Computation of First Cohomology Groups of Finite Covers

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We give several applications of standard methods of group cohomology to some problems arising in model theory concerning finite covers. We prove a conjecture of the author that for \( G \)-finite, \( \aleph_0 \)-categorical structures the kernels of minimal superlinked finite covers have bounded rank. We show that the cohomology groups associated to finite covers of certain structures (amongst them, the primitive, countable, totally categorical structures) have to be finite. From this we deduce that the finite covers of these structures are determined up to finitely many possibilities by their kernels.

This paper contains several applications of standard methods of group cohomology (Shapiro’s lemma and the long exact sequence) to some problems arising in model theory concerning finite covers (see later in this introduction for definitions).

The first collection of results concerns finite covers with finite kernels (superlinked finite covers) and these parts of the paper should be regarded as a sequel to [10]. In particular, we prove a conjecture made in [10] that the kernels of minimal superlinked finite covers of a certain natural class of structures have bounded rank (Theorem 1.5). We also give a cohomological criterion (Theorem 6.1) for the minimal finite covers of a primitive structure with trivial algebraic closure and weak elimination of imaginaries to be superlinked.

In the final section of the paper we show that the cohomology groups associated to finite covers of certain structures (amongst them, the structures obtained by considering the general linear group acting on the

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$k$-dimensional subspaces from a countable vector space over a finite field; see Remark 7.5) have to be finite. From this we deduce that the finite covers of these structures are determined up to finitely many possibilities by their kernels (Theorem 7.9).

The paper is written in group-theoretic terminology throughout with the intention of making it accessible to those with little or no model-theoretic background. For a group theorist, the greatest novelty here is perhaps that there is a natural class of infinite permutation groups coming from automorphism groups of model-theoretically interesting structures to which basic results from the extension theory of groups can be easily adapted to provide nontrivial results. We shall try to emphasize this further in the rest of the introduction.

The results given in Theorems 1.5 and 7.6 were proved in June 1995 and proofs of these and some other results of this paper were sketched in a (slightly different form) in the notes [13] and in a sparsely circulated earlier version of this paper. Theorem 1.5 has also been proved independently by Ehud Hrushovski [17] and Jeffrey Koshan [19]. Hrushovski’s proof is different in that it avoids explicit mention of cohomology groups, and works directly with Baer sums of group extensions corresponding to finite covers with a particular finite kernel. In particular, Hrushovski introduces a modified version of the exact sequence in Lemma 3.1 which allows him to identify directly the group of extensions with the cohomology group $\ker(d_3)/\text{im}(d_2)$, as in Remarks 3.7 (with modified versions of the maps $d_i$).

Our original proof of Theorem 1.5 was more complicated in that the bound on the size of $H^3(G, M_2/d_2(M_1))$ in Corollary 3.3 was obtained by a duality argument, rather than by using more of the exact sequence in 3.1. The current direct proof (and the ensuing computational methods in Section 3.2) was produced only after seeing Hrushovski’s proof.

1. INTRODUCTION

If $W$ is any set then the symmetric group $\text{Sym}(W)$ on $W$ can be considered as a topological group by taking as open sets arbitrary unions of cosets of pointwise stabilisers of finite subsets of $W$. In this topology, closed subgroups are precisely automorphism groups of first-order structures with domain $W$. In fact, if $H$ is a subgroup of $\text{Sym}(W)$ then the closure of $H$ in $\text{Sym}(W)$ is the set of elements of $\text{Sym}(W)$ which, for each $n \in \mathbb{N}$, preserve each $H$-orbit on $W^n$. Thus we employ the following notation and terminology.

**Definition 1.1.** A permutation structure is a pair $\langle W; G \rangle$ where $W$ is a nonempty set (the domain) and $G$ is a closed subgroup of $\text{Sym}(W)$ (the group of automorphisms). We shall usually write $G = \text{Aut}(W)$ and refer
simply to “the permutation structure \( W \).” If \( A \) is a subset of \( W \) and \( B \) a subset of \( W \) (or more generally of some set on which \( \text{Aut}(W) \) is acting in an obvious way), then \( \text{Aut}(A/B) \) denotes the permutations of \( A \) which extend to elements of \( \text{Aut}(W) \) fixing every element of \( B \). We regard \( \text{Aut}(W) \) as a topological group with the subspace topology from \( \text{Sym}(W) \): a base of open neighborhoods of the identity consists of subgroups \( \text{Aut}(W/X) \) for finite \( X \subseteq W \). We shall write permutations on the left of the elements of \( W \).

In practice, the permutation structures we consider are obtained by taking automorphism groups of first-order structures on \( W \), and we often regard a first-order structure as a permutation structure without explicitly saying so (by taking for the group of automorphisms of the permutation structure the automorphism group of the first-order structure). Of course, it is most interesting to do this when some model-theoretic property guarantees that the first-order structure has a “rich” automorphism group. The strongest such property is \( \aleph_0 \)-categoricity, where the Ryll-Nardzewski theorem shows (for countable \( W \)) that the automorphism group is oligomorphic, that is, \( \text{Aut}(W) \) has finitely many orbits on \( W^n \) for all \( n \in \mathbb{N} \).

Automorphism groups of the most natural examples of \( \aleph_0 \)-categorical structures are \( G \)-finite and many satisfy various irreducibility conditions on point stabilisers. We simply give the definitions of these properties here, and ask the reader to believe that they represent “reasonable” hypotheses. A fuller discussion can be found in Section 3.4 of [13].

**Definition 1.2.** Let \( \Sigma \) be a closed subgroup of \( \text{Sym}(W) \). Denote by \( \Sigma^o \) the intersection of the closed subgroups of finite index in \( \Sigma \). We say that \( \Sigma \) is irreducible if \( \Sigma = \Sigma^o \). Suppose \( W \) is a permutation structure. We say that \( W \) (or \( \text{Aut}(W) \)) is \( G \)-finite if \( \text{Aut}(W/X)^o \) is of finite index in \( \text{Aut}(W/X) \), for all finite subsets \( X \) of \( W \).

Thus the groups which interest us in this paper are closed, oligomorphic \( G \)-finite permutation groups, and automorphism groups of countable \( \aleph_0 \)-categorical structures provide a plentiful and diverse supply of these. The main technical part of the paper is to provide an explicit way of computing the cohomology group \( H^1(G, M) \) (defined in Section 2) for such groups \( G = \text{Aut}(W) \) and certain topological \( G \)-modules \( M \). These modules, and the applications of the computations, arise in the context of finite covers of \( W \), which we now define.

**Definition 1.3.** If \( C, W \) are permutation structures then a finite-to-one surjection \( \pi: C \to W \) is a finite cover of \( W \) if its fibres form an \( \text{Aut}(C) \)-invariant partition of \( C \) and the induced map (restriction) \( \rho: \text{Aut}(C) \to \text{Sym}(W) \) given by \( \rho(g)(w) = \pi(g \pi^{-1}(w)) \) (for \( w \in W \) and \( g \in \text{Aut}(C) \)) has image \( \text{Aut}(W) \). The kernel of the finite cover is \( \ker(\rho) = \text{Aut}(C/W) \).
If \( \pi: C \to W \) is a finite cover then the associated restriction map \( \rho: \text{Aut}(C) \to \text{Aut}(W) \) is continuous and so the kernel of the cover \( K = \text{Aut}(C)/W \) is a closed normal subgroup of \( \text{Aut}(C) \). As all \( K \)-orbits on \( W \) are finite, it follows that \( K \) is compact (and, in fact, profinite). By Lemma 1.1 of [10], \( \rho \) maps open sets to open sets and closed subgroups to closed subgroups. In particular, the induced isomorphism \( \text{Aut}(C)/K \to \text{Aut}(W) \) is a homeomorphism. Furthermore, if \( K \) is abelian then conjugation in \( \text{Aut}(C) \) gives an action of \( \text{Aut}(W) \) on \( K \) which makes \( K \) into a topological \( \text{Aut}(W) \)-module (see Lemma 6.2.1 of [13] for a proof).

**Definition 1.4.** Suppose \( \langle C; \text{Aut}(C) \rangle, \langle C'; \text{Aut}(C') \rangle \) are two permutation structures with the same domain \( C = C' \). We say that \( C \) is an expansion of \( C' \), or that \( C' \) is a reduct of \( C \), if \( \text{Aut}(C) \leq \text{Aut}(C') \). In this case, if \( \pi: C \to W \) and \( \pi': C' \to W \) are finite covers with \( \pi(c) = \pi'(c) \) for all \( c \in C = C' \) then we say that \( \pi \) is a covering expansion of \( \pi' \). This is proper if \( \text{Aut}(C) < \text{Aut}(C') \).

The finite cover \( \pi': C' \to W \) is minimal if it has no proper covering expansion.

In other words, the finite cover \( \pi': C' \to W \) is minimal if for all proper closed subgroups \( \Sigma \) of \( \text{Aut}(C') \) we have that \( \rho'(\Sigma) < \text{Aut}(W) \), where \( \rho' \) is the restriction map associated to \( \pi' \). Any finite cover \( \pi: C \to W \) has a covering expansion \( C' \) which is a minimal finite cover of \( W \) (see Lemma 5.2). Thus we can factorise \( \text{Aut}(C) \) as \( \text{Aut}(C/W) \cdot \text{Aut}(C') \), and so, to some extent, the problem of determining the finite covers of \( W \) divides into the problems of determining the minimal finite covers and the possible kernels. This is a meaningful division: for example, it can be shown that for many \( W \) that all finite covers \( \pi: C \to W \) are split (that is, there is a closed complement to \( \text{Aut}(C/W) \) in \( \text{Aut}(C) \)), or, equivalently, the kernel of any minimal finite cover of \( W \) is trivial.

Our main application of the cohomology computations is to prove the following, which appeared as a conjecture in Section 0.3 of [10]. Recall that by the rank of a finite group we mean the minimum size of a generating set.

**Theorem 1.5.** Let \( W \) be a \( G \)-finite oligomorphic permutation structure. Then there exists a natural number \( r \) such that if \( \pi: C \to W \) is a minimal finite cover with finite kernel \( K \), then \( K \) has rank at most \( r \).

The bound \( r \) is obtained fairly explicitly, as can be seen from inspection of the proof. For example, if \( W \) is a transitive structure and stabilisers of 0, 1, and 2 points in \( W \) are irreducible then \( r \) can be taken as the number of orbits on triples of distinct elements of \( W \), minus the number of orbits on pairs. With a bit more effort, the exact value of \( r \) can be obtained (see Remarks 3.2).
It is worth noting at this stage that the proof of Theorem 1.5 can quickly be reduced to the case where $W$ is transitive and irreducible. The latter condition means that if $\pi: C \to W$ is a minimal finite cover with finite kernel $K$, then $C$ is irreducible and $K$ is therefore central in $\text{Aut}(C)$ (as its centraliser is a closed subgroup of finite index in $\text{Aut}(C)$).

We shall need some facts about free finite covers. Full proofs of all these results can be found in Section 2.1 of the notes [13].

Suppose $\pi: C \to W$ is a finite cover and let $w \in W$. Denote by $C(w)$ the fibre $\pi^{-1}(w)$. The fibre group $F(w)$ and the binding group $B(w)$ at $w$ are respectively the permutation groups $\text{Aut}(C(w)/w)$ and $\text{Aut}(C(w)/W)$ on $C(w)$. The latter is a normal subgroup of the former and there is a natural continuous epimorphism $\chi_w: \text{Aut}(W/w) \to F(w)/B(w)$ called the canonical homomorphism (at $w$) given by $\chi_w(g) = (\hat{g} | C(w))B(w)$, for any $\hat{g} \in \rho^{-1}(g)$. Clearly the kernel $\text{Aut}(C/W)$ is a subgroup of $\prod_{w \in W} B(w)$. If there is equality here then we say that $\pi$ is a free finite cover.

**Lemma 1.6.** Let $W$ be a transitive permutation structure and $\pi: C \to W$ a finite cover. Then there exists a free finite cover $\pi_0: C_0 \to W$ such that $\pi$ is a covering expansion of $\pi_0$ and the fibre and binding groups and the canonical homomorphisms are the same in $\pi$ and $\pi_0$. A free finite cover of $W$ is determined up to isomorphism by the fibre and binding groups and the canonical homomorphisms. Moreover, if the fibre group in $\pi_0$ splits over the binding group then $\pi_0$ is a split finite cover.

**2. Derivations**

The first uses of cohomology groups to classify covers of $\mathfrak{K}_\omega$-categorical structures are by Martin [20] and Ahlbrandt and Ziegler [2]. In this section, we follow rather closely the approach of [2] as modified by Hodges and Pillay [15].

Recall that if $G$ is a group and $M$ is a $G$-module, then a derivation from $G$ to $M$ is a map $d: G \to M$ which satisfies $d(gh) = d(g) + gd(h)$ for all $g, h \in G$. An inner derivation is a derivation of the form $d_a$ (for $a \in M$) where $d_a(g) = ga - a$ for all $g \in G$. The set of all such derivations forms an abelian group, and the inner derivations form a subgroup. The quotient group is denoted by $H^1(G, M)$, and is referred to as the first cohomology group of $G$ on $M$. If $M$ is a topological $G$-module then the continuous derivations form a subgroup of the group of all derivations, and this clearly contains all the inner derivations. We denote the quotient group by $H^1_c(G, M)$.

Suppose now that $\pi_0: C_0 \to W$ is a finite cover of the permutation structure $W$, and suppose from now on that the kernel $K_0 = \text{Aut}(C_0/W)$
is abelian. Then conjugation in Aut(C) gives $K_0$ the structure of a topological Aut(W)-module. Let $\rho: \text{Aut}(C_0) \to \text{Aut}(W)$ be the restriction homomorphism. Suppose $K$ is a $G$-invariant subgroup of $K_0$ such that there exists $H_0 \leq \text{Aut}(C_0)$ with $H_0 \cap K_0 = K$ and $\rho(H_0) = G$. Then we have [15, Propositions 16 and 17]:

**Proposition 2.1.** There is a one-to-one correspondence between $H^1(G, K_0/K)$ and the set of Aut($C_0$)-conjugacy classes of subgroups $H$ of Aut($C_0$) satisfying $H \cap K_0 = K$ and $\rho(H) = G$.

It is worth noting how the correspondence in the above is obtained. Note that $G = H_0/H_0 \cap K_0 \leq \text{Aut}(C_0)/K$. So there is an embedding $\sigma_0: G \to \text{Aut}(C_0)/K$ given by $\sigma_0(g) = (\rho^{-1}(g) \cap H_0)K$. Now take any $H$ as in the proposition. We obtain similarly a map $\sigma_H: G \to \text{Aut}(C_0)/K$. Then the map given by $d_H(g) = \sigma_H(g)\sigma_0(g)^{-1}$ has image in $K_0/K$ and is a derivation.

Suppose now that $K$ is a closed subgroup of $K_0$, and $H_0$ is a closed subgroup of Aut($C_0$) with $\rho(H) = G$ and $H_0 \cap K_0 = K$. Then, by Lemma 1.1 of [10], $\sigma_0$ is continuous. We have:

**Corollary 2.2.** There is a one-to-one correspondence between $H^1(G, K_0/K)$ and conjugacy classes of closed subgroups $H$ of Aut($C_0$) which satisfy $\rho(H) = G$ and $H \cap K_0 = K$.

**Proof.** This is proved under the assumption that $C_0$ is countable in Corollary 18 of [15]. The context in which Hodges and Pillay work (symmetric extensions) is more general than ours, and the countability assumption is required to invoke their Lemma 6 in the proof of Theorem 11 of [15]. However, we can substitute Lemma 1.1 of [10] in place of Lemma 6 of [15], and the proofs of Theorem 11 and Corollary 18 of [15] go through without further modification.

In practice, $\pi_0: C_0 \to W$ will be a free finite cover and we will be interested in classifying covering expansions of this which have as kernel some particular $G$-invariant closed subgroup $K$ of $K_0$. Corollary 2.2 indicates that to do this we should compute the cohomology group $H^1(G, K_0/K)$. We now give some modifications of standard results from cohomology of discrete groups results which will assist in the computation of the groups $H^1$ and $H^2$. Proofs can be found in Section 7.1 of [13].

**Definition 2.3.** If $G$ is a group and $M$ a $G$-module then we define the zeroth cohomology group $H^0(G, M)$ to be the elements of $M$ fixed by all elements of $G$. Note that if $G$ is a topological group and $M$ a topological $G$-module, then this is a closed subgroup of $M$. 
Lemma 2.5 (The Long Exact Sequence). Suppose $G$ is a group and

$$0 \to K \to M \to N \to 0$$

is an exact sequence of $G$-modules. Then there is an exact sequence of abelian groups:

$$0 \to H^0(G, K) \to H^0(G, M) \to H^0(G, N)$$
$$\to H^1(G, K) \to H^1(G, M) \to H^1(G, N).$$

If, moreover, $G$ is a topological group and the short exact sequence is a sequence of topological $G$-modules in which the homomorphisms are continuous open maps, then there is a long exact sequence as above in which the $H^1$ terms are replaced by $H^1$.

Remarks 2.5. 1. In our context, the modules which arise are closed subgroups of kernels of finite covers, so are profinite, and a continuous map between profinite groups is automatically an open map.

2. Our main use of the long exact sequence will be to effect a trick known as "dimension shifting": we shift a problem about computing $H^1$ to one about computing $H^0$. The procedure, roughly, is this. We want to compute $H^1(G, K)$ for some $G$-module $K$. Suppose we can embed $K$ in a module $M$ for which we know $H^1(G, M)$. Then, by the long exact sequence, if we can compute $H^0$ of $K, M$, and $M/K$, we can read off $H^1(G, K)$.

All this relies on having a good supply of $G$-modules whose cohomology we know about. In our context, the appropriate modules are kernels of free finite covers. In the group-theoretic terminology (at least if the base $W$ of the cover is transitive) these modules are coinduced from a finite module for the stabiliser of a point: the relevant module is the binding group at that point. The next lemma is then seen as a special case of Shapiro's lemma in group cohomology (cf. [5, Proposition III.6.2]), and it tells us how to compute the cohomology of the coinduced modules.

Lemma 2.6 (Shapiro's Lemma). Let $W$ be a transitive permutation structure and $G = \text{Aut}(W)$. Suppose $\pi : C \to W$ is a free finite cover with abelian kernel $K$. Let $w \in W$, $H = \text{Aut}(W/w)$, and $A = \text{Aut}(C(w)/W)$. Then $A$ is naturally an $H$-module (via the canonical homomorphism $\chi_w$), and for $i = 0$, $1$ we have

$$H^1(G, K) = H^1(H, A).$$

Moreover $H^2(G, K) = H^1(H, A).$
Computation of the groups $H^i(H, A)$ is, in the $G$-finite case, a problem about finite groups:

**Lemma 2.7.** Suppose $\Sigma$ is a topological group and $A$ a finite topological $\Sigma$-module. Suppose further that $\Sigma^0$ is of finite index in $\Sigma$. Then $\Sigma^0$ acts trivially on $A$ and

1. $H^0(\Sigma, A) = H^0(\Sigma/\Sigma^0, A)$;
2. $H^1(\Sigma, A) = H^1(\Sigma/\Sigma^0, A)$.

In particular, these groups are finite.

### 3. Finite Covers with Finite Kernels

Throughout this section $W$ will be an irreducible, transitive permutation structure. Let $G = \text{Aut}(W)$ and let $A$ be a finite abelian group. We wish to classify finite covers $\pi: C \to W$ where the kernel $K$ is isomorphic to $A$ and central in $\text{Aut}(C)$. Such a cover is a covering expansion of a free finite cover $\pi_0: C_0 \to W$ with the same fibre and binding groups and canonical homomorphisms.

Let $K_0$ denote the kernel of $\pi_0$. Let $A^W$ be the module of functions $f: W \to A$ with the product topology and $G$-action $(g \cdot f)(w) = f(g^{-1}w)$ (for $g \in G$), and let $\Delta(A)$ consist of the constant functions $W \to A$. Then $K_0/K$ and $A^W/\Delta(A)$ are isomorphic as topological $G$-modules. In this section we give an explicit method for computing $H^1(G, A^W/\Delta(A))$. By Corollary 2.2 this gives a description of all the covering expansions of $C_0$ with kernel $K$.

#### 3.1. Exact Sequences

For $n \in \mathbb{N}$ let $W^{(n)}$ denote the set of $n$-tuples of distinct elements of $W$. Let $M_n = A^{W^{(n)}}$ be the set of functions $f: W^{(n)} \to A$ considered as a topological $G$-module (as for $A^W$). Define $\Delta: A \to A^W$ so that $\Delta(a)$ is the constant function with image $a$, for $a \in A$. Clearly this is a continuous $G$-module homomorphism, if we regard $A$ as a trivial $G$-module. Define the map $d_n: M_n \to M_{n+1}$ by

$$(d_n f)(w_1, \ldots, w_{n+1}) = \sum_{i=1}^{n+1} (-1)^i f(w_1, \ldots, \hat{w}_i, \ldots, w_{n+1}),$$

where $f \in M_n$, $(w_1, \ldots, w_{n+1}) \in W^{(n+1)}$, and the $\hat{w}_i$ denotes that the $i$th term is to be omitted. Then $d_n$ is a continuous $G$-module homomorphism. The following is well-known.
Lemma 3.1. The sequence
\[ 0 \to A \overset{\Delta}{\to} M_1 \overset{d_1}{\to} M_2 \overset{d_2}{\to} M_3 \overset{d_3}{\to} \cdots \]
is exact.

From now on assume that \( G \) has \( t \) orbits on \( W^{(3)} \) for some finite \( t \). Then the number \( s \) of \( G \)-orbits on \( W^{(2)} \) is also finite. Let \((x_i, y_i)\) (for \( i = 1, \ldots, s \)) be representatives of these orbits. Suppose further that the group
\[ X_i = \text{Aut}(W/x_i, y_i)/(\text{Aut}(W/x_i, y_i))^g \]
is finite for \( i = 1, \ldots, s \). (Note that all of these hypotheses are satisfied if \( W \) is a \( G \)-finite oligomorphic permutation structure.) Below, \( \text{Hom}(X_i, A) \) denotes the group of all homomorphisms from \( X_i \) to \( A \).

Lemma 3.2. There is an exact sequence of abelian groups
\[ 0 \to A^t \to H^0(G, M_2/d_1(M_1)) \to H^1_c(G, A^W/\Delta(A)) \to \bigoplus_{i=1}^s \text{Hom}(X_i, A). \]

Proof. By Lemma 3.1 we have a short exact sequence
\[ 0 \to A^W/\Delta(A) \to M_2 \to M_2/d_1(M_1) \to 0. \]

The maps in this sequence are continuous, open, \( G \)-module homomorphisms. The lemma follows from the long exact sequence (Lemma 2.4) and the following observations.

Claim 1. \( H^0(G, A^W/\Delta(A)) = 0 \). Indeed, suppose \( f + \Delta(A) \) is fixed by \( G \). Then \( \langle f, \Delta(A) \rangle \) is a finite submodule of \( A^W \). As \( G \) is irreducible, \( G \) acts trivially on this, so \( f \) is fixed by \( G \). But the only functions in \( A^W \) fixed by \( G \) are the constant functions, so \( f \in \Delta(A) \).

Claim 2. \( H^0(G, M_2) \equiv A^t \). A function in \( M_2 \) is fixed by \( G \) if and only if it is constant on each \( G \)-orbit on \( W^2 \). The claim then follows immediately.

Claim 3. \( H^1_c(G, M_2) \equiv \bigoplus_{i=1}^s \text{Hom}(X_i, A) \). Let \( O_1, \ldots, O_s \) be the \( G \)-orbits on \( W^{(2)} \). Then
\[ H^1_c(G, M_2) \equiv \bigoplus_{i=1}^s H^1_c(G, A^{O_i}). \]

By Shapiro’s lemma (Lemma 2.6),
\[ H^1_c(G, A^{O_i}) \equiv H^1_c(\text{Aut}(W/x_i, y_i), A). \]
By Lemma 2.7, this latter group is isomorphic to $H^1(X, A)$, and as $A$ here is a trivial module, this is the same as $\text{Hom}(X, A)$.

**Corollary 3.3.** The group $H^2_c(G, A^w/\Delta(A))$ is finite. In fact,

$$|H^2_c(G, A^w/\Delta(A))| \leq |A|^{r-1} \prod_{i=1}^{s} |\text{Hom}(X_i, A)|.$$

**Proof.** This follows immediately from Lemma 3.2 once we show that the size of $H^0(G, M_2/d_3(M_2))$ is bounded by $|A|^r$. But, by the exact sequence in Lemma 3.1, $M_2/d_3(M_2)$ is isomorphic to a submodule of $M_3$, and the submodule of fixed points of $G$ on $M_3$ is isomorphic to $A'$ (as in the proof of Claim 2 in the above).

### 3.2. Computation

We can be more precise about the group $H^0(G, M_2/d_3(M_2))$ by using more of the exact sequence in Lemma 3.1. By the exactness, we have the isomorphisms

$$M_2/d_3(M_2) \cong d_3(M_2) \cong \ker(d_3).$$

Thus, we want to known the fixed points of $G$ on the kernel of $d_3$. This is the same as the kernel of $d_3$ restricted to the fixed points of $G$ on $M_3$. We outline an algorithm for computing this.

List the $G$-orbits on $W^{(3)}$ as $P_1, \ldots, P_t$ and let $(Q_i; i \in I)$ be the $G$-orbits on $W^{(4)}$. Define an $|I| \times t$ matrix of integers $M$ as follows. For $i \in I$ choose $(w_1, w_2, w_3, w_4) \in Q_i$. For $j = 1, \ldots, t$ let

$$m_{ij} = \sum \left\{ (-1)^k : (w_1, \ldots, \hat{w}_k, \ldots, w_4) \in P_j \right\}.$$

This depends only on $i$ and $j$. Note that if $f \in M_3$ is fixed by $G$ and takes the value $a$ on $P_j$ then

$$(d_3f)(w_1, \ldots, w_4) = \sum_{j=1}^{t} m_{ij}a_j.$$

Let $e_1, \ldots, e_t$ be the invariants of the integer matrix $M$ (so these are non-negative integers, and $e_j$ divides $e_{j+1}$ for $j = 1, \ldots, t - 1$). Then it follows that the kernel of $d_3$ restricted to the $G$-fixed points in $M_3$ is isomorphic to

$$\bigoplus_{j=1}^{t} \{ a \in A : e_ja = 0 \}.$$
We summarise this as:

**Corollary 3.4.** Let \( e_1, \ldots, e_t \) be the invariants of the matrix \( M \) as above. Then:

1. Writing \( U = \bigoplus_{j=1}^{t} \mathbb{Z}/e_j \mathbb{Z} \), we have
   \[
   H^0(G, M_2/d_3(M_1)) = A \otimes_{\mathbb{Z}} U.
   \]
2. Suppose stabilizers of pairs of points in \( W \) are irreducible, let \( m \in \mathbb{N} \), and let \( A \) be cyclic of order \( m \). Then
   \[
   |H^1(G, A^w/\Delta(A))| = \frac{\prod_{j=1}^{t} (e_j, m)}{m^s}
   \]
   (where \( (m, n) \) denotes the highest common factor of \( m \) and \( n \)).

**Proof.** (i) follows from the above remarks. Then (ii) follows from (i) and Lemma 3.2. 

The following lemma is straightforward, but useful.

**Lemma 3.5.** Let \( 1 \leq n \leq t \). Suppose it is possible to label the \( G \)-orbits on \( W^{(3)} \) as \( P_1, \ldots, P_t \) so that if \( j > n \) and \( (x_2, x_3, x_4) \in P_j \), then there exists \( x_1 \in W \) such that \( (x_1, x_2, x_3), (x_1, x_2, x_4), \) and \( (x_3, x_3, x_4) \) lie in \( P_1, \ldots, P_{j-1} \). Then at least \( t - n \) of the invariants of \( M \) are equal to \( 1 \).

**Example 3.6.** Suppose \( W \) is a binary homogeneous structure with an \( \text{Aut}(W) \)-orbit \( Q \) on pairs which has the property that for all \( (x, y, z) \in W^{(3)} \) there exists \( w \in W \) such that \( (w, x), (w, y), (w, z) \in Q \). Then the hypotheses of the above lemma hold with \( n = s \), so (at least) \( t - s \) of the invariants are equal to \( 1 \). So that rank of the group \( U \) in Corollary 3.4 is at most \( s \), and if stabilisers of pairs of points in \( W \) are irreducible then the cohomology groups \( H^1(G, A^w/\Delta(A)) \) are trivial for all finite abelian groups \( A \).

**Remarks 3.7.** In the exact sequence in Lemma 3.2, if, as above, we identify the third group with the kernel of \( d_3 \) restricted to the \( G \)-fixed points of \( M_2 \), then the image of the second group in this is the image under \( d_2 \) of the \( G \)-fixed points of \( M_2 \). Thus we have the following alternative description of the image of \( H^3(G, M_2/d_3(M_1)) \) in \( H^2(G, A^w/\Delta(A)) \). Denote by \( M_i^G \) the fixed points of \( G \) on \( M_i \). The maps \( d_i \) induce maps \( \bar{d}_i : M_i^G \to M_{i+1}^G \) and \( \bar{d}_{i+1} \circ \bar{d}_i = 0 \). So we have a cochain complex, and the image of the group \( H^3(G, M_2/d_3(M_1)) \) in \( H^2(G, A^w/\Delta(A)) \) is isomorphic to the cohomology group \( \ker(\bar{d}_3)/\text{im}(\bar{d}_2) \). In the case where stabilisers of pairs of points of \( W \) are irreducible, it follows that

\[
H^2(G, A^w/\Delta(A)) \cong \ker(\bar{d}_3)/\text{im}(\bar{d}_2).
\]
4. PROOF OF THE MAIN THEOREM

The main part of the work is in proving the following.

**Theorem 4.1.** Let $W$ be a transitive irreducible, permutation structure with automorphism group $G = \text{Aut}(W)$. Suppose that $G$ has finitely many orbits on triples from $W$, and that for all $x, y \in W$, each of $\text{Aut}(W/\langle x \rangle)$ and $\text{Aut}(W/\langle x, y \rangle)$ has a smallest closed subgroup of finite index. Then there is a natural number $r$ such that if $\pi : C \to W$ is a minimal finite cover with finite kernel $K$, then $K$ can be generated by $r$ elements.

The result follows from the next two lemmas, and for these the hypotheses of Theorem 4.1 which relate to $W$ will be in force.

**Lemma 4.2.** Suppose there is a nontrivial, minimal finite cover $\pi : C \to W$ with finite kernel $K$ of rank $r$. Then for some prime $p$ there is a minimal finite cover of $W$ whose kernel is an elementary abelian $p$-group of rank $r$.

**Proof.** By the fundamental theorem of abelian groups, the rank of a finite abelian group is equal to the maximum of the ranks of its Sylow subgroups. Let $p$ be a prime such that the rank of a Sylow $p$-subgroup of $K$ is as large as possible. Let $K_1$ consist of the product of the Sylow $p'$-subgroups of $K$ (for $p' \neq p$) and the Frattini subgroup of the Sylow $p$-subgroup. Then $K/K_1$ is an elementary abelian $p$-group of rank $r$. Let $C_1$ consist of the $K_1$-orbits on $C$ and consider this as a permutation structure with automorphisms those permutations induced by elements of $\text{Aut}(C_1)$ (this is a closed subgroup of $\text{Sym}(C_1)$) by Lemma 1.1 of [10]). Then $\pi$ induces a map $\pi_1 : C_1 \to W$ which is a finite cover. The kernel of the finite cover is $K/K_1$ and the cover is irreducible (and as the kernel is finite, it is therefore minimal). 

For a prime $p$, let $F_p$ denote the cyclic group of order $p$ (and also think of this as the field with $p$ elements). We use the notation of the previous section.

**Lemma 4.3.** There exists a natural number $n$ (depending only on $W$) with the following property. Suppose that $H^2(G, F_p^W/\Delta(F_p))$ is finite, of cardinality $p^k$. Let $\pi : C \to W$ be an irreducible finite cover whose kernel is a finite elementary abelian $p$-group of rank $r$. Then $r \leq k + n$.

**Proof.** First, we clarify what $n$ is here. By assumption, there is a number $m$ such that any continuous finite image of the stabiliser of a point in $W$ has size at most $m$. By [8, 2.1] there exists an integer $n$ such that if $T$ is a finite group of size at most $m$ and $\phi : S \to T$ is a Frattini cover with kernel $Z$, then $Z$ has rank at most $n$.

Suppose now that the kernel $A$ of $\pi : C \to W$ has rank greater than $k + n$. We shall show, for a contradiction, that $\pi$ is not a minimal cover,
and so cannot be irreducible. For \( w \in W \), the group \( \text{Aut}(C(w)/w)/\text{Aut}(C(w)/W) \) is a continuous homomorphic image of \( \text{Aut}(W/w) \). Furthermore, \( \text{Aut}(C/W, C(w)) = 1 \) so we may identify \( A \) with \( \text{Aut}(C(w)/W) \).

By the definition of \( n \) there exists a proper subgroup \( X \) of \( A \) (of rank at most \( n \)) such that \( \text{Aut}(C(w)/w)/X \) splits over \( A/X \). By factoring out by \( X \) we can assume therefore that \( \pi: C \to W \) is an irreducible finite cover whose kernel is an elementary abelian \( p \)-group of rank \( r \), where \( r > k \), and such that \( \text{Aut}(C(w)/w) \) splits over \( \text{Aut}(C(w)/W) \). Let \( \rho: \text{Aut}(C) \to \text{Aut}(W) \) be the restriction homomorphism.

Let \( \pi_0: C_0 \to W \) be the free finite cover which is a reduct of \( \pi \), with the same fibre and binding groups and canonical homomorphisms as in \( \pi \) (see Lemma 1.6). Note that this free finite cover is split. The kernel \( K_0 \) of \( \pi_0 \) can be identified with \( A^\infty \), and with this identification, the kernel of \( \pi \) corresponds to \( \Delta(A) \) the constant functions in \( K_0 \). Let \( d: G \to K_0/\Delta(A) \) be the continuous derivation obtained from the cover \( \pi \), as in Lemma 2.2.

For any \( B \leq A \) we can form the cover \( \pi^B: C^B \to W \), where \( C^B \) consists of the \( B \)-orbits on \( C \). This has automorphism group \( \text{Aut}(C)/B \) and has as a reduct the free cover \( \pi^B_0: C^B_0 \to W \) with kernel \( (A/B)^W \). (Note that this is split, and it can also be seen as the cover resulting from factoring out the action of \( \Delta(B) \) on \( \pi_0 \).) The derivation \( d^B_\rho: G \to (A/B)^W/\Delta(A/B) \) resulting from \( \pi^B \) can be described as the result of composing \( d \) with the natural map \( \nu: A^W/\Delta(A) \to (A/B)^W/\Delta(A/B) \).

Suppose we can find a proper subgroup \( B \) of \( A \) such that \( d_B \) is inner. Then, because \( \pi^B_0 \) is split, it follows that \( \pi^B \) is split. So there exists a closed subgroup \( H \) of \( \text{Aut}(C) \) with \( H \cap A = B < A \) and \( \rho(H) = \text{Aut}(W) \). This contradicts the minimality of \( \pi \).

Write \( A \) as \( F_p^r \), that is, \( r \)-tuples of elements of \( F_p \). An element of \( K_0 \) is a function \( f: W \to A \), which we can therefore write componentwise as \( (f_1, \ldots, f_r) \), where each \( f_i \in F_p^W \). Moreover, we can identify \( f + \Delta(A) \in K_0/\Delta(A) \) with \( (f_1 + \Delta(F_p), \ldots, f_r + \Delta(F_p)) \). Thus \( d: G \to K_0/\Delta(A) \) can be identified with an \( r \)-tuple \( (d_1, \ldots, d_r) \) of continuous derivations \( d_i: G \to F_p^W/\Delta(F_p) \). By our assumption on \( H^1 \) (which is, of course, elementary abelian, of rank \( t \)) and the fact that \( r > k \), some nontrivial \( F_p \)-linear combination of \( d_1, \ldots, d_r \) is equal to an inner derivation \( d^B_\rho: G \to F_p^W/\Delta(F_p) \), for some \( a \in F_p^W/\Delta(F_p) \). So, without loss of generality, there exist \( \alpha_1, \ldots, \alpha_r \in F_p \) such that \( d_1 = d^B_\rho + \sum_{i=2}^r \alpha_i d_i \). Let

\[
B = \left\{ \left( \sum_{i=2}^r \alpha_i x_i, x_2, \ldots, x_r \right) : x_2, \ldots, x_r \in F_p \right\}.
\]

This is a proper subgroup of \( A \) and it is easy to verify that \( d_B \) is inner.

This concludes the proof. 


Theorem 4.1 now follows from Lemmas 4.2 and 4.3 and Corollary 3.3.

Proof of Theorem 1.5. Let \( \pi : C \to W \) be a minimal finite cover with finite kernel \( K \). Let \( \Gamma = \text{Aut}(C) \), \( \Sigma = \text{Aut}(W) \), \( \Gamma_1 = \Gamma^\circ \), and \( \Sigma_1 = \Sigma^\circ \). Consider \( \Gamma_1 \) and \( \Sigma_1 \) as automorphism groups of irreducible expansions \( C_1 \) and \( W_1 \) of \( C \) and \( W \), and note that \( \pi_1 = \pi : C_1 \to W_1 \) is an irreducible finite cover of \( W_1 \) with finite kernel \( K_1 = K \cap \Gamma_1 \).

By the isomorphism theorems, \( \Gamma_1 / \Gamma_1^\circ \) is an extension of \( \Sigma_1 / \Sigma_1^\circ \) by \( K_1 / K_1^\circ \), and minimality of \( \pi \) implies that this is a Frattini extension. Thus its rank is bounded by a function of \( \vert \Sigma / \Sigma_1 \vert \) (and this is independent of \( \pi \), of course).

It remains to bound the rank of \( K_1 \). Clearly \( W_1 \) is \( G \)-finite and oligomorphic. We replace \( W_1 \) by a transitive structure as follows (cf. [16]). Take a tuple \( \bar{w} \) of elements of \( W_1 \) consisting of one element from each \( \text{Aut}(W_1) \)-orbit on \( W_1 \), and let \( W'_1 \) be the orbit of this under \( \text{Aut}(W_1) \). The group of permutations induced on this by \( \text{Aut}(W'_1) \) is closed (and topologically isomorphic to \( \text{Aut}(W'_1) \)), and so we may regard \( W'_1 \) with this group as a permutation structure. Now \( W'_1 \) satisfies the hypothesis of Theorem 4.1, and so there is a bound on the rank of the kernel an irreducible finite cover of \( W'_1 \) with finite kernel. But it is easy to construct an irreducible finite cover \( C'_1 \to W'_1 \) with kernel \( K_1 \) and automorphism group topologically isomorphic to \( \Gamma_1 \) (see Lemma 1.9 of [12]). The result now follows. \( \square \)

5. MINIMAL COVERS AND FRATTINI COVERS

We digress slightly to prove some general results about minimal finite covers. First, we put them in a wider context.

Definition 5.1. If \( G \) is a topological group, the Frattini subgroup, \( \Phi(G) \), of \( G \) is the intersection of the maximal open subgroups of \( G \). A continuous epimorphism of topological groups \( \phi : G \to H \) is a Frattini cover if for every closed proper subgroup \( G_1 \) of \( G \) we have \( \phi(G_1) \neq H \).

Remark. This use of terminology requires some explanation. If \( G, H \) are profinite groups, then the condition that a continuous epimorphism \( \phi : G \to H \) is a Frattini cover is equivalent to \( \ker \phi \) being contained in \( \Phi(G) \) (see [14, Section 20.6]). However, this is not true in general, even if \( \ker \phi \) is profinite. Certainly, the kernel of a Frattini cover is contained in the Frattini subgroup. However, the converse fails, essentially because a proper closed subgroup need not be contained in a proper open subgroup.

So a finite cover \( \pi : C \to W \) is minimal if the restriction map \( \rho : \text{Aut}(C) \to \text{Aut}(W) \) is a Frattini cover. The argument proving the following lemma
is due to Cossey, Kegel, and Kovács [8], and guarantees that an arbitrary finite cover can be expended to a minimal one.

**Lemma 5.2.** If $\phi: G \rightarrow H$ is a continuous surjection of Hausdorff topological groups which has compact kernel, then there exists a closed subgroup $G_1 \leq G$ such that

(i) $\phi(G_1) = H$;

(ii) if $G_2$ is a proper closed subgroup of $G_1$, then $\phi(G_2) \neq H$.

**Proof.** See Lemma 1.5(i) of [9].

**Lemma 5.3.** Suppose $\phi: G \rightarrow H$ is a Frattini cover of Hausdorff topological groups with $K = \ker \phi$ profinite. Then $K$ is pronilpotent.

**Proof.** The proof is the classic “Frattini argument,” which works because Sylow’s theorems hold for profinite groups (see [14, Section 20.10, p. 306]). We show that for each prime $p$, there is a unique Sylow $p$-subgroup of $K$, which is thus normal in $G$. Indeed, let $P$ be any Sylow $p$-subgroup of $K$. Then the normaliser $N_G(P)$ is a closed subgroup of $G$ (in general, the normaliser of a closed subgroup of a Hausdorff topological group is closed). Let $g \in G$. Then $P^g$ is also a Sylow $p$-subgroup of $K$, and so (by [14, 20.43]) there exists $k \in K$ with $P^g = P^k$. It follows that $G = KN_G(P)$, and so, as $\phi$ is a Frattini cover, we get $N_G(P) = G$. In other words, $P$ is normal in $G$. As $K$ is topologically generated by its Sylow subgroups, it follows that $K$ is the direct product of Sylow subgroups. Each of these is pronilpotent, and so the same is true of $K$.

**Corollary 5.4.** If $\pi: C \rightarrow W$ is a minimal finite cover with kernel $K$ and $\text{Aut}(W)$ has finitely many orbits on $W$, then $K$ is nilpotent.

**Proof.** By Lemma 5.3, $K$ is pronilpotent. So each binding group $B(w)$ is nilpotent. There are only finitely many isomorphism types of binding group, so there is a bound on the nilpotency class of these. As $K \leq \prod_{w \in W} B(w)$, it follows that $K$ is nilpotent.

As an application of this we give the following, which shows how splitting questions can be reduced to consideration of finite covers with abelian kernel.

**Corollary 5.5.** Let $W$ be a permutation structure. The following are equivalent:

1. every finite cover of $W$ splits;

2. every finite cover of $W$ with elementary abelian kernel splits.

**Proof.** One direction is trivial. So suppose (2) holds. Let $\pi: C \rightarrow W$ be a finite cover with kernel $K$. By 5.2 we may assume (for a contradiction)
that \( \pi \) is minimal and nontrivial. By 5.3 each binding group \( G_w = Aut(C(w)/W) \) is nilpotent. Choose a prime \( p \) dividing the order of some \( G_w \), and let \( K_p \) be the closure in \( K \) of the subgroup generated by \( p \)-th powers and commutators in \( K \). Then \( K_p \) is a proper, closed normal subgroup of \( K \), and \( K/K_p \) is elementary abelian (this subgroup is proper, because the subgroup it induces on \( C(w) \) is a proper subgroup of \( G_w \)).

Now let \( C_1 \) be the set of \( K \)-orbits on \( C \). As \( K \) is normal in \( Aut(C) \) we get an action \( \rho_1 : Aut(C) \to Sym(C_1) \) of \( Aut(C) \) on this. The natural map \( C \to C_1 \) is finite-to-one, and so by Lemma 1.1 of [10], the group of permutations of \( C_1 \) induced by \( Aut(C) \) is closed. Thus we may consider \( C_1 \) as a permutation structure with \( Aut(C_1) = Aut(C)/ker \rho_1 \), and \( \pi \) gives us a finite cover \( \pi_1 : C_1 \to W \) with kernel \( K_1/K \), where \( K_1 = ker \rho_1 \). As \( K_1 \geq K_p \), this is elementary abelian. We claim that \( K_1 \neq K \). In fact, the following shows that \( K \) and \( K_p \) have different orbits on \( C(w) \).

**Lemma 5.6.** Let \( G \) be a nilpotent group of permutations on a finite set \( X \). Let \( p \) be a prime dividing the order of \( G \), and let \( G_p \) be the subgroup generated by \( p \)-th powers and commutators in \( G \). Then there exists \( x \in X \) such that the \( G \)-orbit containing \( x \) is bigger than the \( G_p \)-orbit containing \( x \).

**Proof of Lemma 5.6.** Note that \( G \) is the direct product of its Sylow subgroups, and \( G_p \) is the product of the Frattini subgroup of the Sylow \( p \)-subgroups of \( G \) with the Sylow \( p' \)-subgroups (for primes \( p' \neq p \)). Take \( x \in X \) which is not fixed by the Sylow \( p \)-subgroup of \( G \). Let \( H \) be the stabiliser of \( x \) in \( G \). So, by considering the \( p \)-parts of the orders, the projection of \( H \) to the Sylow \( p \)-subgroup of \( G \) has a proper subgroup of the Sylow \( p \)-subgroup of \( G \) as its image. Suppose the \( G \)-orbit and \( G_p \)-orbit containing \( x \) are identical. Then \( HG_p = G \). But consider projecting this equation to the Sylow \( p \)-subgroup: it says that this Sylow \( p \)-subgroup is generated by its Frattini subgroup and a proper subgroup. This is impossible, as the Frattini subgroup of a finite group consists of the nongenerators of the group. \( \square \)

Now we finish off the proof of 5.5. By (2), \( \pi_1 \) splits. So there is a closed subgroup \( H \) of \( Aut(C) \) with \( \rho(H) = Aut(W) \) and \( H \cap K = ker \rho_1 \). But then \( H < Aut(C) \), and this contradicts the minimality of \( \pi \). \( \square \)

6. **FINITENESS OF KERNELS OF MINIMAL COVERS**

The paper [10] gives extensive information about finite covers with finite kernels. In particular, the results given there enable us to describe such covers of the primitive homogeneous graphs and digraphs (see the paper [11] for the case of Cherlin's homogeneous "local partial order"). But to
describe all the finite covers of these structures, we need to know in particular the minimal finite covers. In this section we give a criterion which shows that under restrictive conditions, any minimal finite cover has finite kernel.

**Theorem 6.1.** Let \( W \) be an irreducible, primitive permutation structure such that:

- the stabiliser of any point in \( W \) is irreducible;
- algebraic closure in \( W \) is trivial;
- if \( A \) is a cyclic finite abelian group then

\[
H^1(\text{Aut}(W), A^w/\Delta(A)) \cong A.
\]

Then a minimal finite cover of \( W \) has finite kernel.

The second itemized condition here means that \( \text{Aut}(W/X) \) has no finite orbits on \( W \setminus X \), for all finite \( X \subseteq W \). The third condition is obviously the most technical, but recall that we gave in Remarks 3.2 an explicit way of calculating these cohomology groups, under the additional assumption that stabilisers of pairs of points are irreducible.

For the rest of this section, let \( W \) be as above, and let \( \pi: C \rightarrow W \) be a minimal finite cover with kernel \( K \). Let \( \rho: \text{Aut}(C) \rightarrow \text{Aut}(W) \) be the restriction map. Let \( G = \text{Aut}(W) \). We shall suppose, for a contradiction, that \( K \) is infinite. Without loss of generality we may assume that \( \pi \) is a regular finite cover (that is, each fibre group acts regularly on the corresponding fibre: see Lemma 1.8 of [12] or Lemma 3.1.1 of [13]). By Corollary 5.4, \( K \) is nilpotent, and we may assume, without loss of generality, that it is a pro-\( p \)-group, for some prime \( p \). As point stabilisers are irreducible, the canonical homomorphisms for \( \pi \) are trivial and so the fibre and binding groups in \( \pi \) are the same, and isomorphic to some finite \( p \)-group. Let \( \pi_0: C_0 \rightarrow W \) be a reduct of \( \pi \) which is a free finite cover with fibre groups isomorphic to \( F \) (as in Lemma 1.6). The kernel of this can be identified with \( F^w \) and so \( K \) can be thought of as a closed subgroup of this (and, in fact, \( \text{Aut}(C_0) \cong F W r \text{ut}(W) \)). Let \( \Phi(F) \) be the Frattini subgroup of \( F \).

**Lemma 6.2.** \( K/K \cap \Phi(F)^w \) is finite.

**Proof.** Suppose not. Let \( F_1, \ldots, F_k \) be the maximal subgroups of \( F \). So each \( F_i \) is normal in \( F \), of index \( p \) and \( \bigcap_{i=1}^k F_i = \Phi(F) \). It follows that we may assume that \( K_1 = K \cap F_1^w \) is of infinite index in \( K \). Note that \( F_1^w \trianglelefteq \text{Aut}(C_0) \) so \( K_1 \trianglelefteq \text{Aut}(C) \). Let \( C_1 \) be the set of \( K_1 \)-orbits on \( C \). Then we get a finite cover \( \pi_1: C_1 \rightarrow W \) with kernel \( K/K_1 \) (by the regularity of \( \pi \)). The fibre group in \( \pi_1 \) is \( F/F_1 \), which is cyclic of order \( p \). As \( W \) is
primitive and has trivial algebraic closure and $\pi$ has infinite kernel, it follows from Theorem 5.8 of [12] that $\pi_1$ is a free cover. By Lemma 1.6 this splits. So there is a closed subgroup $H$ of $\text{Aut}(C)$ with $\rho(H) = \text{Aut}(W)$ and $H \cap K = K_1$. This contradicts the minimality of $\pi$.

**Lemma 6.3.** If the finite group $A$ is the kernel of a minimal finite cover of $W$ then $A$ is cyclic.

**Proof.** By minimality, the cover is irreducible, and so, as usual, $A$ is central in the automorphism group of the cover. In particular, $A$ is abelian. As point stabilisers in $W$ are irreducible, the lemma follows from Lemmas 4.2 and 4.3 and the hypotheses about $H$. 

**Proof of Theorem 6.1.** Let $K_2 = K \cap \Phi(F)$, let $C_2$ be the set of $K_2$-orbits on $C$, and let $\pi_2: C_2 \to W$ be the induced finite cover (as in the construction of $\pi_1$ in Lemma 6.2). Then $\pi_2$ is a minimal finite cover with fibre group $F/\Phi(F)$ and kernel $K/K_2$, which by Lemma 6.2 is finite. By Lemma 6.3, $F/\Phi(F)$ is cyclic and so (by the Burnside basis theorem) $F$ is cyclic.

Let $F$ have order $p^n$. By factoring out the action of some $K \cap (p^nF)^W$ if necessary, we may assume that $K/K \cap p^{n-1}F$ is finite (and therefore cyclic of order $p^{n-1}$). Thus $(p^{n-1}F)^W \cap K$ is infinite and so (as in the proof of 6.2) is equal to $K' = (p^{n-1}F)^W$. So $K = \Delta(F) + K'$.

Now consider all automorphism groups of covering expansions of the free cover $\pi_0$ which have kernel $K$. By Corollary 2.2 the number of conjugacy classes of these in $\text{Aut}(C_0)$ is the cardinality of $H^1_2(G, F^W/K)$, and as $F^W/K \cong (F/p^{n-1}F)^W/\Delta(F/p^{n-1}F)$, this is $p^{n-1}$. Some of these can be obtained as reducts of finite covers $\pi_2: C_2 \to W$ with kernel $\Delta(F)$ by taking as automorphism group $\text{Aut}(C_2) \cdot K'$. Clearly such covers are not minimal. We count the number of conjugacy classes of these in $\text{Aut}(C_2)$.

Denote by $\mathcal{C}$ the set of automorphism groups of covering expansions of $\pi_0$ with kernel $\Delta(F)$. The number of $\text{Aut}(C_0)$-conjugacy classes of these is $|H^1_2(G, F^W/\Delta(F))|$. On the other hand, if $H \in \mathcal{C}$ then the number of $HK'$-conjugacy classes of subgroups of $HK'$ in $\mathcal{C}$ is $|H^1_2(G, K/\Delta(F))|$. Thus, the number of conjugacy classes of automorphism groups of nonminimal covering expansions of $\pi_0$ with kernel $K$ is at least

$$|H^1_2(G, F^W/\Delta(F))/H^1_2(G, K/\Delta(F))|.$$ 

Now $K/\Delta(F) \cong (p^{n-1}F)^W/\Delta(p^{n-1}F)$, so (using our hypothesis on $H^1_2$), this number is $p^{n-1}$. But this is the total number of conjugacy classes of automorphism groups of covering expansions of $\pi_0$ with kernel $K$. So all of these are nonminimal. This is a contradiction.

As an example of the use of this, we give the following (cf. [9], Theorem 5.8 and [10, Example 2.8]).
Corollary 6.4. Suppose $W$ is the countable universal homogeneous local order. Then any minimal finite cover of $W$ has finite kernel.

Proof. Algebraic closure in $W$ is trivial, and pointwise stabilisers of finite subsets are irreducible, so the result follows from Theorem 6.1 if we can check the condition on $H^1_c$. We use the notation of Section 3.2. By Corollary 3.4 and Lemma 3.2 it will suffice to show that $s+1$ of the invariants of the matrix $M$ are 0 and the rest 1.

First, we show that at least $s+1$ of the invariants are zero. Otherwise, pick a prime $p$ not dividing $e_j$ for $j \geq t-s-1$ and let $A$ be the cyclic group of order $p$. Then (by Corollary 3.4) the rank of $H^0(G, M_2/d_1(M_1))$ is at most $s$. As stabilisers of pairs of points in $W$ are irreducible, Lemma 3.2 shows that $H^2(G, M) = 0$. But this is impossible, as $W$ has a nonsplit finite cover with kernel $A$ (see [9]).

It is now straightforward to use Lemma 3.5 to show that the number of invariants equal to 1 is at least $t-s-1$. This completes the proof.

7. Finiteness of $H^1_c$

Associated to any finite cover of a permutation structure $W$, we have certain canonical data: the fibre and binding groups and the canonical homomorphisms. Given these, we can regard the finite cover as a covering expansion of a free finite cover with the same canonical data in a unique way (see Lemma 1.6). The kernel $K$ of the finite cover is of course a subgroup of the direct product of the binding groups (which is also the kernel of the free cover). In this section we shall be concerned with the question as to whether this information about the finite cover (the fibre and binding groups, the canonical homomorphisms, and the kernel) determines the cover up to finitely many possibilities.

In the context of totally categorical structures and working also with affine covers as well as finite covers, the statement that this question has an affirmative answer has become known as “Ziegler’s finiteness conjecture.” In fact, the conjecture is now known to hold more generally for smoothly approximated structures $W$. The proof of this is an elegant compactness argument using the quasi-finite axiomatizability of these structures. Full details will appear in the final version of [7], but a sketch can be found in the notes [6].

In this section we show how dimension-shifting and appropriate chain conditions also give Ziegler’s finiteness conjecture for finite covers with abelian kernel, and then deduce the results for the case of general kernels. It should be noted however, that neither of these approaches gives effective bounds on the number of covers with a particular kernel, and both rely heavily on the existence of particular enumerations of the base $W$. 
To make this more precise, we should clarify the notion of the isomorphism of finite covers which we wish to use.

**Definition 7.1.** We shall say that two permutation structures $(C; \text{Aut}(C))$ and $(C'; \text{Aut}(C'))$ are isomorphic if there is a bijection $\varphi: C \rightarrow C'$ for which the induced map $\text{Sym}(C) \rightarrow \text{Sym}(C')$ sends $\text{Aut}(C)$ to $\text{Aut}(C')$. Two finite covers $\pi: C \rightarrow W$ and $\pi': C' \rightarrow W'$ are isomorphic if there is such a bijection which sends the fibres of $\pi$ to the fibres of $\pi'$. If $W = W'$ the covers are isomorphic over $W$ if they are isomorphic via a bijection sending $\pi^{-1}(w)$ to $\pi'^{-1}(w)$ for all $w \in W$. It is straightforward to show that if $\text{Aut}(W)$ is self-normalising in $\text{Sym}(W)$ then these are the same notations. Also note that if $C, C'$ are covering expansions of $\pi_0: C_0 \rightarrow W$ which are conjugate in $\text{Aut}(C_0)$ then $C, C'$ are isomorphic over $W$. Finally, if $W = W'$, we say that the kernels of the covers $\pi$ and $\pi'$ are isomorphic over $W$ if the permutation structures $(C; \text{Aut}(C/W))$ and $(C'; \text{Aut}(C'/W))$ are isomorphic via an isomorphism sending $\pi^{-1}(w)$ to $\pi'^{-1}(w)$ for all $w \in W$.

Suppose $\pi: C \rightarrow W$ is a finite cover with abelian kernel $K$. Let $\pi_0: C_0 \rightarrow W$ be the free reduct of $\pi$ with the same fibre and binding groups and canonical homomorphisms. Then by Proposition 2.1 and Lemma 1.6 the number of isomorphism classes of finite covers $W$ having the same data as $\pi$ is at most the cardinality of $H^1(\text{Aut}(W), K_0/K)$ (and if $\text{Aut}(W)$ is self-normalising in $\text{Sym}(W)$ it is exactly this). We now give some conditions which allow us to deduce (rather cheaply) that this cohomology group is finite. First, we introduce some more terminology and notation.

Suppose $C$ is a transitive permutation structure and $x \in C^{(n)}$ for some $n \in \mathbb{N}$. Denote by $C_x$ the $\text{Aut}(C)$-orbit on $C^{(n)}$ containing $x$. As $\text{Aut}(C)$ is transitive on $C$ it follows that the group of permutations induced by $\text{Aut}(C)$ on $C_x$ is closed, and so we may regard $C_x$ as a permutation structure with this as automorphism group.

**Definition 7.2** [12, Definition 5.3]. Suppose $C, W$ are permutation structures and $\pi: C \rightarrow W$ is a finite cover with kernel $K$. We say that $\pi$ has $q\text{dcc}$ if any chain $K > K_1 > K_2 > \cdots$ of closed normal subgroups of $\text{Aut}(C)$ in $K$ is finite.

**Remarks 7.3.** In [1] Ahlbrandt and Ziegler identified a combinatorial condition on $W$ (in terms of having a particular sort of enumeration) which guarantees that all finite covers of $W$ have $q\text{dcc}$ (see Proposition 5.4 of [12] for a modification of this). In particular, if $W$ is either a countable disintegrated set or projective geometry over a finite field and $x$ is an enumeration of a finite algebraically closed subset of $W$ then combining results in [1] with Proposition 5.4 of [12] shows that all finite covers of $W_x$ have $q\text{dcc}$. Similarly results in [7] show this is also true if $W$ is a Lie
geometry over a finite field and a result of Albert and Chowdhury [4] shows this if \( W \) is the rationals (as an ordered set).

We shall assume the following hypotheses.

**Hypotheses 7.4.** Assume \( W \) is a transitive permutation structure such that any finite tuple of elements of \( W \) can be extended to a finite tuple \( \tilde{y} \) with the property that all finite covers of \( W_{\tilde{y}} \) have qdcc and are \( G \)-finite.

**Remark 7.5.** These appear rather technical, but it is worth noting that they are satisfied if \( W \) is the permutation structure consisting of the \( k \)-sets from an infinite set \( D \), with automorphism group \( \text{Sym}(D) \); or the set of \( k \)-dimensional subspaces from an infinite vector space over a finite field, with automorphism group given by the corresponding general linear group; or an orbit of a classical group over a finite field on totally isotropic finite-dimensional subspaces. (References for the chain conditions have been given above; the \( G \)-finiteness follows from the structure theory of totally categorical and smoothly approximated structures.)

**Theorem 7.6.** Suppose \( W \) satisfies 7.4. Let \( \pi_0: C_0 \to W \) be a finite cover of \( W \) with abelian kernel \( K_0 \), and let \( K \) be a closed subgroup of \( K_0 \) which is normal in \( \text{Aut}(C_0) \). Then \( H^1(\text{Aut}(W), K_0/K) \) is finite.

The proof involves dimension shifting. This involves embedding the quotient module \( K_0/K \) as a submodule of the kernel of some free cover. To do this, we need the following lemma.

**Lemma 7.7.** Assume that \( W \) is a transitive permutation structure. Suppose \( \pi_0: C_0 \to W \) is a finite cover with abelian kernel \( K_0 \), and \( K \) is a closed subgroup of \( K_0 \) which is normal in \( \text{Aut}(C_0) \). Suppose \( \pi_0 \) has qdcc on covers of \( W \). Then there exists a finite subset \( X \) of \( C_0 \) such that

\[
K = K(X) := \bigcap_{g \in \text{Aut}(C_0)} K \cdot \text{Aut}(C_0 / gX).
\]

**Proof.** Throughout \( X \) will denote a nonempty finite subset of \( C_0 \).

Clearly \( K \leq K(X) \leq K_0 \) and \( K(X) \) is a closed normal subgroup of \( \text{Aut}(C_0) \). Also, if \( X_1 \subseteq X \) then \( K(X_1) \geq K(X) \). Suppose \( g \in \bigcap_X K(X) \).

Then \( g \in K \cdot \text{Aut}(C_0 / X) \) so \( K \cap g \cdot \text{Aut}(C_0 / X) \neq \emptyset \). Thus \( g \) is in the closure of \( K \). But \( K \) is closed so \( g \in K \). This shows that \( \bigcap_X K(X) = K \), and the statement now follows from the assumption of qdcc.

**Proof of Theorem 7.6.** Let \( \tilde{x} \) be an enumeration of the set \( X \) given by the above lemma. By Hypotheses 7.4 we may assume that \( \tilde{y} = \pi(\tilde{x}) \) has the property that all finite covers of \( W_{\tilde{y}} \) have qdcc and are \( G \)-finite. Let \( C_1 \),
be the set of left cosets of $\Sigma = \text{Aut}(C_0/X)$ in $\Gamma = \text{Aut}(C_0)$ considered as a permutation structure with automorphisms those permutations induced by multiplication by elements of $\Gamma$ (there is a finite-to-one map from $(C_0)_x$ to $C_1$ inducing the required map $\text{Aut}(C_0) \to \text{Sym}(C_1)$ and so closedness of the image follows from Lemma 1.1 of [10]). Note that by the lemma the kernel of the action of $\text{Aut}(C_0)$ on $C_1$ is $K$. The map $\pi_1: C_1 \to W_1$ given by $\pi_1(g\Sigma) = g^\Sigma$ is a finite cover with kernel $K_1$ topologically isomorphic to $K_0/K$.

Let $\pi_2: C_2 \to W_2$ be a free finite cover of $\pi_1$ which has the same canonical data as $\pi_1$ (Lemma 1.6). Denote its kernel by $K_2$. Applying Lemma 2.4 to the exact sequence

$$0 \to K_1 \to K_2 \to K_2/K_1 \to 0,$$

we get an exact sequence

$$H^1(G, K_2/K_1) \to H^1(G, K_1) \to H^1(G, K_2),$$

where $G = \text{Aut}(W)$. Now, $K_2$ is the kernel of a free finite cover so Shapiro’s lemma (Lemma 2.6) and Lemma 2.7, together with the $G$-finiteness of $\text{Aut}(W)$, give that $H^1(G, K_2)$ is finite. But also $H^1(G, K_2/K_1)$ is the set of $G$-fixed elements of $K_2/K_1$ and if this were infinite then $K_2$ would have an infinite descending chain of closed $G$-submodules (containing $K_1$), which would contradict qdcc on finite covers of $W$. Thus $H^1(G, K_2)$ is finite, as required.

**Remark 7.8.** It would be very interesting to have an explicit bound on the size of the cohomology groups here for the structures mentioned in Remark 7.5.

We now consider finite covers in general (not necessarily with abelian kernel). The notion of the isomorphism of kernels in the following is given in Definition 7.1.

**Theorem 7.9.** Let $W$ be a permutation structure satisfying Hypotheses 7.4. Let $\pi: C \to W$ be a finite cover with kernel $K$. Then there are only finitely many isomorphism types over $W$ of finite covers $\pi': C' \to W$ with kernel isomorphic over $W$ to $K$.

**Proof.** The proof is by contradiction. Take a counterexample $\pi: C \to W$ to the statement of the theorem where the fibre groups $\text{Aut}(C(w))$ are as small as possible (amongst all $W$ satisfying 7.4).

Isomorphism over $W$ of the kernels of $\pi$ and $\pi'$ means that we may assume that $C = C'$ and $\pi^{-1}(w) = \pi'^{-1}(w)$ for all $w \in W$, and that the kernels are actually equal. Pick some $w_0 \in W$. As $\text{Aut}(W/w_0)$ is of finite index in $\text{Aut}(W/w_0)$ (by the $G$-finiteness of $W$) there are only finitely
many possibilities for the canonical data of $\pi'$ at $w_0$. As $W$ is transitive
these data determine up to isomorphism (over $W'$) a free cover with these
canonical data at $w_0$ (see Lemma 1.6 or, more accurately, Lemma 2.1.2 of
[13]), and $\pi'$ is a covering expansion of this. Thus, we may assume that
there are infinitely many isomorphism classes (over $W'$) of finite covers
with kernel $K$ having the same canonical data at $\pi$. Let $\pi_0: C_0 \to W$
be the free cover with the same canonical data as $\pi$. Thus, there are infinitely
many isomorphism classes over $W$ of covering expansions of $\pi_0$ with
kernel $K$. It follows from Theorem 7.6 and Proposition 2.1 that $K$ is
nonabelian.

We next claim that we may assume that $\pi$ is a regular finite cover: that
is, each fibre group $F(w)$ acts regularly on the corresponding fibre $C(w)$
[12, Definition 1.1]. Indeed, by [12, Lemma 1.8(i)] if $W$ is transitive then
any finite cover $\pi': C \to W$ can be converted to a regular finite cover
$\pi': C \to W$ where $s$ is an enumeration of a fibre of $C$. Under this
process two finite covers of $W$ with the same canonical data and kernels
convert to regular finite covers with the same canonical data and kernels
and the regular finite covers are isomorphic if and only if the original
covers are.

We now show that the fibre groups $F(w)$ in $\pi$ are simple. Suppose there
is a proper normal subgroup $1 \neq E(w) \triangleleft F(w)$. By [13, Lemma 1.8(ii)] we
can factorise $\pi$ as $\pi_1 \circ \pi_2$ where $\pi_1: C_1 \to W$ and $\pi_2: C \to C_1$ are regular
finite covers with fibre groups $F(w)/E(w)$ and $E(w)$ respectively. The
kernels of $\pi_1$ and $\pi_2$ are determined by $K$, and the canonical data of $\pi_1$
and $\pi_2$ are determined by the canonical data of $\pi$. Now observe that $C_1$
also satisfies Hypotheses 7.4. By construction, $C_1$ is transitive. The rest of
the hypotheses follow by noting that any finite cover of $(C_1)_z$ gives a finite
cover of $W_{\pi_1(s)}$, for all finite tuples $z$ of elements of $C_1$. But it then follows
that at least one of $\pi_1$ or $\pi_2$ is a counterexample to the statement of the
theorem, and we have a contradiction to the minimality of the size of the
fibre group in $\pi$. Note that we have now proved the required result for
finite covers with a solvable fibre group.

Thus, $\pi: C \to W$ is a regular finite cover whose fibre group is non-
abelian simple group $S$. The binding group is a normal subgroup of this, so
is trivial or is equal to the fibre group. In the former case we must have
$K = 1$ so $\pi$ is a trivial cover. But transitive trivial covers are determined by
their canonical data (they are coset spaces on closed subgroups of finite
index in $\text{Aut}(W/w_0)$). So we must be in the latter case, where the fibre and
binding groups in $\pi$ are equal.

We can identify $C$ with $S \times W$, $\pi$ with projection to the second
coordinate, and $\text{Aut}(C)$ with a subgroup of the wreath product
$S \wr W \text{Aut}(W)$ (cf. [18, p. 68]). Thus the kernel $K$ of $\pi$ can be thought of
as a closed subgroup of $S^W$. Define a relation $\sim$ on $W$ as follows. Write
$x \sim y$ if and only if $\text{Aut}(C(x)/W \cup C(y)) = 1$. Then by [12, Lemma 5.7 and Theorem 5.8], this is an invariant equivalence relation on $W$ determined by $K$, for each equivalence class $Y$ the set $K \upharpoonright Y$ of functions in $K$ restricted to $Y$ is a diagonal subgroup of $S^Y$, and $K = \Pi_{Y \in W/\sim} K \upharpoonright Y$. Now, any two diagonal subgroups of $S^Y$ are conjugate in $(\text{Aut}(S))^Y$ and so (by conjugating by a suitable element of $(\text{Sym}(S))^W$) we may assume that

$$K = \{ f \in S^W : f \text{ constant on each } \sim \text{ class} \}.$$ 

By Lemma 5.2 there is a minimal covering expansion $\pi_1: C_1 \rightarrow W$ of $\pi$ and by Corollary 5.4 its kernel $K_1$ is nilpotent. For $w \in W$ let $T_w = \{ f(w) : f \in K_1 \}$. So this is a nilpotent subgroup of $S$ which depends only on the $\sim$-class containing $w$. Let

$$K_2 = \{ f \in K : f(w) \in T_w \forall w \in W \}.$$ 

Then $K_1 \leq K_2 \leq K$ and it is easy to see that $\text{Aut}(C_1)$ normalises $K_2$. Thus $K_2 \text{ Aut}(C_1)$ is a closed subgroup of $\text{Aut}(C)$ and so may be regarded as the automorphism group of a covering expansion $\pi_2: C_2 \rightarrow W$ of $\pi$ with kernel $K_2$.

The subgroups $T_w$ are all conjugate in $\text{Aut}(S)$ and so we can conjugate by an element of $(\text{Sym}(S))^W$ which normalises $K$ and assume that

$$K_2 = \{ f \in K : F(w) \in T \forall w \in W \}$$

for some nilpotent subgroup $T$ of $S$. Then $\text{Aut}(C) = K \text{ Aut}(C_2)$, so, in particular, $\pi_2$ determines $\pi$. But we have already proved our result in the solvable case, so for each $T$ there are only finitely many possibilities for $\pi_2$. This is the final contradiction.]

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