result is sharp. If, however, $\partial\Omega \in C^1$ then we may take $1<p<\infty$.

Proof. Here the departure point is to consider the operator

$$L^p(\Omega) \ni u \mapsto \nabla(\mathrm{div}\, \Pi_\Omega(u) - \psi) \in \mathrm{grad}\, L^p_{1,0}(\Omega), \tag{11.11}$$

where ψ is the unique solution to the Dirichlet problem

$$\begin{cases} \Delta\psi = 0 & \text{in } \Omega, \\ \mathrm{Tr}\,\psi = \mathrm{Tr}\,\mathrm{div}\,\Pi_\Omega(u) \in B^p_{1-1/p}(\partial\Omega), \\ \psi \in L^p_1(\Omega). \end{cases} \tag{11.12}$$

The results in Section 10 guarantee that this assignment is well defined, linear and bounded if $\frac{3}{2}-\varepsilon<p<3+\varepsilon$ for some $\varepsilon=\varepsilon(\Omega)>0$. Using this and paralleling the argument in Theorem 11.1 yields the desired conclusion; we omit the details. The sharpness of the range of p's is proved in Section 12. ∎

12. COUNTEREXAMPLES

In this section we shall prove, by means of counterexamples that our main results in Sections 8–12 are sharp in the class of Lipschitz domains. To begin with, a useful observation is contained in the following lemma.

LEMMA 12.1. *For any bounded Lipschitz domain Ω in $\mathbb{R}^n$ and $1<q<\infty$ the mapping in* (11.1) *is onto. That is, for any $\xi \in B^q_{-1/q}(\partial\Omega)$ there exists a* (*not necessarily unique*) *vector field $U \in L^q(\Omega)$ such that*

$$\operatorname{div} U \in L^q(\Omega), \qquad U \cdot N = \xi \tag{12.1}$$

and

$$\|\operatorname{div} U\|_{L^q(\Omega)} + \|U\|_{L^q(\Omega)} \leqslant C(\Omega, q)\, \|\xi\|_{B^q_{-1/q}(\partial\Omega)}. \tag{12.2}$$

Proof. With ξ as in the statement of the lemma, define $\Phi \in (L^p_1(\Omega))^*$ by $\langle \Phi, \phi \rangle := \langle \xi, \operatorname{Tr} \phi \rangle$ for all $\phi \in L^p_1(\Omega)$. It is then well known that there exist some (not necessarily unique) functions $h_i \in L^q(\Omega)$, $1/p + 1/q = 1$, $0 \leqslant i \leqslant n$, so that

$$\langle \Phi, \phi \rangle = \int_\Omega h_0 \phi + \sum_{i=1}^n \int_\Omega h_i D_i \phi, \qquad \forall \phi \in L^p_1(\Omega), \tag{12.3}$$

and $\sum \|h_j\|_{L^q(\Omega)}$ is controlled by $\|\Phi\|_{(L^p_1(\Omega))^*}$. Then one can easily check that the field $U := (h_1, h_2, ..., h_n) \in L^q(\Omega)$ satisfies $\operatorname{div} U = h_0$ and $U \cdot N = \xi$, i.e., the desired properties. ▮

THEOREM 12.2. *For any $p \notin [\frac{3}{2}, 3]$ there exists a bounded Lipschitz domain Ω in $\mathbb{R}^n$ for which the L^p-Helmholtz decomposition* (11.5) *fails. A similar statement is valid for the Helmholtz decomposition* (11.10).

Proof. First we claim that, for any bounded Lipschitz domain Ω and any $1 < p \leqslant 2$, the validity of the L^p-Helmholtz decomposition (11.5) in Ω entails the solvability (with naturally accompanying estimates) of the boundary problem

$$\begin{cases} u \in L^p_1(\Omega), \\ \Delta u = g - \dfrac{1}{|\Omega|} \langle g, 1 \rangle, \\ \dfrac{\partial u}{\partial N} = 0, \\ \displaystyle\int_\Omega u = 0, \end{cases} \tag{12.4}$$

for any $g \in L^p_{-1,0}(\Omega)$. To see this, fix such an arbitrary g and set $g' := g - (1/|\Omega|)\langle g, 1 \rangle$. By Proposition 2.1, we have that $\Pi_\Omega(g') \in L^p_1(\Omega)$.

Let U be the vector field from Lemma 12.1 corresponding to $\xi := \partial \Pi_\Omega(g')/\partial N \in B^p_{-1/p}(\partial\Omega)$. Since, by assumption, the L^p-Helmholtz decomposition holds, we can find unique $\varphi \in L^p_1(\Omega)$ and $w \in L^p_{\mathrm{div},0}(\Omega)$ with norms controlled in terms of the $L^p(\Omega)$ norm of U and such that

$$U - \nabla \Pi_\Omega(\mathrm{div}\, U) = \nabla\varphi + w \tag{12.5}$$

Consider now $f := \partial \Pi_\Omega(\mathrm{div}\, U)/\partial N \in L^p(\partial\Omega)$ and observe that $\int_{\partial\Omega} f\, d\sigma = 0$. Since $1 < p \leqslant 2$, there exists a unique harmonic function v in Ω so that $\partial v/\partial n = f$ and such that the non-tangential maximal function of (∇v) lies in $L^p(\partial\Omega)$; cf. [DaKe]. In particular, $h := \Pi_\Omega(g') - \varphi - v \in L^p_1(\Omega)$ and it is not difficult to check now that $u := h - (1/|\Omega|) \int_\Omega h$ solves (12.4). Uniqueness of the solution follows from the uniqueness in the corresponding Helmholtz decomposition, while estimates are derived from the explicit form of the solution and estimates for the corresponding Helmholtz decomposition. This completes the proof of the claim.

Let now

$$T\colon (L^2_1(\Omega))^* = L^2_{-1,0}(\Omega) \to L^2_1(\Omega) \tag{12.6}$$

be the solution operator for the problem (12.4), mapping g into u. Clearly, this is well defined, linear and bounded. Moreover, by Green's formula, T also satisfies

$$\langle g_1, Tg_2 \rangle = \langle g_2, Tg_1 \rangle, \qquad g_1, g_2 \in L^2_{-1,0}(\Omega), \tag{12.7}$$

for the natural pairing between functionals $(L^2_1(\Omega))^*$ and elements in $L^2_1(\Omega)$.

From what we have proved so far, the solvability of the boundary problem (12.4) for some $p \in (1, 2]$ implies that T above extends to a bounded operator from $L^p_{-1,0}(\Omega)$ into $L^p_1(\Omega)$. Given (12.7), we can further conclude that, under the same hypotheses, T also extends as a bounded mapping of $L^q_{-1,0}(\Omega)$ into $L^q_1(\Omega)$, where $q \geqslant 2$ is the conjugate exponent of p.

However, given $q > 3$, there exist a bounded (cone-like) Lipschitz domain Ω in $\mathbb{R}^n$ and a function $u \in L^2_1(\Omega)$ such that $\Delta u \in C^\infty(\bar{\Omega})$, $\partial u/\partial N = 0$ but $u \notin L^q_1(\Omega)$ (a construction is sketched in [JeKel]). In the light of our discussion, this implies the failure of the L^p-Helmholtz decomposition (11.5) for $1 < p < \frac{3}{2}$ on such domains. Now, the remark following the proof of Theorem 11.1 gives that there are counterexamples to the L^p-Helmholtz decomposition (11.5) for the dual range, $3 < p < \infty$, also. This contradicts the extendibility of T as above and, hence, concludes the proof of the first part of the theorem.

The optimality of Theorem 11.2 follows from the counterexamples in [JeKe]. ∎

COROLLARY 12.3. *The range* $[\frac{3}{2}, 3]$ *of validity for Corollary* 9.3 *is sharp in the class of Lipschitz domains.*

Proof. As seen in the proof of Theorem 11.1, the solvability of the problem (9.12) entails the validity of the Helmholtz decomposition (11.5). Thus, the statement is a direct consequence of the Theorem 12.2. ∎

COROLLARY 12.4. *The range of validity for the Theorem* 9.2 *in the class of Lipschitz domains is in the nature of best possible.*

Proof. By interpolation, the optimal range of solvability for the problem (9.8)–(9.9) must be a convex subset of the unit square $[-1, 0] \times [0, 1]$ which contains the parallelogram with vertices at $(-1, 0)$, $(0, \frac{1}{2})$, $(0, 1)$, $(-1, \frac{1}{2})$, and whose trace on the main diagonal of this square coincides (by Corollary 12.3) with the segment joining $(-\frac{1}{3}, \frac{1}{3})$ and $(-\frac{2}{3}, \frac{2}{3})$. From this, the conclusion follows by elementary geometrical considerations. ∎

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