Bicritical domination

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Abstract

A graph $G$ is domination bicritical if the removal of any pair of vertices decreases the domination number. Properties of bicritical graphs are studied. We show that a connected bicritical graph has domination number at least 3, minimum degree at least 3, and edge-connectivity at least 2. Ways of constructing a bicritical graph from smaller bicritical graphs are presented.

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1. Introduction

For many graph parameters, criticality is a fundamental issue. Much has been written about graphs for which a parameter (such as connectedness or chromatic number) increases or decreases whenever an edge or vertex is removed or added. For domination number, Brigham et al. [2] began the study of graphs where the domination number decreases on the removal of any vertex. Further properties of these graphs were explored in [2,3,5], but

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they have not been characterized. Other types of domination critical graphs have also been studied, for example, see [4,9–12].

In this paper, we introduce and study those graphs where the domination number decreases on the removal of any set of \( k \) vertices. Recall that for a graph \( G = (V, E) \), the open neighborhood of a vertex \( v \in V \) is \( N(v) = \{x \in V \mid vx \in E\} \). The closed neighborhood is \( N[v] = N(v) \cup \{v\} \). A set \( S \subseteq V \) is a dominating set if every vertex in \( V \) is either in \( S \) or is adjacent to a vertex in \( S \), that is, \( V = \bigcup_{s \in S} N[s] \). The domination number \( \gamma(G) \) is the minimum cardinality of a dominating set of \( G \), and a dominating set of minimum cardinality is called a \( \gamma(G) \)-set. For a set \( S \), a vertex \( v \) is a private neighbor of \( u \) (with respect to \( S \)) if \( N[v] \cap S = \{u\} \); and the private neighbor set of \( u \), with respect to \( S \), is the set \( \text{pn}[u, S] = \{v \mid N[v] \cap S = \{u\}\} \). We denote the subgraph induced by \( S \) in \( G \) by \( G[S] \). We denote the distance between two vertices \( x \) and \( y \) in \( G \) by \( d_G(x, y) \). For a detailed discussion of domination and for notation not defined here, see [6,7].

Note that removing a vertex can increase the domination number by more than one, but can decrease it by at most one. It is useful to write the vertex set of a graph as a disjoint union of three sets according to how their removal affects \( \gamma(G) \). Let \( V(G) = V^0 \cup V^+ \cup V^- \) where

\[
V^0 = \{v \in V \mid \gamma(G - v) = \gamma(G)\},
\]

\[
V^+ = \{v \in V \mid \gamma(G - v) > \gamma(G)\},
\]

and

\[
V^- = \{v \in V \mid \gamma(G - v) < \gamma(G)\}.
\]

It is possible for a single graph to have all of the sets \( V^0, V^- \), and \( V^+ \) nonempty. For example, if \( k \geq 3 \) and \( T \) is the tree obtained from a star \( K_{1,k} \) with center \( u \) by subdividing an edge \( uw \) of this star once, then \( V^+ = \{u\} \), \( V^- = \{w\} \), and \( V^0 = V(T) - \{u, w\} \).

Brigham et al. [2] defined a vertex \( v \) to be critical if \( v \in V^- \), and a graph \( G \) to be domination critical if every vertex of \( G \) is critical. A generalization of this concept was presented in [8]. Here we consider a different generalization. We define a graph \( G \) to be \((\gamma, k)\)-critical, if \( \gamma(G - S) < \gamma(G) \) for any set \( S \) of \( k \) vertices. Obviously, a \((\gamma, k)\)-critical graph \( G \) has \( \gamma(G) \geq 2 \). For instance, \( K_n \) is \((\gamma, k)\)-critical for all \( k \leq n - 1 \). The \((\gamma, 1)\)-critical graphs are precisely the domination critical graphs introduced by Brigham, Chinn, and Dutton. In the special case of \( k = 2 \), we say that \( G \) is domination bicritical, or just bicritical.

In this paper, we call a graph critical (respectively, bicritical) if it is domination critical (respectively, domination bicritical). Further, we call a graph \( \gamma \)-critical (respectively, \( \gamma \)-bicritical) if it is domination critical (respectively, \( \gamma \)-bicritical) with domination number \( \gamma \). For example, the self-complementary Cartesian product \( G = K_3 \square K_3 \), where \( \gamma(G) = 3 \), is 3-critical and 3-bicritical, since removing any vertex or any pair of vertices decreases the domination number. However, critical graphs are not necessarily bicritical. For instance, the cycles \( C_n \) for \( n \equiv 1 \pmod{3} \) are critical, but not bicritical. On the other hand, bicritical graphs are not necessarily critical. For example, the graph \( H \) formed from the Cartesian product \( K_3 \square K_3 \) (where the vertices of the \( i \)th copy of \( K_3 \) are labelled \( v_{ij} \) for \( 1 \leq j \leq 3 \)) by adding a new vertex \( x \) adjacent to \( v_{11}, v_{12}, v_{23}, \) and \( v_{33} \) is bicritical and not critical (since \( x \in V^0 \)).
2. Examples of bicritical graphs

In this section, we present three examples of bicritical graphs. We begin with the circulant graph $C_8\langle 1, 4 \rangle$ (shown in Fig. 1), i.e., the graph with vertex set \{v_0, v_1, \ldots, v_7\} and edge set \{v_iv_i+j (\text{mod } 8) \mid i \in \{0, 1, \ldots, 7\} \text{ and } j \in \{1, 4\}\}.

**Proposition 1.** The circulant $C_8\langle 1, 4 \rangle$ is 3-critical and 3-bicritical.

**Proof.** Let graph $G = C_8\langle 1, 4 \rangle$ be labelled as in Fig. 1. It has domination number 3 and is vertex-transitive. Since \{v_3, v_5\} dominates $G-\{v_0\}$, $G$ is critical. Furthermore, since each of $G-\{v_0, v_1\}$, $G-\{v_0, v_2\}$ and $G-\{v_0, v_4\}$ is dominated by \{v_3, v_5\}, while $G-\{v_0, v_3\}$ is dominated by \{v_5, v_6\}, it follows from vertex-transitivity that $G$ is also bicritical. □

Our second example is the Cartesian product $G_t = K_t \square K_t$. We can think of $G_t$ as having $t$ disjoint copies of $K_t$ in “rows” and $t$ disjoint copies of $K_t$ in columns. In other words, we consider the vertices of $G_t$ as a matrix, where vertex $v_{ij}$ is in the $i$th row (copy of $K_t$) and the $j$th column (copy of $K_t$). For ease of discussion, we will use the words row and column to mean a “copy of $K_t$”.

**Proposition 2.** The Cartesian product $G_t = K_t \square K_t$ for $t \geq 3$ is $t$-critical and $t$-bicritical.

**Proof.** We show first that $\gamma(G_t) = t$. Since \{v_{11}, v_{21}, \ldots, v_{t1}\} is a dominating set, we have $\gamma(G_t) \leq t$. Suppose $\gamma(G_t) \leq t - 1$. Then any $\gamma(G_t)$-set $S$ does not have a vertex in row $i$ for some $i$. Now any vertex in $S$ can dominate only one vertex of row $i$ implying that at most $t - 1$ of the $t$ vertices of row $i$ are dominated. Thus, $\gamma(G_t) = t$.

To see that $G_t$ is critical, without loss of generality, consider $G_t-\{v_{11}\}$. Then $\{v_{ss} \mid 2 \leq s \leq t\}$ is a dominating set of cardinality $t - 1$. Consider removing two vertices $v_{ij}$ and $v_{ij'}$ from $G_t$. Within symmetry, there are two possibilities: suppose $s = i$ (or, equivalently, $r = j$). Without loss of generality, let the vertices be $v_{11}$ and $v_{12}$. Then $\{v_{ss} \mid 2 \leq s \leq t\}$ is a dominating set of cardinality $t - 1$. Suppose $s \neq i$ and $r \neq j$. Without loss of generality, let the vertices be $v_{11}$ and $v_{22}$. Then $\{v_{23}, v_{32}\} \cup \{v_{ss} \mid 4 \leq s \leq t\}$ is a dominating set of cardinality $t - 1$. Thus, for any two vertices $u$ and $v$ of $G_t$, $\gamma(G_t - \{u, v\}) \leq t - 1$ implying $G_t$ is bicritical. □
Proposition 3. Any graph formed from the complete bipartite graph $K_{2t,2t}$ where $t \geq 3$ by removing the edges of $t$ disjoint 4-cycles is 4-critical and 4-bicritical.

Proof. Let $K_{2t,2t}$ have partite sets $\mathcal{L}$ and $\mathcal{R}$, and let $H$ be formed from $K_{2t,2t}$ by removing the edges of $t$ disjoint 4-cycles. Let $\{1, 2, 3, 4, 5, 6\} \subseteq \mathcal{L}$ and $\{a, b, c, d, e, f\} \subseteq \mathcal{R}$. For notational convenience, we may assume that the edges of the 4-cycles induced by each of the sets $\{1, 2, a, b\}, \{3, 4, c, d\}$ and $\{5, 6, e, f\}$ of vertices in $K_{2t,2t}$ have been removed when forming $H$. Then $\gamma(H)$ = 4, and by symmetry the following four cases cover all possibilities: 

1. $\{2, b\}$ dominates $H - \{1, a\}$, 
2. $\{2, 3, a\}$ dominates $H - \{1, c\}$, 
3. $\{3, 5, a\}$ dominates $H - \{1, 2\}$, 
4. $\{2, a, b\}$ dominates $H - \{1, 3\}$. 

Therefore, $H$ is 4-bicritical. Also, $\{1, 2, b\}$ dominates $H - a$, so $H$ is 4-critical. \qed

3. Basic properties

In this section, we investigate some basic properties of bicritical graphs. Since removing a vertex can decrease the domination number by at most one, we make a straightforward, but useful observation.

Observation 4. For a bicritical graph $G$ and $x, y \in V(G)$, $\gamma(G) - 2 \leq \gamma(G - \{x, y\}) \leq \gamma(G) - 1$.

Our next observation holds for a general graph.

Observation 5. If $G$ is any graph and $x, y \in V(G)$ such that $\gamma(G - \{x, y\}) = \gamma(G) - 2$, then $d_G(x, y) \geq 3$.

Proof. Let $S$ be a $\gamma(G - \{x, y\})$-set. Then, $|S| = \gamma(G) - 2$. If $xy \in E(G)$, then $S \cup \{x\}$ dominates $G$, while if $z \in N(x) \cap N(y)$, then $S \cup \{z\}$ dominates $G$. Both cases produce a contradiction. Therefore, $d_G(x, y) \geq 3$. \qed

We observe two immediate consequences of Observation 5. First, if $\gamma(G - \{u, v\}) = \gamma(G) - 2$ for every pair of distinct vertices $u$ and $v$ in a graph $G$, then $G$ has no edges. Secondly, if $G$ is a connected, bicritical graph having diameter two, then for every pair of distinct vertices $u$ and $v$ in $G$, $\gamma(G - \{u, v\}) = \gamma(G) - 1$. Note that the graph $H$ in the proof of Proposition 3 is a connected, bicritical graph having some pairs of vertices whose deletion reduces the domination number by two.

By Observation 5, removing $v$ and any neighbor $u$ of $v$ from a bicritical graph $G$ reduces the domination number of $G$ by one. Thus adding $v$ to any $\gamma(G - \{u, v\})$-set produces a $\gamma(G)$-set. This yields the following observation.

Observation 6. If $G$ is a bicritical graph, then every vertex of $G$ belongs to a $\gamma(G)$-set.

It is also easy to see that if $G$ is a bicritical graph and $x$ and $y$ are vertices of $G$ such that $\gamma(G - \{x, y\}) = \gamma(G) - 2$, then $G$ has a $\gamma(G)$-set containing both $x$ and $y$. 

[86x627]
If $G$ is a graph and $v \in V^+$, then $\gamma(G-v) > \gamma(G)$. Since removing a vertex can decrease the domination number of a graph by at most one, $\gamma((G-v)-u) \geq \gamma(G)$ for all $u \in V(G-v)$. Thus, we have the following observation.

**Observation 7.** If $G$ is a bicritical graph, then $V = V^- \cup V^0$, that is, $V^+ = \emptyset$. Furthermore, either $G$ is critical, or $G-v$ is critical for all $v \in V^0$.

Brigham et al. [2] established an upper bound on the order of a critical graph in terms of its maximum degree and domination number.

**Proposition 8** (Brigham et al. [2]). If $G$ is a critical graph of order $n$, then $n \leq (\Delta(G) + 1)(\gamma(G) - 1) + 1$.

**Proposition 9** (Fulman et al. [5]). If $G$ is a critical graph of order $n = (\Delta(G) + 1)(\gamma(G) - 1) + 1$, then $G$ is regular.

By Observation 7 and Proposition 8, we have the following upper bound.

**Proposition 10.** If $G$ is a bicritical graph of order $n$, then $n \leq (\Delta(G) + 1)(\gamma(G) - 1) + 2$.

If a bicritical graph $G$ attains the upper bound of Proposition 10, then $G$ is not critical and $G-v$ is both critical and, by Proposition 9, regular for each $v \in V^0(G)$. The upper bound of Proposition 10 can be improved slightly if $G$ is regular.

**Proposition 11.** If $G$ is a regular bicritical graph of order $n$, then $n \leq (\Delta(G) + 1)(\gamma(G) - 1) + 1$.

**Proof.** If $G$ is critical, then the result holds from Proposition 8. On the other hand, if $G$ is not critical, then, by Observation 7, $G-v$ is critical for some $v \in V^0(G)$. Since $\gamma(G) \geq 2$, $v$ does not dominate $G$, and so $G-v$ is not regular and $\Delta(G-v) = \Delta(G)$. Hence, by Propositions 8 and 9, $n-1 = |V(G-v)| \leq (\Delta(G-v) + 1)(\gamma(G-v) - 1) = (\Delta(G) + 1)(\gamma(G) - 1)$, and the result follows. □

The 2-critical graphs were characterized in [2].

**Proposition 12** (Brigham et al. [2]). A graph $G$ is 2-critical if and only if $G$ is $K_{2t}$, $t \geq 1$, minus a perfect matching.

Our next result shows that there are no connected 2-bicritical graphs.

**Proposition 13.** If $G$ is a connected bicritical graph, then $\gamma(G) \geq 3$.

**Proof.** Suppose that $\gamma(G) = 2$. By Observation 7, either $G$ is critical or $G-v$ is critical for every $v \in V^0$. Suppose $G$ is critical. Then, by Proposition 12, $G = K_{2t} - M$ where $M$ is a perfect matching. Let $xy \in M$. (Note that $t \geq 2$ since $G$ is connected.) Then $G' = G - \{x, y\} = K_{2t-2} - M'$ where $M'$ is a perfect matching of $K_{2t-2}$, and so $\gamma(G') = 2 = \gamma(G)$, contradicting
Again, if \( t \geq 2 \) since \( G \) is connected. Label the edges of \( M, u_iv_i \) for \( 1 \leq i \leq t \). Since \( \gamma(G) = 2 \), we may assume that \( u_1 \) (say) is not adjacent to \( v \). But then \( \gamma(G - \{u_2, v_2\}) = 2 = \gamma(G) \), a contradiction. \( \square \)

**Proposition 14.** If \( G \) is a connected bicritical graph, then \( \delta(G) \geq 3 \).

**Proof.** Assume that \( \delta(G) < 2 \). Since \( \gamma(G) \geq 3 \), \( G \) has order at least 4. Let \( v \) be a vertex of minimum degree in \( G \). If \( \deg v = 1 \), let \( u \) be the neighbor of \( v \) in \( G \) and let \( w \) be a neighbor of \( u \) different from \( v \). If \( \deg v = 2 \), let \( u \) and \( w \) denote the two neighbors of \( v \). Let \( G' = G - \{u, w\} \) and let \( S' \) be a \( \gamma(G') \)-set. Since \( G \) is bicritical, \( \gamma(G') < \gamma(G) \). Now \( v \) is an isolate in \( G' \), and hence \( v \in S' \). If \( \deg v = 1 \), let \( S = (S' - \{v\}) \cup \{u\} \), while if \( \deg v = 2 \), let \( S = S' \). In both cases, \( S \) is a dominating set of \( G \) of cardinality less than \( \gamma(G) \), a contradiction. \( \square \)

By Proposition 1, the bounds in Propositions 13 and 14 are sharp.

### 4. Constructions

In this section, we give two ways of constructing a bicritical graph from smaller bicritical graphs. The second construction is used to determine additional properties of bicritical graphs.

#### 4.1. Expansion of a graph

A simple construction from Favaron et al. [4] makes it possible to extend a bicritical graph to a larger one provided the graph is also critical.

Let \( G = (V, E) \) be any graph and let \( v \in V \) and \( v' \notin V \). The expansion of \( G \) via \( v \), which we shall denote by \( G_{[v]} \), is defined in [4] to be the graph with vertex set \( V \cup \{v'\} \) and edge set \( E \cup \{v'x \mid x \in N_G(v)\} \). Thus, \( G_{[v]} \) is obtained from \( G \) by adding a new vertex \( v' \) that has the same closed neighborhood as \( v \).

**Theorem 15.** If \( v \) is a vertex of a graph \( G \) that is both critical and bicritical, then the graph \( G_{[v]} \) is bicritical.

**Proof.** Note that \( \gamma(G_{[v]}) = \gamma(G) \). Let \( x, y \in V(G_{[v]}) \). There are three cases to consider depending on whether \( |\{x, y\} \cap \{v, v'\}| \) is 0, 1 or 2.

Suppose \( |\{x, y\} \cap \{v, v'\}| = 0 \). Let \( D \) be a \( \gamma(G - \{x, y\}) \)-set. Since \( G \) is bicritical, \( |D| < \gamma(G) \).

Since \( D \) dominates \( v \) in \( G - \{x, y\} \), it also dominates \( v' \) in \( G_{[v]} - \{x, y\} \). Thus, \( D \) is a dominating set of \( G_{[v]} - \{x, y\} \), and so \( \gamma(G_{[v]} - \{x, y\}) \leq |D| < \gamma(G) = \gamma(G_{[v]}) \).

Suppose \( |\{x, y\} \cap \{v, v'\}| = 1 \). Since \( N_{G_{[v]}}(v) = N_{G_{[v]}}(v') \), we assume, without loss of generality, that \( x = v \) and \( y \in V(G) - \{v\} \). But then \( G_{[v]} - \{x, y\} = G_{[v]} - \{v, y\} \equiv G_{[v]} - \{v', y\} = G - y \). Since \( G \) is also a critical graph, it follows that \( \gamma(G_{[v]} - \{x, y\}) = \gamma(G - y) = \gamma(G) - 1 < \gamma(G_{[v]}) \).
Suppose, finally, \( \{x, y\} = \{v, v'\} \). Then \( G_{\{v\}} - \{x, y\} = G_{\{v\}} - \{v, v'\} = G - v \), and again since \( G \) is critical, we have \( \gamma(G_{\{v\}} - \{x, y\}) = \gamma(G - v) = \gamma(G) - 1 < \gamma(G_{\{v\}}) \).

Therefore, in all three cases \( \gamma(G_{\{v\}} - \{x, y\}) < \gamma(G_{\{v\}}) \) and \( G_{\{v\}} \) is bicritical. \( \square \)

We note that under the assumptions of Theorem 15, the graph \( G_{\{v\}} \) is not critical (\( v \) and \( v' \) are in \( V^0 \)), so the procedure can not be repeated.

### 4.2. Coalescence of two graphs

In this subsection, we give a simple construction from Brigham et al. [2] that makes it possible to build a bicritical graph from two smaller ones.

Suppose \( F \) and \( H \) are nonempty graphs. Let \( u \) and \( w \) be non-isolated vertices of \( F \) and \( H \), respectively. Then \((F \cdot H)(u, w : v)\) denotes the graph obtained from \( F \) and \( H \) by identifying \( u \) and \( w \) in a vertex labelled \( v \). We call \((F \cdot H)(u, w : v)\) the coalescence of \( F \) and \( H \) via \( u \) and \( w \).

Brigham et al. [2] proved the following result.

**Proposition 16** (Brigham et al. [2]). Let \( G \) be a coalescence of two graphs \( F \) and \( H \). Then, \( G \) is critical if and only if both \( F \) and \( H \) are critical. Furthermore, if \( G \) is critical, then \( \gamma(G) = \gamma(F) + \gamma(H) - 1 \).

**Lemma 17.** Let \( u \) and \( w \) be non-isolated vertices of distinct nonempty graphs \( F \) and \( H \), respectively, and let \( G = (F \cdot H)(u, w : v) \) be a coalescence of \( F \) and \( H \). Then,

\[
\gamma(F) + \gamma(H) - 1 \leq \gamma(G) \leq \gamma(F) + \gamma(H).
\]

**Proof.** Let \( D_F \) and \( D_H \) be a \( \gamma(F) \)-set and \( \gamma(H) \)-set, respectively. If \( u \notin D_F \) and \( w \notin D_H \), then \( D_F \cup D_H \) is a dominating set of \( G \). Otherwise, \((D_F - \{u\}) \cup (D_H - \{w\}) \cup \{v\}\) is a dominating set of \( G \). In either case, we see that the right inequality in the statement of the lemma follows.

To establish the left inequality, let \( D \) be a \( \gamma(G) \)-set. If \( v \in D \), then \( D_F = V(F) \cap (D - \{v\}) \cup \{u\} \) and \( D_H = V(H) \cap (D - \{v\}) \cup \{w\} \) are dominating sets of \( F \) and \( H \), respectively. So \( \gamma(F) + \gamma(H) \leq |D_F| + |D_H| \leq |D| + 1 \) and the left inequality holds in this case. Suppose now that \( v \notin D \). Then \( v \) is adjacent to a vertex \( x \), say, of \( D \). We may assume that \( x \) is a vertex of \( F \). Then, \( D_F = D \cap V(F) \) is a dominating set of \( F \). Also, \( D_H = (D \cap V(H)) \cup \{w\} \) is a dominating set of \( H \). Thus, \( \gamma(F) + \gamma(H) \leq |D_F| + |D_H| \leq |D| + 1 \) and once again the left inequality of the lemma follows. \( \square \)

**Proposition 18.** Let \( G \) be a coalescence of two graphs \( F \) and \( H \). Then, \( G \) is critical and bicritical if and only if both \( F \) and \( H \) are critical and bicritical.

**Proof.** Let \( G = (F \cdot H)(u, w : v) \), and let \( \gamma(F) = r \) and \( \gamma(H) = s \). Suppose first that \( G \) is critical and bicritical. By Proposition 16, \( \gamma(G) = r + s - 1 \) and both \( F \) and \( H \) are critical. We show that \( F \) is bicritical. Let \( x, y \in V(F) \). Since \( G \) is bicritical, \( r + s - 2 \geq \gamma(G - \{x, y\}) \).

If \( u \in \{x, y\} \), say \( u = x \), then, since \( H \) is critical, \( r + s - 2 \geq \gamma(G - \{x, y\}) = \gamma(F - \{x, y\}) + \gamma(H - w) = \gamma(F - \{x, y\}) + s - 1 \), and so \( \gamma(F - \{x, y\}) \leq r - 1 \). On the other
For every integer $s$, there exists a connected bicritical graph $G$ with $|V(G)| = 3s + 1$. Suppose $u$ is not isolated in $G = \{x, y\}$. If $u$ is isolated in $G = \{x, y\}$, then by Lemma 17, $\gamma(G - \{x, y\}) = \gamma(\{x, y\}) + \gamma(H) = \gamma(\{x, y\}) + \gamma(H) - 1 = \gamma(G - \{x, y\}) + s - 1$, and so $\gamma(G - \{x, y\}) \leq r - 1$. Suppose $u$ is isolated in $G - \{x, y\}$. Let $G - \{x, y\} = K \cup \{u\}$. Then $G = \{x, y\} = K \cup u$, and $\gamma(G - \{x, y\}) = \gamma(K) + 1$. But then $r + s - 2 \geq \gamma(G - \{x, y\}) = \gamma(K) + \gamma(H) = \gamma(G - \{x, y\}) - 1 + s$, and so once again $\gamma(G - \{x, y\}) \leq r - 1$. Hence, $F$ is bicritical. Similarly, $H$ is bicritical.

For the converse, suppose both $F$ and $H$ are critical and bicritical. By Proposition 16, $\gamma(T) = r + s - 1$ and $G$ is critical. We show that $G$ is bicritical. Let $x$ and $y$ be distinct vertices in $G$. Suppose that $x \in V(F) - \{u\}$ and $y \in V(F)$ (possibly, $u = y$). Since $F$ is bicritical, there is a dominating set $D_F$ of $F - \{x, y\}$ such that $|D_F| \leq r - 1$, and because $H$ is critical, there is a dominating set $D_H$ of $H - w$ such that $|D_H| = s - 1$. The set $D_F \cup D_H$ dominates $G - \{x, y\}$, and so $\gamma(G - \{x, y\}) = |D_F| + |D_H| < r + s - 2 < \gamma(G)$. Similarly, if $x \in V(H) - \{w\}$ and $y \in V(H)$, then $\gamma(G - \{x, y\}) < \gamma(G)$. Hence, we may assume that $x \in V(F) - \{u\}$ and $y \in V(F) - \{w\}$. Since $F$ is critical, there is a dominating set $D_F$ of $F - x$ such that $|D_F| = r - 1$. Since $H$ is bicritical, there is a dominating set $D_H$ of $H - \{w\}$ such that $|D_H| = s - 1$. The set $D_F \cup D_H$ dominates $G - \{x, y\}$, and so $\gamma(G - \{x, y\}) \leq |D_F| + |D_H| < r + s - 2 < \gamma(G)$. Hence, $G$ is bicritical.

Proposition 18 immediately yields a relationship between the domination number of a bicritical graph and the domination number of its blocks.

**Corollary 19.** A graph $G$ is critical and bicritical if and only if each block of $G$ is critical and bicritical. Further, if $G$ is critical and bicritical with blocks $G_1, G_2, \ldots, G_k$, then

$$\gamma(G) = \left(\sum_{i=1}^{k} \gamma(G_i)\right) - k + c(G),$$

where $c(G)$ is the number of components of $G$.

We believe that if $G$ is a connected bicritical graph, then $\text{diam}(G) \leq \gamma(G) - 1$. If this is the case, then Observation 20 shows that the bound is sharp. The proof of Observation 20 serves to illustrate the existence of bicritical graphs that contain cut-vertices.

**Observation 20.** For every integer $\gamma \geq 3$, there exists a connected graph $G_\gamma$ that is both critical and bicritical satisfying $\gamma(G_\gamma) = \gamma$ and $\text{diam}(G_\gamma) = \gamma - 1$.

**Proof.** Let $F$ be the circulant $C_8(1, 4)$. Then, $\text{diam}(F) = 2$ and, by Proposition 1, $F$ is 3-critical and 3-bicritical. Let $H$ be formed from the complete bipartite graph $K_{6,6}$ by removing the edges of three disjoint 4-cycles. Then, $\text{diam}(H) = 3$ and, by Proposition 3, $H$ is 4-critical and 4-bicritical. If $\gamma = 3$ or $\gamma = 4$, then we can take $G_\gamma = F$ or $G_\gamma = H$, respectively. Hence we may assume that $\gamma \geq 5$. We consider two possibilities, depending on whether $\gamma$ is odd or even.

Suppose $\gamma = 2k + 1$, where $k \geq 2$. Let $u$ and $w$ be any two nonadjacent vertices of $F$. Let $B_1, B_2, \ldots, B_k$ be $k$ disjoint copies of $F$. For $i = 1, 2, \ldots, k$, let $u_i$ and $w_i$ denote the vertices of $B_i$ corresponding to $u$ and $w$, respectively, in $F$. Let $G_\gamma$ be obtained by identifying $w_i$
and \( u_{i+1} \) for \( i = 1, \ldots, k - 1 \). Then \( B_1, B_2, \ldots, B_k \) are the blocks of \( G_γ \). Since each \( B_i \) is critical and bicritical with \( γ(B_i) = 3 \), we know from Corollary 19 that \( G_γ \) is critical and bicritical with \( γ(G_γ) = 2k + 1 = γ \). Furthermore, \( diam(G_γ) = 2k = γ - 1 \).

Suppose \( γ = 2k \), where \( k \geq 3 \). In the construction of \( G_γ \) in the preceding paragraph, replace \( B_{k-1} \) and \( B_k \) with a copy \( L \) of \( H \). Then \( B_1, \ldots, B_{k-2}, L \) are the blocks of \( G_γ \). By Corollary 19, \( G_γ \) is critical and bicritical with \( γ(G_γ) = 2k = γ \). Furthermore, \( diam(G_γ) = 2k - 1 = γ - 1 \). □

5. Edge connectivity

As illustrated in the previous section, there exist connected bicritical graphs that contain cut-vertices. In this section we show that the edge connectivity \( λ(G) \) of a bicritical graph \( G \) is at least two.

**Proposition 21.** If \( G \) is a connected bicritical graph, then \( λ(G) \geq 2 \).

**Proof.** Suppose that \( uv \) is a bridge of \( G \). Let \( G_u \) be the component of \( G - uv \) containing \( u \) and \( G_v \) be the component containing \( v \). By Proposition 14, \( δ(G) \geq 3 \), and so each of \( G_u \) and \( G_v \) has order at least 3. Clearly, \( γ(G) \leq γ(G_u) + γ(G_v) \). Let \( x \in V(G_u) \cap N(u) \). By Observation 5, removing adjacent vertices can decrease the domination number by at most one, and so \( γ(G) - 1 = γ(G - \{u, x\}) = γ(G_u - \{u, x\}) + γ(G_v) \geq γ(G_u) - 1 + γ(G_v) \) implying that \( γ(G) = γ(G_u) + γ(G_v) \).

If \( u \in V^0(G_u) \) and \( v \in V^0(G_v) \), then \( γ(G) - 1 = γ(G - \{u, v\}) = γ(G_u - u) + γ(G_v - v) = γ(G_u) + γ(G_v) = γ(G) \), a contradiction. Hence, \( u \in V^-(G_u) \) or \( v \in V^-(G_v) \). Without loss of generality, assume that \( u \in V^-(G_u) \). Let \( S_u \) be a \( γ(G_u - u) \)-set. Then, \( |S_u| = γ(G_u) - 1 \). Moreover, we claim that \( v \) is in some \( γ(G_v) \)-set. To see this, let \( y \in V(G_v) \cap N(v) \) and consider \( G - \{v, y\} \). Then \( γ(G) - 1 = γ(G - \{v, y\}) = γ(G_u) + γ(G_v) - 1 \) implying that a subset \( S' \) of \( γ(G_v) - 1 \) vertices in \( G_v \) dominates \( G_v - \{v, y\} \). Hence, \( S_v = S' \cup \{v\} \) is a \( γ(G_v) \)-set. But then \( S_u \cup S_v \) is a dominating set of \( G \) with cardinality \( γ(G_u) - 1 + γ(G_v) < γ(G) \), a contradiction. □

It can also be shown that if \( G \) is a connected critical graph, then \( λ(G) \geq 2 \). We omit the proof.

If we restrict the graph in Proposition 21 to be a cubic graph or a claw-free graph, then we show that its edge-connectivity is at least three. First we prove the following general lemma.

**Lemma 22.** Suppose that \( G \) is a connected bicritical graph with \( λ(G) = 2 \) and an edge-cut \( \{ab, cd\} \). Let \( G_1 \) and \( G_2 \) be the two components of \( G - ab - cd \), with \( a, c \in V(G_1), b, d \in V(G_2) \) and \( a \neq c \). Then the following must all be true.

(i) \( γ(G) = γ(G_1) + γ(G_2) \).
(ii) \( a, c \notin V^+(G_1) \) and \( b, d \notin V^-(G_2) \).
(iii) \( b \neq d \).
(iv) Without loss of generality, \( a, c \in V^-(G_1) \) and \( b, d \in V^0(G_2) \).
(v) Neither $b$ nor $d$ is in a $\gamma(G_2)$-set.
(vi) $\gamma(G_2 - \{b, d\}) = \gamma(G_2) - 1$, and a $\gamma(G_2 - \{b, d\})$-set dominates neither $b$ nor $d$.
(vii) There is a $\gamma(G_2 - d)$-set containing $b$, and there is a $\gamma(G_2 - b)$-set containing $d$.
(viii) There is no $\gamma(G_1)$-set containing both $a$ and $c$.
(ix) There is no $\gamma(G_1 - a)$-set containing $c$, and there is no $\gamma(G_1 - c)$-set containing $a$.
(x) $\gamma(G_1) \geq 3$.

Proof. Let $\gamma = \gamma(G)$. For $i = 1, 2$, let $V_i = V(G_i)$ and let $\gamma_i = \gamma(G_i)$. It is possible that $b = d$.

(i) Clearly, $\gamma \leq \gamma_1 + \gamma_2$. It suffices to show that $\gamma \geq \gamma_1 + \gamma_2$. Suppose $b = d$. Let $x \in V(V(G_2) \cap N(b))$. By Observation 5 and the fact that $G$ is bicritical, $\gamma - 1 = \gamma(G - \{b, x\}) = \gamma_1 + \gamma(G_2 - \{b, x\}) \geq \gamma_1 + \gamma_2 - 1$, and so $\gamma \geq \gamma_1 + \gamma_2$.

Suppose $b \neq d$. If $\gamma(G_2 - \{b, d\}) = \gamma_2 - 2$, there is a $\gamma(G_2)$-set which includes both $b$ and $d$. Now, $\gamma(G - \{a, c\}) \leq \gamma_1 - 1$. Let $D$ be a $\gamma(G - \{a, c\})$-set. Then, $|D| \leq \gamma - 1$ and $D = D_1 \cup D_2$ where $D_1$ is a $\gamma(G_1 - \{a, c\})$-set and $D_2$ is a $\gamma(G_2)$-set. Take $D_2$ to include $b$ and $d$. Then $D$ dominates $G$, a contradiction. Thus, $\gamma(G_2 - \{b, d\}) \geq \gamma_1 - 2$. Now, $\gamma - 1 \geq \gamma(G - \{b, d\}) = \gamma_1 + \gamma(G_2 - \{b, d\}) \geq \gamma_1 + \gamma_2 - 1$, and so $\gamma \geq \gamma_1 + \gamma_2$.

(ii) If $a \in V^+(G_1)$, then $\gamma - 1 \geq \gamma(G - \{a, d\}) = \gamma(G_1 - a) + \gamma(G_2 - d) \geq (\gamma_1 + 1) + (\gamma_2 - 1) = \gamma_1 + \gamma_2 = \gamma$, a contradiction. The results for $b, c$, and $d$ follow by a similar argument.

(iii) Suppose $b = d$. By (ii), $b \notin V^+(G_2)$. We show there is a $\gamma(G_2)$-set containing $b$. If $b \in V^-(G_2)$, then there is a $\gamma(G_2)$-set $D_2$ which contains $b$. If $b \in V^0(G_2)$, let $x \in N(b) \cap V(G_2)$. Then, $\gamma - 1 = \gamma(G - \{b, x\}) = \gamma_1 + \gamma(G_2 - \{b, x\})$ which implies $\gamma(G_2 - \{b, x\}) = \gamma_2 - 1$. Let $D_2'$ be a $\gamma(G_2 - \{b, x\})$-set. Then, in this case we also have $D_2 = D_2' \cup \{b\}$ is a $\gamma(G_2)$-set which contains $b$. Now, $\gamma - 1 \geq \gamma(G - \{a, c\}) = \gamma(G_1 - \{a, c\}) + \gamma_2$, and so $\gamma(G_1 - \{a, c\}) \leq \gamma_1 - 1$. Let $D_1$ be a $\gamma(G_1 - \{a, c\})$-set. Then, $D_1 \cup D_2$ is a dominating set of $G$ of cardinality $|D_1 \cup D_2| \leq \gamma - 1$, a contradiction.

(iv) By (ii), none of $a, b, c, d$ are in $V^+(G_i)$ for the appropriate $i$. Suppose $a \in V^-(G_1)$ and $b \in V^-(G_2)$. Then, $\gamma(G - \{a, b\}) \leq \gamma(G_1 - a) + \gamma(G_2 - b) = (\gamma_1 - 1) + (\gamma_2 - 1) = \gamma - 2$ which is impossible since $a$ and $b$ are adjacent. Thus at least one of $a \in V^0(G_1)$ or $b \in V^0(G_2)$ is true (as is one of $c \in V^0(G_1)$ or $d \in V^0(G_2)$). Next suppose $a \in V^0(G_1)$ and $d \in V^0(G_2)$. Then, $\gamma - 1 \geq \gamma(G - \{a, d\}) = \gamma(G_1 - a) + \gamma(G_2 - d) = \gamma_1 + \gamma_2 = \gamma$, a contradiction. Thus at least one of $a \in V^-(G_1)$ or $d \in V^-(G_2)$ is true (as is one of $c \in V^-(G_1)$ or $b \in V^-(G_2)$). It follows that exactly two of $a, b, c, d$ are in $V^-(G_i)$ for the appropriate $i$. Without loss of generality assume, $a \in V^-(G_1)$. Then the above comments imply $b \in V^0(G_2)$, $c \in V^0(G_1)$, and $d \in V^0(G_2)$.

(v) Suppose $b$ is in $\gamma(G_2)$-set $D_2$. Let $D_1$ be a $\gamma(G_1 - a)$-set. Since $a \in V^-(G_1)$, $|D_1| = \gamma_1 - 1$. Now, $D = D_1 \cup D_2$ dominates $G$ and $|D| = (\gamma_1 - 1) + \gamma_2 = \gamma - 1$, a contradiction. Hence, $b$ is not in any $\gamma(G_2)$-set. The result for $d$ follows from an identical argument.

(vi) Since $G$ is bicritical, $\gamma - 1 \geq \gamma(G - \{b, d\}) = \gamma_1 + \gamma(G_2 - \{b, d\})$, and so $\gamma(G_2 - \{b, d\}) \leq \gamma_2 - 1$. Let $D_2$ be a $\gamma(G_2 - \{b, d\})$-set. If $|D_2| = \gamma_2 - 2$, then $D_2 \cup \{b, d\}$ is a
Theorem 24. Let $G$ be a connected bicritical graph. If $G$ is cubic or claw-free, then $\lambda(G) \geq 3$.

Proof. By Proposition 21, $\lambda(G) \geq 2$. Suppose that $\lambda(G) = 2$. In what follows, we adopt the notation introduced in the statement of Lemma 22. Let $a_1$ and $a_2$ be two neighbors of $a$ in $G_1$. Since $G$ is bicritical and $d(a_1, a_2) \leq 2$, $\gamma = 1 = \gamma(G - \{a_1, a_2\})$. Let $D$ be a $\gamma(G - \{a_1, a_2\})$-set and let $D_i = D \cap V_i$ for $i = 1, 2$. Then, $|D| = \gamma - 1$. If $G$ is a cubic graph, then $a$ is adjacent only to $b$ in $G - \{a_1, a_2\}$. On the other hand if $G$ is a claw-free graph, then $N(a) - \{b\}$ induces a clique, and so any vertex of $G - \{a_1, a_2\}$ different from $b$ that dominates $a$ also dominates $a_1$ and $a_2$. In both cases, it follows that since $D$ is not a dominating set of $G$, $N_G[a] \cap D = \{b\}$.

If $|D| \geq \gamma_2 + 1$, then $|D_1| = |D| - |D_2| \leq \gamma_1 - 2$. But then $D_1 \cup \{a, c\}$ is a $\gamma(G_1)$-set, contradicting Lemma 22(viii). Hence, $|D_2| \leq \gamma_2$. Thus, since $b \in D_2$, it follows by Lemma 22(v) that $|D_2| = \gamma_2$, $D_2$ is a dominating set of $G_2 - d$ and $D_2$ does not dominate $d$. Hence, $c \in D_1$ in order to dominate $d$ and $|D_1| = \gamma_1 - 1$. But then $D_1 \cup \{a\}$ is a $\gamma(G_1)$-set, contradicting Lemma 22(viii). $\square$
6. 3-bicritical graphs

As shown in Corollary 23, a connected 3-bicritical graph has edge-connectivity at least three. We show here that a connected 3-bicritical graph has vertex-connectivity $\kappa(G)$ at least three.

**Proposition 25.** If $G$ is a connected 3-bicritical graph, then $\kappa(G) \geq 3$.

**Proof.** We show first that $G = (V, E)$ has no cut-vertex.

**Claim 1.** $\kappa(G) \geq 2$.

**Proof.** Suppose that $G$ has a cut-vertex $v$. Since $\delta(G) \geq 3$, each component of $G - v$ has order at least three. By Observation 7, $v \notin V^+(G)$. Suppose $v \in V^0(G)$. By Observation 7, $G - v$ is critical. Since $\gamma(G - v) = 3$, $G - v$ has at least two and at most three components, one of which, say $F$, has $\gamma(F) = 1$. But then for any vertex $z$ in $F$, $\gamma(G - \{v, z\}) = \gamma(F - z) + \gamma((G - v) - V(F)) = \gamma(F - z) + 2$ implying that $\gamma(F - z) \leq 0$, a contradiction.

Since $v \in V^-(G)$, $\gamma(G - v) = 2$ and $G - v$ has two components, $G_1$ and $G_2$ say, each of which is dominated by one vertex. For $i = 1, 2$, let $v_i$ be a vertex that dominates $G_i$. Since $\gamma(G) = 3$, no neighbor of $v$ dominates $G_1$ (respectively, $G_2$). In particular, neither $v_1$ nor $v_2$ is adjacent to $v$.

Let $S$ be a $\gamma(G - \{v_1, v_2\})$-set. Since $G$ is bicritical, $|S| \leq 2$. If $v \in S$, then $v$ dominates $G_1 - v_1$ or $G_2 - v_2$ (or both), say the former. But then since no neighbor of $v$ dominates $G_1$, $\gamma(G_1 - v_1) \geq 2$, and so $\gamma(G - \{v, v_1\}) = \gamma(G_1 - v_1) + \gamma(G_2) \geq 3$, a contradiction. Hence, $v \notin S$. Thus, $S = \{u_1, u_2\}$ where $u_i \in V(G_i)$ for $i = 1, 2$. For $i = 1, 2$, $u_i$ dominates $G_i - v_i$, and so since $v_i$ and $u_i$ are adjacent, $u_i$ dominates $G_i$. In order to dominate $v$, we may assume that $u_1 \in N(v)$. But this contradicts our earlier observation that no neighbor of $v$ dominates $G_1$. Hence, $G$ has no cut-vertex. $\square$

By Claim 1, $\kappa(G) \geq 2$. Suppose that $\kappa(G) = 2$. Then there exist vertices $a$ and $b$ such that $G - \{a, b\}$ is disconnected. Since $G$ is bicritical, $\gamma(G - \{a, b\}) = 2$ and $G - \{a, b\}$ has two components, $G_1$ and $G_2$ say, each of which is dominated by one vertex. For $i = 1, 2$, let $V_i = V(G_i)$. Let $\{v_1, v_2\}$ be a $\gamma(G - \{a, b\})$-set, where $v_1 \in V_1$ (and so, $v_2 \in V_2$). Since $\gamma(G) = 3$, at least one of $a$ and $b$ is not dominated by $\{v_1, v_2\}$, say $a$. We proceed further with the following claim.

**Claim 2.** (i) The vertex $b$ dominates neither $V_1$ nor $V_2$.

(ii) The vertex $b$ dominates either $V_1 - \{v_1\}$ nor $V_2 - \{v_2\}$.

(iii) The set $\{v_1, v_2\}$ can be chosen to dominate $b$.

**Proof.** (i) Suppose $b$ dominates $V_2$. Let $S$ be a $\gamma(G - \{b, v_1\})$-set. In order to dominate $v_2$, the set $S$ contains a vertex $s \in V_2$ since $a$ and $v_2$ are not adjacent (possibly, $s = v_2$), and so $s$ dominates $b$. The remaining vertex of $S$ cannot be adjacent to $v_1$, for otherwise $S$ also dominates $G$. Thus, $S = \{a, s\}$ and $a$ dominates $V_1 - \{v_1\}$. But then no $\gamma(G - \{a, v_1\})$-set $S^*$ can contain a vertex in $V_1 - \{v_1\}$ or $S^*$ would dominate $G$, and so $b$ must dominate
We may assume $v_3$. Since any vertex in $V_1 - \{v_1\}$ dominates both $a$ and $v_1$, it follows that $\gamma(G) = 2$, a contradiction. Hence, $b$ does not dominate $V_2$. Similarly, $b$ does not dominate $V_1$.

(ii) Suppose $b$ dominates $V_1 - \{v_1\}$. Then, by (i), $b$ is not adjacent to $v_1$. An identical proof as in (i) shows that $a$ dominates $V_1 - \{v_1\}$. Let $x_1 \in V_1 - \{v_1\}$. If $x_1$ dominates $V_1$, then $\{v_2, x_1\}$ dominates $G$, a contradiction. Hence every vertex in $V_1 - \{v_1\}$ is not adjacent to at least one other vertex in $V_1 - \{v_1\}$. Let $y_1 \in V_1 - N[x_1]$ and consider a $\gamma(G - \{x_1, y_1\})$-set $R$. If $a \in R$ or $b \in R$, then $R$ does not dominate $v_1$, a contradiction. Hence, $R$ contains a vertex $z_1$ of $V_1$ that dominates $V_1 - \{x_1, y_1\}$ and is not adjacent to at least one of $x_1$ or $y_1$.

We may assume $y_1$ and $z_1$ are not adjacent.

We show now that $x_1$ dominates $V_1 - \{y_1, z_1\}$. If there exists a vertex $x_2 \in V_1 - \{v_1, x_1, y_1, z_1\}$ that is not adjacent to $x_1$, then a $\gamma(G - \{x_2, z_1\})$-set contains neither $a$ nor $b$ and therefore contains a vertex $x_3 \in V_1 - \{v_1, x_2, z_1\}$ that dominates $V_1 - \{x_2, z_1\}$ and is not adjacent to at least one of $x_2$ or $z_1$. Since neither $x_1$ nor $y_1$ dominates $V_1 - \{x_2, z_1\}$, $x_3 \notin \{x_1, y_1\}$. Hence, $x_3$ and $z_1$ are adjacent, and so $x_3$ dominates $V_1 - \{x_2\}$ and $x_2x_3 \notin E(G)$.

Now a $\gamma(G - \{x_3, z_1\})$-set contains neither $a$ nor $b$ and therefore contains a vertex $y \in V_1 - \{x_3, z_1\}$ that dominates $V_1 - \{x_3, z_1\}$ and is notadjacent to at least one of $x_3$ or $z_1$. Hence, $y \in \{x_1, x_2, y_1\}$.

We now return to the proof of Proposition 25. By Claim 2(iii), we may assume that $b$ and $v_2$ are adjacent. Suppose $a$ dominates $V_2 - \{v_2\}$. Then, every vertex of $V_2 - \{v_2\}$ is
adjacent to both $a$ and $v_2$. Hence no $\gamma(G - \{a, v_2\})$-set contains a vertex in $V_2 - \{v_2\}$ for otherwise such a set also dominates $G$. But then every $\gamma(G - \{a, v_2\})$-set contains $b$ in order to dominate $V_2 - \{v_2\}$. Since $b$ and $v_2$ are adjacent, $b$ therefore dominates $G_2$, contradicting (i). Hence, $a$ does not dominate $V_2 - \{v_2\}$. Let $x_2$ be a vertex in $V_2 - \{v_2\}$ not adjacent to $a$.

Let $Y = \{y_1, y_2\}$ be a $\gamma(G - \{b, v_2\})$-set. In order to dominate $x_2$, we may assume $y_2 \in V_2$ (possibly, $x_2 = y_2$). Hence, $y_1 \in V_1$ in order to dominate $v_1$ (possibly, $y_1 = v_1$) and $y_2$ dominates $V_2$. If $y_2$ is not adjacent to $a$, then $y_1$ dominates $V_1 \cup \{a\}$. But then $\{v_2, y_1\}$ dominates $G$, a contradiction. Hence, $a$ and $y_2$ are adjacent. In particular, $y_2 \notin \{v_2, x_2\}$. Since $y_2$ dominates $V_2$, every vertex of $V_2 - \{v_2, y_2\}$ is adjacent to both $v_2$ and $y_2$. Since $Y$ does not dominate $G$, $y_2$ is not adjacent to $b$. Let $D$ be a $\gamma(G - \{v_2, y_2\})$-set. Then $D$ cannot contain a vertex in $V_2 - \{v_2, y_2\}$, for otherwise $D$ also dominates $G$. Hence in order to dominate $V_2 - \{v_2, y_2\}$, $\{a, b\} \cap D \neq \emptyset$. Since $a$ is not adjacent to $x_2$, $b \in D$. If $a \notin D$, then $b$ dominates $V_2 - \{y_2\}$. Hence every vertex of $V_2 - \{y_2\}$ is adjacent to both $b$ and $y_2$. But since every dominating set of $G - \{b, y_2\}$ contains a vertex of $V_2 - \{y_2\}$ (in order to dominate $v_2$), a $\gamma(G - \{b, y_2\})$-set also dominates $G$, a contradiction. Hence, $D = \{a, b\}$. Thus we have shown that $D = \{a, b\}$ dominates $G - \{v_2, y_2\}$. But $b$ dominates $v_2$ and $a$ dominates $y_2$, and so $D$ dominates $G$, which is not possible since $\gamma(G) = 3$ and $|D| = 2$. We deduce, therefore, that our supposition that $\kappa(G) = 2$ was false. Hence by Claim 1, $\kappa(G) \geq 3$. □

We close this section by observing that there is a unique 3-bicritical cubic graph.

**Observation 26.** A cubic graph $G$ is 3-bicritical if and only if $G$ is isomorphic to the circulant $C_8(1, 4)$.

**Proof.** Proposition 1 shows that the circulant $C_8(1, 4)$ is 3-bicritical. For the converse, let $G$ be 3-bicritical. Note that $n \geq 8$ because every cubic graph with $n \leq 6$ has domination number at most two. Proposition 11 implies that $n \leq 9$ and hence, $n = 8$ since $G$ is cubic. Since there are only two cubic graphs of order 8 with domination number three (namely, the two non-planar cubic graphs of order 8), and only one of these is bicritical, the desired result follows. □

### 7. Summary and open problems

As a consequence of Propositions 13, 14, and 21, we summarize some basic properties of bicritical graphs established thus far.

**Proposition 27.** If $G$ is a connected bicritical graph, then $\gamma(G) \geq 3$, $\delta(G) \geq 3$, and $\lambda(G) \geq 2$.

We close with some open questions and problems.

1. Is it true that every connected bicritical graph has a minimum dominating set containing any two specified vertices of the graph?

2. If $G$ is a connected bicritical graph, is it true that $\lambda(G) \geq 3$? In particular, if $G$ is a connected 5-bicritical graph, is it true that $\lambda(G) \geq 3$?
3. Characterize the 3-bicritical graphs.
4. Characterize the connected cubic bicritical graphs.
5. Is it true that if $G$ is a connected bicritical graph, then $\text{diam}(G) \leq \gamma(G) - 1$? If this is the case, then Observation 20 shows that the bound is sharp.
6. Is it true that if $G$ is a connected bicritical graph, then $\gamma(G) = i(G)$, where $i(G)$ is the independent domination number?

References