Abstract

Consider the Cauchy problem in odd dimensions for the dissipative wave equation: \((\Box + \partial_t^2)u = 0\) in \(\mathbb{R}^{2n+1} \times (0, \infty)\) with \((u, \partial_t u)|_{t=0} = (u_0, u_1)\). Because the \(L^2\) estimates and the \(L^{\infty}\) estimates of the solution \(u(t)\) are well known, in this paper we pay attention to the \(L^p\) estimates with \(1 \leq p < 2\) (in particular, \(p = 1\)) of the solution \(u(t)\) for \(t \geq 0\). In order to derive \(L^p\) estimates we first give the representation formulas of the solution \(u(t) = \partial_t S(t)u_0 + S(t)(u_0 + u_1)\) and then we directly estimate the exact solution \(S(t)g\) and its derivative \(\partial_t S(t)g\) of the dissipative wave equation with the initial data \((u_0, u_1) = (0, g)\). In particular, when \(p = 1\) and \(n \geq 1\), we get the \(L^1\) estimate:

\[
\|u(t)\|_{L^1} \leq Ce^{-t/4}(\|u_0\|_{W^{n+1,1}} + \|u_1\|_{W^{n-1,1}}) + C(\|u_0\|_{L^1} + \|u_1\|_{L^1}) \quad \text{for } t \geq 0.
\]

\(\Box\) and \(\partial_t\) denote the wave operator and its time derivative, respectively. The \(L^p\) estimates for \(p < 2\) are obtained using the technique of energy estimates, which is a standard method in the theory of partial differential equations.

Keywords: Dissipative wave equation; Decay; Cauchy problem; Odd dimensions

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1. Introduction and results

We investigate the $L^p$ ($p \geq 1$) estimates of the solution $u = u(x, t)$ to the Cauchy problem for the dissipative wave equation

$$
\begin{cases}
(\Box + \partial_t)u = 0 & \text{in } \mathbb{R}^N \times (0, \infty), \\
(u, \partial_t u)|_{t=0} = (u_0, u_1),
\end{cases}
$$

(1.1)

where $\Box = \partial^2_t + \partial^2_x - \Delta_x$ is the dissipative wave operator with Laplacian $\Delta_x = \sum_{j=1}^N \partial^2_{x_j}$. Equation (1.1) is often called the damped wave equation or the telegraph equation.

Matsumura [6] has shown the $L^2$ estimates and the $L^\infty$ estimates of the solution $u(t)$ of (1.1) (see Proposition 4.1 below), by using the Fourier transform method. Then, in this paper we pay attention to the $L^p$ estimates with $1 \leq p < \infty$ (in particular, $p = 1$) of the solution $u(t)$ of the dissipative wave equation (1.1).

By applying the Fourier transformation in the space variable together with the $L^\infty$ decay estimates of the solution $u(t)$ in [6], Milani and Han [7] derived the following $L^1$ type estimates of the solution $u(t)$ for large time $t$:

$$
\left\| \int_t^\infty D_x^\beta u(t) \right\|_{L^1} \leq C \tilde{\alpha}_t e^{-|\beta|/2} \quad \text{for } t \gg 1.
$$

Here, $\tilde{\alpha}_t = \|u_0\|_{H^{k+|\beta|/2} + 1} + \|u_1\|_{H^{k+|\beta|+N/2} + 1} + \|u_0\|_{L^1} + \|u_1\|_{L^1}$ with integers $s_0 > (N + k + |\beta| + 1)(N + 1) - 1$ and $s_1 > (N + k + |\beta|)(N + 1) - 1$, where the first and second terms come from the $L^\infty$ estimates of the solution and the third and fourth terms come from the inverse Fourier transformation of the solution, and the decay rate for large time $t$ seems to be sharp (cf. [15] for heat equations). However, their estimates should be relaxed the regularity conditions on the initial data and also estimated near the origin in time.

Concerning the $L^1$ estimates of the solution $u(t)$ for $t \geq 0$ in lower dimensions, there are a few results (see Marcati and Nishihara [5] for $N = 1$, Nishihara [10] for $N = 3$, and Ono [11,12] for $N \leq 3$, and also see [13] for exterior domains), e.g., $\|u(t)\|_{L^1} \leq C(\|u_0\|_{L^1} + \|u_1\|_{L^1})$ if $N = 1$ and $\|u(t)\|_{L^1} \leq C(\|u_0\|_{W^{1,1}} + \|u_1\|_{L^1})$ if $N = 2, 3$ for $t \geq 0$. On the other hand, in higher-dimensional cases, the analysis for the $L^1$ estimates of the solution becomes complicated. Recently, when dimension $N = 2n$ is even, we gave the $L^p$ ($p \geq 1$) estimates of the solution $u(t)$ as in Theorem 1.1 below, e.g., $\|u(t)\|_{L^p} \leq C(\|u_0\|_{W^{n,p}} + \|u_1\|_{W^{n-1,p}})$ for $t \geq 0$. In order to get such estimates, we used the representation formulas of the solution $u(t)$ in even dimensions. Therefore, we should consider the $L^p$ estimates with $1 \leq p < 2$ of the solution $u(t)$ in any odd dimensions. Here, we note that the representation formulas of the solution $u(t)$ in odd dimensions are different from ones in even dimensions.

Our main results are as follows.

**Theorem 1.1.** Let $N = 2n + 1 \geq 3$ be odd and let $1 \leq p \leq \infty$. Suppose that the initial data $(u_0, u_1)$ belong to $W^{n,p} \times W^{n-1,p}$ ($L^p \times L^p$ if $N = 1$). Then the solution $u(t)$ of (1.1) satisfies that

$$
\|u(t)\|_{L^p} \leq \begin{cases}
C(\|u_0\|_{L^p} + \|u_1\|_{L^p}), & \text{if } N = 1, \\
C(\|u_0\|_{W^{n,p}} + \|u_1\|_{W^{n-1,p}}), & \text{if } N = 2n + 1 \geq 3.
\end{cases}
$$

(1.2)
for \( t \geq 0 \).
Moreover, when \( p = 1 \) and \((u_0, u_1) \in L^2 \times H^{-1}\) in addition, for \( 1 \leq q < 2 \),
\[
\|u(t)\|_{L^q} \leq \mathcal{C} d_{0,n,1}(1 + t)^{-N/(2)(1-1/q)}, \quad t \geq 0,
\]
where \(d_{0,n,1}\) is given by (1.4) with \( m = 0, \ p = 1\).

Here, \(d_{m,n,p}\) stands for the norms of the initial data defined by
\[
d_{m,n,p} = \left\{ \begin{array}{ll}
\|u_0\|_{W^m} + \|u_1\|_{H^{m-1}} + \|u_0\|_{L^p} + \|u_1\|_{L^p}, & \text{if } n = 0, \\
\|u_0\|_{H^m} + \|u_1\|_{H^{m-1}} + \|u_0\|_{W^{n,p}} + \|u_1\|_{W^{n-1,p}}, & \text{if } n \geq 1.
\end{array} \right.
\]

**Remark.** When \( N = 2n + 1 \geq 3 \), we know from Propositions 3.1 and 3.2 (or Proposition A.1) below that for \( 1 \leq p \leq \infty \),
\[
\|u(t)\|_{L^p} \leq C e^{-t/4} \left( \|u_0\|_{W^{n,p}} + \|u_1\|_{W^{n-1,p}} \right) + C \left( \|u_0\|_{L^p} + \|u_1\|_{L^p} \right), \quad t \geq 0.
\]

**Theorem 1.2.** Let \( N = 2n + 1 \geq 3 \) be odd. Let \( m \geq 0 \) and \( 1 \leq p < 2 \). Suppose that the initial data \((u_0, u_1)\) belong to \((H^{m+1} \cap W^{n,p}) \times (H^m \cap W^{n-1,p})\) if \( N = 1 \). Then the solution \( u(t) \) of (1.1) satisfies that for \( 0 \leq k + |\beta| \leq m \) and \( k \neq m \),
\[
\|\partial_t^k \partial_x^\beta u(t)\|_{L^p} \leq C d_{m+1,n,p}(1 + t)^{-k-|\beta|/2}, \quad t \geq 0,
\]
where \(d_{m+1,n,p}\) is given by (1.4).

Moreover, when \( p = 1 \), for \( 1 \leq q < 2 \) and for \( 0 \leq k + |\beta| \leq m \) and \( k \neq m \),
\[
\|\partial_t^k \partial_x^\beta u(t)\|_{L^q} \leq C d_{m+1,n,1}(1 + t)^{-k-|\beta|/2-(N/(2)(1-1/q))}, \quad t \geq 0.
\]

We note that Marcati and Nishihara [5] for \( N = 1 \) and Nishihara [10] for \( N = 3 \) derived the \(L^p-L^q\)-type estimates with \( 1 \leq p \leq q \leq \infty \) of the solution \( u(t) \) (cf. Hayashi, Kaikina, and Naumkin [3], Hosono and Ogawa [4], Narazaki [8] for \(L^p-L^q\)-type estimates with some \( p \neq 1\)).

In order to derive the \(L^p \) \((p \geq 1)\) estimates of the solution \(u(t) = \partial_t S(t) u_0 + S(t) \times (u_0 + u_1)\), we first show the representation formula of the solution \(S(t)g\) of (1.1) with the initial data \((u_0, u_1) = (0, g)\) in Section 2. Next we derive the \(L^p\) estimates of \(S(t)g\) and \(\partial_t S(t)g\) in Section 3. By using these estimates, we give the proofs of Theorems 1.1 and 1.2 in Section 4. In Appendix A, we improve the \(L^p\) estimate of \(\partial_t S(t)g\).

We use only standard function spaces \(L^p = L^p(\mathbb{R}^N), W^{k,p} = W^{k,p}(\mathbb{R}^N)\), and \(H^s = W^{s,2} (W^{0,p} = L^p, H^0 = L^2)\) with usual norms \(\| \cdot \|_{L^p}, \| \cdot \|_{W^{n,p}}, \text{ and } \| \cdot \|_{H^s}\), respectively. Positive constants will be denoted by \(C\) and will change from line to line.

### 2. Representation formulas

By Courant and Hilbert's book [1], we obtain the representation formula of the solution \(w(t)\) to the Cauchy problem for the following wave equation (cf. [16,17]):
\[
\begin{cases}
\Box w = a^2 w \quad \text{in } \mathbb{R}^N \times (0, \infty), \\
(w, \partial_t w)|_{t=0} = (0, g).
\end{cases}
\]
For $N = 2n + 1 \geq 1$ being odd, the solution $w(t)$ is expressed as

$$w(t) = U(N-1/2)(t),$$

(2.2)

where $U_\lambda(t)$ is given inductively by

$$U_0(t) = tF_{2n+1}(t) \quad \text{and} \quad U_\lambda(t) = \frac{1}{2\lambda + 1} (t^\partial + 2\lambda) U_{\lambda-1}(t) = \frac{1}{2\lambda + 1} \frac{1}{t^{2\lambda-1}} \partial_t (t^{2\lambda} U_{\lambda-1}(t))$$

(2.3)

with

$$F_{2n+1}(t) \equiv F_{2n+1}(x, t) = \frac{2n+1}{t^{2n+1}} \int_0^t I_0(a \sqrt{t^2 - \rho^2}) \rho^{2n} Q_{2n+1}(x, \rho) d\rho.$$ (2.4)

Here, $Q_{2n+1}(\cdot, \rho)$ is the mean value of $g$, that is,

$$Q_{2n+1}(x, \rho) = \begin{cases} \frac{1}{b_{2n+1}} \int_{S^{2n}} g(x + \rho \omega) d\omega, & \text{if } n \geq 1, \\ \frac{1}{b_{2n+1}} \int_{S^{2n}} g(x + \rho \omega) d\omega, & \text{if } n = 0, \end{cases}$$

(2.5)

with the surface area $b_{2n+1} = 2\pi^{n+1/2}/\Gamma(n + 1/2)$ of the unit sphere in $\mathbb{R}^{2n+1}$ and $S^{2n} = \{ \omega \in \mathbb{R}^{2n+1} | |\omega| = 1 \}$, and $I_v(y)$ is the modified Bessel function of order $v$ and is given by

$$I_v(y) = \sum_{m=0}^{\infty} \frac{1}{m! \Gamma(m + 1 + v)} \left( \frac{y}{2} \right)^{2m+v}$$

(2.6)

with the Gamma function $\Gamma(\cdot)$ and satisfies the following properties:

$$I_{v+1}(y) = I_v'(y) - \frac{v}{y} I_v(y),$$

(2.7)

$$I_v(y) = \frac{e^y}{\sqrt{2\pi y}} \left( 1 + O(y^{-1}) \right), \quad \text{as } y \to \infty,$$ (2.8)

$$I_v(y) = \frac{1}{\Gamma(v+1)} \left( \frac{y}{2} \right)^v + O(y^{v+2}), \quad \text{as } y \to 0.$$ (2.9)

From the definition (2.3) of the function $U_\lambda(t)$, we observe that

$$U_\lambda(t) = \frac{1}{(2\lambda + 1)!!} \frac{1}{t^{2\lambda+2}} (t^3 \partial_t)^{\lambda} \left( t^3 F_{2n+1}(t) \right).$$ (2.10)

where $k!! = k(k - 2) \cdots 2$ if $k$ is even and $k!! = k(k - 2) \cdots 3 \cdot 1$ if $k$ is odd. Indeed, assuming the case $\lambda = 1$, we observe that for the case $\lambda$,

$$U_\lambda(t) = \frac{1}{2\lambda + 1} \frac{1}{t^{2\lambda-1}} \partial_t (t^{2\lambda} U_{\lambda-1}(t)) = \frac{1}{(2\lambda + 1)!!} \frac{1}{t^{2\lambda+1}} \partial_t (t^3 \partial_t)^{\lambda-1} \left( t^3 F_{2n+1}(t) \right)$$

$$= \frac{1}{(2\lambda + 1)!!} \frac{1}{t^{2\lambda+2}} (t^3 \partial_t)^{\lambda} \left( t^3 F_{2n+1}(t) \right).$$
Let \( v(t) = S(t)g \) be the solution to the Cauchy problem for the following dissipative wave equation:

\[
\begin{align*}
\left\{ \begin{array}{l}
(\Box + 2a\partial_t)v = 0 \quad \text{in } \mathbb{R}^N \times (0, \infty), \\
(v, \partial_t v)|_{t=0} = (0, g).
\end{array} \right.
\end{align*}
\]

Then, for the solution \( w(t) \) of (2.1) and the solution \( S(t)g \) of (2.11), we know that \( S(t)g = e^{-at}w(t) \), and hence, we obtain from (2.2)–(2.10) that

\[
S(t)g = e^{-at} t^{-2n+2}(t^3 \partial_t)^n (t^{2(1-n)} R(t)),
\]

where

\[
R(t) \equiv R_{2n+1}(x, t) = \int_0^t I_0\left(a\sqrt{t^2 - \rho^2}\right) \rho^{2n} G(\rho) \, d\rho
\]

and

\[
G(\rho) \equiv G_{2n+1}(x, \rho) = \begin{cases} 
\frac{1}{2}(g(x + \rho) + g(x - \rho)), & \text{if } n = 0, \\
\frac{1}{2}\left(\frac{1}{2\pi}\right)^n \int_{S^2} g(x + \rho\omega) \, d\omega, & \text{if } n \geq 1.
\end{cases}
\]

Moreover, by the Duhamel principle (e.g., [2]) together with the representation formula (2.12), the solution \( u(t) \) of (1.1) is expressed as

\[
u(t) = \partial_t S(t)u_0 + S(t)(u_0 + u_1)
\]

with \( a = 1/2 \).

Thus, it is enough to estimate the \( L^p \) (\( p \geq 1 \)) norms of \( S(t)g \) and \( \partial_t S(t)g \) in order to get the \( L^p \) estimate (1.2).

By an elementary calculation, we observe that

\[
(t^3 \partial_t)^n (t^{2(1-n)} R(t))
\]

\[
= (t^3 \partial_t)^{n-1} (2(1-n)t^{2(2-n)} R(t) + t^{2(2-n)+1} \partial_t R(t))
\]

\[
= (t^3 \partial_t)^{n-2} (2(1-n)(2-n)t^{2(3-n)} R(t) + c t^{2(3-n)+1} \partial_t R(t) + t^{2(3-n)+2} \partial_t^2 R(t))
\]

\[\cdots\]

\[
= 0 + ct^3 \partial_t R(t) + ct^4 \partial_t^2 R(t) + \cdots + ct^{n+1} \partial_t^{n-1} R(t) + t^{n+2} \partial_t^n R(t),
\]

where \( c \) is some constants which are usually different. Then, it follows from (2.12) that

\[
S(t)g = e^{-at} t^{-n} \left( c t^{-(n+1)} \partial_t R(t) + c t^{-n+2} \partial_t^2 R(t) + \cdots + c t^{-1} \partial_t^{n-1} R(t) + \partial_t^n R(t) \right).
\]

(2.16)

Moreover, differentiating \( R(t) \) defined by (2.13) in time \( t \), we have that for \( k \geq 1 \),

\[
\partial_t^k R(t) = \sum_{j=1}^{k} \partial_t^{k-j} f_j(t) + \int_0^t \partial_t^k I_0(a\sqrt{t^2 - \rho^2}) \rho^{2n} G(\rho) \, d\rho.
\]

(2.17)

where for \( 1 \leq j \leq k \),

\[
f_j(t) \equiv f_j(x, t) = \left( \partial_t^{j-1} I_0(a\sqrt{t^2 - \rho^2}) \right)|_{\rho=t} t^{2n} G(t)
\]

(2.18)
with $G(t) = G_{2n+1}(x, t)$ given by (2.14).

Inductively, we define $A_k(y)$ by

$$A_0(y) = I_0(y) \quad \text{and} \quad A_k(y) = A'_{k-1}(y) \frac{1}{y}. \quad (2.19)$$

Then, noting $\partial_t A_k(a\sqrt{t^2 - \rho^2}) = a^2 t A_{k+1}(a\sqrt{t^2 - \rho^2})$, we observe that

$$\partial_t I_0(a\sqrt{t^2 - \rho^2}) = a^2 t A_1(a\sqrt{t^2 - \rho^2}),$$

$$\partial_t I_0(a\sqrt{t^2 - \rho^2}) = (a^2 t)^2 A_2(a\sqrt{t^2 - \rho^2}) + c A_1(a\sqrt{t^2 - \rho^2}),$$

$$\partial_t I_0(a\sqrt{t^2 - \rho^2}) = (a^2 t)^3 A_3(a\sqrt{t^2 - \rho^2}) + c t A_2(a\sqrt{t^2 - \rho^2}),$$

$$\partial_t I_0(a\sqrt{t^2 - \rho^2}) = (a^2 t)^4 A_4(a\sqrt{t^2 - \rho^2})$$

$$+ c t^2 A_3(a\sqrt{t^2 - \rho^2}) + c A_2(a\sqrt{t^2 - \rho^2}),$$

$$\partial_t I_0(a\sqrt{t^2 - \rho^2}) = (a^2 t)^5 A_5(a\sqrt{t^2 - \rho^2}) + c t^3 A_4(a\sqrt{t^2 - \rho^2})$$

$$+ c t A_3(a\sqrt{t^2 - \rho^2}),$$

and, in general,

$$\partial_t^\ell I_0(a\sqrt{t^2 - \rho^2}) = (a^2 t)^\ell A_{\ell}(a\sqrt{t^2 - \rho^2})$$

$$+ \cdots + c t^{\ell-1} A_{\ell-1}(a\sqrt{t^2 - \rho^2}) + c A_\ell(a\sqrt{t^2 - \rho^2}) \quad (2.20)$$

and

$$\partial_t^{2\ell+1} I_0(a\sqrt{t^2 - \rho^2}) = (a^2 t)^{2\ell+1} A_{2\ell+1}(a\sqrt{t^2 - \rho^2})$$

$$+ \cdots + c t^{2\ell} A_{2\ell}(a\sqrt{t^2 - \rho^2}) + c A_{2\ell}(a\sqrt{t^2 - \rho^2}). \quad (2.21)$$

In order to estimate the function $A_k(y)$ defined by (2.19), we prepare the following lemma which plays an important role in the next section.

**Lemma 2.1.** The function $A_k(y)$ ($k = 0, 1, 2, \ldots$) satisfies that

$$A_k(y) = I_k(y) \frac{1}{y^k} \quad (2.22)$$

and

$$A_k(0) = \frac{1}{2^k k!}. \quad (2.23)$$

**Proof.** Assuming the case $k - 1$, we observe from (2.7) and (2.19) that for the case $k$,
\[ \Lambda_k(y) = \Lambda'_k(y) \frac{1}{y} = \left( I_{k-1}(y) \frac{1}{y^{k-1}} \right)' \frac{1}{y} = I'_k(y) \frac{1}{y^k} + I_{k-1}(y) \frac{k+1}{y^{k+1}} \]

Moreover, by the property (2.9) of the modified Bessel function, we have that
\[ \Lambda_k(0) = I_k(y) \frac{1}{y^k} \bigg|_{y=0} = \frac{1}{2^k \Gamma(k+1)} \]
which implies (2.23).

3. \( L^p \) estimates of \( S(t)g \) and \( \partial_t S(t)g \)

The following estimates of the function \( \Lambda_k(a \sqrt{t^2 - \rho^2}) \) are crucial for the \( L^p \) estimates of \( S(t)g \) and \( \partial_t S(t)g \).

Lemma 3.1. For \( t \geq 2 \), it holds that
\[ e^{-at} \Lambda_k(a \sqrt{t^2 - \rho^2}) \leq Ct^{-k-1/2} e^{-a \rho^2 [(2t)^{3/4}}}, \quad \text{if } 0 \leq \rho < t^{1/4}, \tag{3.1} \]
\[ e^{-at} \Lambda_k(a \sqrt{t^2 - \rho^2}) \leq Ct^{-1/2} e^{-a \rho^2}, \quad \text{if } t^{1/4} \leq \rho < \sqrt{t^2 - 1}, \tag{3.2} \]
\[ e^{-at} \Lambda_k(a \sqrt{t^2 - \rho^2}) \leq Ce^{-at}, \quad \text{if } \sqrt{t^2 - 1} \leq \rho \leq t \tag{3.3} \]
with some constant \( C \).

Proof. Let \( t \geq 2 \). Since
\[ e^{-at} e^{a \sqrt{t^2 - \rho^2}} \leq e^{-a \rho^2 [(2t)^{3/4}}}, \tag{3.4} \]
we observe from (2.6) and (2.8) that
\[ e^{-at} I_k(a \sqrt{t^2 - \rho^2}) \leq \begin{cases} \sqrt{t^{-1/2}} e^{-a \rho^2}, & \text{if } 0 \leq \rho < t^{1/4}, \\ Ct^{1/2} e^{-a \rho^2}, & \text{if } \rho > t^{1/4}, \end{cases} \]
and hence, from (2.22) we obtain (3.1) and (3.2). On the other hand, noting the fact (2.23) we know (3.3). \( \square \)

In order to the \( L^p \) estimate of the solution \( u(t) \), first we shall estimate the \( L^p \) norm of \( S(t)g \) given by (2.16).

Proposition 3.1. For \( 1 \leq p \leq \infty \),
\[ \| S(t)g \|_{L^p} \leq C t^{-1/2} \| w_{n-1, p} \|_{L^p} + C \| g \|_{L^p}, \quad t \geq 0. \tag{3.5} \]

Proof. From (2.16) and (2.17), it follows that
\[ S(t)g_L^p \leq Ce^{-at} t^{-n} \sum_{k=1}^{n} t^{-n+k} \| \partial_t^k R(t) \|_{L^p} \]
\[ \leq Ce^{-at} t^{-n} \sum_{k=1}^{n} t^{-n+k} \left( \sum_{j=1}^{k} \| \partial_t^{k-j} f_j(t) \|_{L^p} \right) \]
\[ + \int_0^t \| \partial_t^k I_0(a \sqrt{t^2 - \rho^2}) \|_{L^p} \rho^{2n} \, d\rho \|_{L^p} \].
\[ (3.6) \]

For \( f_j(t) \) in (2.18), noting (2.20), (2.21), (2.14) and using (2.23) together with the Leibniz rule, we obtain that when \( 1 \leq k \leq n \), for \( 1 \leq j \leq k \),
\[ e^{-at} \| \partial_t^{k-j} f_j(t) \|_{L^p} \leq Ce^{-at/2} \| g \|_{W^{n-1,p}}, \quad t \geq 2, \]
\[ (3.7) \]

and
\[ t^{-2n+k} \| \partial_t^{k-j} f_j(t) \|_{L^p} \leq C \| g \|_{W^{n-1,p}}, \quad t \leq 2. \]
\[ (3.8) \]

Using Lemma 3.1 together with the inequality
\[ \int_0^\delta e^{-\rho^2/\mu} \rho^m \, d\rho \leq Ct^{(m+1)/2}, \quad m \geq 0, \quad \delta > 0 \]
\[ (3.9) \]

for \( t > 0 \), we obtain that for \( 1 \leq k \leq n \),
\[ e^{-at} t^{-n} \int_0^t \| \partial_t^k I_0(a \sqrt{t^2 - \rho^2}) \|_{L^p} \rho^{2n} \, d\rho \leq C, \quad t \geq 2. \]
\[ (3.10) \]

Indeed, when \( k = 2\ell \) is even, for \( \ell \leq j \leq 2\ell \),
\[ e^{-at} t^{-n} \int_0^t \rho^{2(j-\ell)} |A_j(a \sqrt{t^2 - \rho^2})| \rho^{2n} \, d\rho = \int_0^{t^{1/4}} + \int_{t^{1/4}}^{\sqrt{t^2-1}} + \int_{\sqrt{t^2-1}}^t \]
\[ \leq Ce^{-at} t^{-n} t^{2(j-\ell)} \left( \int_0^{t^{1/4}} e^{-a\rho^2/2} \rho^{2n} \, d\rho + \int_{t^{1/4}}^{\sqrt{t^2-1}} e^{-a\sqrt{t^2-1}/2} \rho^{2n} \, d\rho \right) \]
\[ + \int_{\sqrt{t^2-1}}^{t^{1/2}} e^{-a\sqrt{t^2-1}/2} \rho^{2n} \, d\rho + \int_{t^{1/2}}^t e^{-at} \rho^{2n} \, d\rho \]
\[ \leq Ct^{-n} t^{2(j-\ell)} (t^{j-1/2} e^{-a\rho^2/2} + t^{-1/2} e^{-a\sqrt{t^2-1}/2} \rho^{2n} + e^{-at} \rho^{2n} + t \rho^{2n}) \]
\[ \leq Ct^{-2\ell+j} \leq C, \quad t \geq 2. \]
and when $k = 2\ell + 1$ is odd, for $\ell + 1 \leq j \leq 2\ell + 1$,
\[
e^{-at} t^{-n} \int_0^t t^{2(j-\ell-1)} |A_j(a\sqrt{t^2 - \rho^2})| \rho^{2n} d\rho \leq Ct^{-2\ell+1} \leq C, \quad t \geq 2,
\]
and hence, noting (2.20) and (2.21), we have (3.10).

Using Lemma 2.1 when $t \leq 2$, we obtain that for $1 \leq k \leq n$,
\[
e^{-at} t^{-n} \int_0^t |\partial^k t I_0(a\sqrt{t^2 - \rho^2})| \rho^{2n} d\rho \leq C, \quad t \leq 2.
\] (3.11)

Indeed, when $k = 2\ell$ is even, for $\ell \leq j \leq 2\ell$,
\[
t^{-2n+k} \int_0^t t^{2(j-\ell-1)} |A_j(a\sqrt{t^2 - \rho^2})| \rho^{2n} d\rho \leq Ct^{-2n+2j} \int_0^t \rho^{2n} d\rho \leq C, \quad t \leq 2,
\]
and when $k = 2\ell + 1$ is odd, for $\ell + 1 \leq j \leq 2\ell + 1$,
\[
t^{-2n+k} \int_0^t t^{2(j-\ell-1)} |A_j(a\sqrt{t^2 - \rho^2})| \rho^{2n} d\rho \leq C, \quad t \leq 2,
\]
and hence, noting (2.20) and (2.21), we have (3.11).

Therefore, (3.6)–(3.11) yield (3.5).

Next, we shall estimate the $L^p$ norm of $\partial_t S(t) g$.

**Proposition 3.2.** For $1 \leq p \leq \infty$,
\[
\|\partial_t S(t) g\|_{L^p} \leq Ce^{-at/2} \|g\|_{W^{n,p}} + C \|g\|_{L^p}, \quad t \geq 0.
\] (3.12)

**Proof.** From (2.16), it follows that
\[
\partial_t S(t) g + aS(t) g = e^{-at} t^{-n} \left( ct^{-n} \partial_t S(t) + ct^{-n+1} \partial_t^2 R(t) + \cdots + ct^{-1} \partial_t^n R(t) + \partial_t^{n+1} R(t) \right).
\] (3.13)

By the same way as the proof of Proposition 3.1, we observe that
\[
\|\partial_t S(t) g + aS(t) g\|_{L^p} \leq Ce^{-at} t^{-n} \sum_{k=1}^{p+1} t^{-n-1+k} \|\partial_t^k R(t)\|_{L^p}
\]
\[
\leq Ce^{-at} t^{-n} \sum_{k=1}^{p+1} t^{-n-1+k} \left( \sum_{j=1}^k \|\partial_t^{k-j} f_j(t)\|_{L^p} + \int_0^t |\partial_t^k I_0(a\sqrt{t^2 - \rho^2})| \rho^{2n} d\rho \|g\|_{L^p} \right)
\]
\[
\leq Ce^{-at/2} \|g\|_{W^{n,p}} + C \|g\|_{L^p}, \quad t \geq 0,
\]
and hence, noting (3.5), we obtain the desired estimate (3.12).
4. Proofs of Theorems 1.1 and 1.2

In this section we give the proofs of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Using (3.5) and (3.12) together with (2.15), we have that
\[
\|u(t)\|_{L^p} \leq \|\partial_t S(t)u_0\|_{L^p} + \|S(t)(u_0 + u_1)\|_{L^p} \\
\leq Ce^{-at/2}\left(\|u_0\|_{W^{n,p}} + \|u_1\|_{W^{n-1,p}}\right) + C\left(\|u_0\|_{L^p} + \|u_1\|_{L^p}\right)
\]
for \(t \geq 0\) which implies the desired estimate (1.2).

Also, the decay (1.3) is given by the interpolation of (1.2) with \(p = 1\) and \(L^2\) decay (4.1) with \(m = p = 1\) in Proposition 4.1 below. \(\square\)

In order to prove Theorem 1.2, we state the \(L^2\) type estimates of the solution \(u(t)\) of (1.1) given by Matsumura [6] (see also Ono [14], cf. Hayashi, Kaikina, and Naumkin [3], Hosono and Ogawa [4], Narazaki [8], Todorova and Yordanov [18]).

Proposition 4.1. Let \(m \geq 0\) and \(1 \leq p \leq 2\). Suppose that the initial data \((u_0, u_1)\) belong to \((H^m \cap L^p) \times (H^{m-1} \cap L^p)\). Then the solution \(u(t)\) of (1.1) satisfies that for \(0 \leq k + |\beta| \leq m\),
\[
\|\partial^k_t D^\beta x u(t)\|_{L^2} \leq \widetilde{d}_{m,p}(1 + t)^{-k - |\beta|/2 - \eta_p}, \quad \eta_p = \frac{N}{2}\left(\frac{1}{p} - \frac{1}{2}\right),
\]
for \(t \geq 0\) with
\[
\widetilde{d}_{m,p} = \|u_0\|_{H^m} + \|u_1\|_{H^{m-1}} + \|u_0\|_{L^p} + \|u_1\|_{L^p}.
\]

Proof of Theorem 1.2. Using the estimates (1.2) and (4.1) together with the Gagliardo–Nirenberg inequality, we have that
\[
\|D^\beta x u(t)\|_{L^p} \leq C\|u(t)\|^{1-\theta}_{L^p}\|\nabla^{\beta+1} u(t)\|^{\theta}_{L^2} \leq Cd_{|\beta|+1,n,p}(1 + t)^{-\omega_0} \tag{4.2}
\]
with
\[
\theta = \left(\frac{|\beta|}{N}\right)/\left(\frac{1}{p} - \frac{1}{2} + \frac{|\beta| + 1}{N}\right) \quad \text{and} \quad \omega_0 = \theta\left(\frac{|\beta| + 1}{2} + \eta_p\right) = \frac{|\beta|}{2}.
\]

Since the solution \(u(t)\) of (1.1) satisfies that \(\partial_t(e^t\partial_t u) = e^t(\partial_t^2 u + \partial_x u) = e^t \Delta_x u\), it holds that
\[
\|\partial_t u(t)\|_{L^p} \leq e^{-t}\|u_1\|_{L^p} + \int_0^t e^t e^{-(t-s)}\|\Delta_x u(s)\|_{L^p} ds \\
\leq Cd_{3,n,p}(1 + t)^{-1}, \tag{4.3}
\]
where we used the estimates \(\|\nabla \Delta_x u(t)\|_{L^p} \leq Cd_{3,n,p}(1 + t)^{-1}\) in (4.2).
Using the estimates (4.3) and (4.1) together with the Gagliardo–Nirenberg inequality again, we have that
\[
\| D^\beta_x \partial_t u(t) \|_{L^p} \leq C \| \partial_t u(t) \|_{L^2}^{1-\theta} \| \nabla^{|\beta|+1} \partial_t u(t) \|_{L^2}^\theta \leq C d_{|\beta|+2,n,p} (1+t)^{-\omega_1}
\]
with \( \omega_1 = (1-\theta) + \theta (1 + (|\beta| + 1)/2 + \eta_p) = 1 + |\beta|/2. \)

Repeating the same process as the above for \( \partial^k_t u(t) \) we obtain the desired estimate (1.5) (see [13,14] for details). Also, from (1.5) and (4.1) we know (1.6). \( \square \)

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Appendix A

We can improve the \( L^p \) estimate of \( \partial_t S(t) g \) in Proposition 3.2.

**Proposition A.1.** For \( 1 \leq p \leq \infty \),
\[
\| \partial_t S(t) g \|_{L^p} \leq C e^{-at/2} \| g \|_{W^{n,p}} + C (1+t)^{-1} \| g \|_{L^p}, \quad t \geq 0.
\]

**Proof.** It is enough to prove (4.3) for \( t \geq 2. \) From (2.16) and (3.13), it follows that
\[
\partial_t S(t) g = e^{-at} t^{-n} \left( \sum_{k=1}^n c t^{-n-1+k} \partial^k_t R(t) - a \sum_{k=1}^{n-1} c t^{-n+k} \partial^k_t R(t) \right)
+ (\partial^{n+1}_t R(t) - a \partial^n_t R(t)).
\]

We observe from (2.17) that
\[
\| \partial_t S(t) g \|_{L^p} \leq C e^{-at} t^{-n} \left( \sum_{k=1}^n t^{-n-1+k} \| \partial^k_t R(t) \|_{L^p} + \sum_{k=1}^{n-1} t^{-n+k} \| \partial^k_t R(t) \|_{L^p} \right)
+ \| \partial^{n+1}_t R(t) - a \partial^n_t R(t) \|_{L^p}
\leq C e^{-at} t^{-n} \left( \sum_{k=1}^{n+1} t^{-n-1+k} \sum_{j=1}^k \| \partial_j^{k-j} f_j(t) \|_{L^p} \right)
+ \sum_{k=1}^n t^{-n+k} \sum_{j=1}^k \| \partial_j^{k-j} f_j(t) \|_{L^p}
\]
and by the same way as the proof of Proposition 3.1, we have that
\[
\|\partial_t S(t)g\|_{L^p} \leq C e^{-at^2} \|g\|_{W^{s,p}} + C(1 + t)^{-1} \|g\|_{L^p}
\]
and
\[
e^{-at} t^{-n} \left( \int_{t^{1/4}}^{t^{1/4}} + \int_{t^{1/4}}^{t} \right) \left( |\partial_{\rho}^{n+1} I_0(a \sqrt{t^2 - \rho^2})| + |\partial_{\rho}^{n} I_0(a \sqrt{t^2 - \rho^2})| \right) \rho^{2n} d\rho
\]

\[
\leq C t^{-n} \left( t^{-1/2} e^{-a \sqrt{t^{1/2} t^{2n+1}}} + e^{-at} t^{2n+1} \right) \leq C t^{-1}, \quad t \geq 2.
\]

Thus, in order to get (4.3) we need to estimate the integral term
\[
e^{-at} t^{-n} \int_{t^{1/4}}^{t} \left( |\partial_{\rho}^{n+1} I_0(a \sqrt{t^2 - \rho^2})| - a \partial_{\rho}^{n} I_0(a \sqrt{t^2 - \rho^2}) \right) \rho^{2n} d\rho.
\]

Here, since it follows from (2.20) and (2.21) that for even \(n = 2\ell\),
\[
\partial_{\rho}^{2\ell+1} I_0(a \sqrt{t^2 - \rho^2}) - a \partial_{\rho}^{2\ell} I_0(a \sqrt{t^2 - \rho^2}) = a(a^2 n)^{\ell} \left( \Lambda_{2\ell+1}(a \sqrt{t^2 - \rho^2}) - \Lambda_{2\ell}(a \sqrt{t^2 - \rho^2}) \right)
\]
\[
+ \sum_{j=\ell+1}^{\ell} c t^{2(j-\ell-1)+1} \Lambda_j(a \sqrt{t^2 - \rho^2}) - a \sum_{j=\ell}^{2\ell-1} c t^{2(j-\ell)} \Lambda_j(a \sqrt{t^2 - \rho^2})
\]
and for odd \(n = 2\ell+1\),
\[
\partial_{\rho}^{2\ell+2} I_0(a \sqrt{t^2 - \rho^2}) - a \partial_{\rho}^{2\ell+1} I_0(a \sqrt{t^2 - \rho^2}) = a(a^2 n)^{\ell+1} \left( \Lambda_{2\ell+2}(a \sqrt{t^2 - \rho^2}) - \Lambda_{2\ell+1}(a \sqrt{t^2 - \rho^2}) \right)
\]
\[
+ \sum_{j=\ell+1}^{2\ell+1} c t^{2(j-\ell-1)+1} \Lambda_j(a \sqrt{t^2 - \rho^2}) - a \sum_{j=\ell+1}^{2\ell} c t^{2(j-\ell-1)+1} \Lambda_j(a \sqrt{t^2 - \rho^2}).
\]
we have that
\[
\begin{align*}
e^{-at}t^{-n} & \int_{0}^{\frac{t}{4}} \left| \partial_t I_0(a\sqrt{t^2 - \rho^2}) - aI_0(a\sqrt{t^2 - \rho^2}) \right| \rho^{2n} d\rho \\
& \leq Ce^{-at} \int_{0}^{\frac{t}{4}} \left| at \Lambda_n(a\sqrt{t^2 - \rho^2}) - \Lambda_n(a\sqrt{t^2 - \rho^2}) \right| \rho^{2n} d\rho \\
& \quad + \text{error term}, \quad (A.4)
\end{align*}
\]
and by the same way as the proof of Proposition 3.1,
\[
\text{error term} \leq Ct^{-1} \quad \text{for} \quad t \geq 2.
\]
Moreover, in order to estimate the first term of (A.4), we use the following property of the modified Bessel function \( I_\nu(y) \) (see Nikiforv and Ouvarov [9] for details).

**Lemma A.1.**
\[
I_\nu(y) = \frac{e^y}{\sqrt{2\pi y}} \left( 1 - \frac{(\nu - 1/2)(\nu + 1/2)}{2y} + \frac{(\nu - 1/2)(\nu - 3/2)(\nu + 3/2)(\nu + 1/2)}{2! 2^2 y^2} \right.
\]
\[
\left. - \cdots + (-1)^k \frac{(\nu - 1/2) \cdots (\nu - (k - 1/2))(\nu + (k - 1/2)) \cdots (\nu + 1/2)}{k! 2^k y^k} + O(y^{-k}) \right), \quad \text{as} \quad y \to \infty. \quad (A.5)
\]

We continue to estimate the first term of (A.4). Since it holds that
\[
\begin{align*}
at \Lambda_{n+1}(a\sqrt{t^2 - \rho^2}) - \Lambda_n(a\sqrt{t^2 - \rho^2})
& = \frac{1}{(a\sqrt{t^2 - \rho^2})^{n+1}} (at I_{n+1}(a\sqrt{t^2 - \rho^2}) - I_n(a\sqrt{t^2 - \rho^2})) \\
& \quad + \frac{ap^2}{t + \sqrt{t^2 - \rho^2}} I_n(a\sqrt{t^2 - \rho^2})
\end{align*}
\]
by (2.22), we observe from (A.5) and (3.4) that
\[
e^{-at} \left| at \Lambda_{n+1}(a\sqrt{t^2 - \rho^2}) - \Lambda_n(a\sqrt{t^2 - \rho^2}) \right| \\
\leq Ct^{-n-1}\left( t^{-3/2} e^{-ap^2/(2t)} + t^{-1} \rho_{1/2}^{-1/2} e^{-ap^2/(2t)} \right)
\]
and hence, from (3.9) that
\[
e^{-at} \int_{0}^{\frac{t}{4}} \left| at \Lambda_{n+1}(a\sqrt{t^2 - \rho^2}) - \Lambda_n(a\sqrt{t^2 - \rho^2}) \right| \rho^{2n} d\rho
\]
\[ \leq C t^{-n-3/2} \int_0^t e^{-a \rho^2/(2t)} \rho^{2n} d\rho + C t^{-n-2-1/2} \int_0^t e^{-a \rho^2/(2t)} \rho^{2n+2} d\rho \]

\[ \leq C t^{-n-3/2} t^{(2n+1)/2} + C t^{-n-2-1/2} t^{(2n+3)/2} \leq C t^{-1} \quad (A.6) \]

for \( t \geq 2 \).

Summing up the above (A.2)–(A.6), we arrive at the desired estimate (A.1). \( \square \)

Applying Proposition A.1 to the solution \( u(t) (= \partial_t S(t) g \) given by (2.15)) of the dissipative wave equation (1.1) under the condition \( u_0 + u_1 = 0 \), immediately we have the following

**Proposition A.2.** In addition to the assumption of Theorem 1.1, suppose that the initial data \((u_0, u_1)\) satisfy \( u_0 + u_1 = 0 \). Then, the solution \( u(t) \) of (1.1) satisfies that

\[ \|u(t)\|_{L^p} \leq C (1 + t)^{-1} \|u_0\|_{W^{n,p}}, \quad t \geq 0. \]

**References**
