MATHEMATICS

INVARIANCE OF PARTIAL ORDER OF RECURSIVE EQUIVALENCE TYPES UNDER FINITE DIVISION

BY

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We wish to prove the following

THEOREM [1]). If n is a positive integer and A and B are recursive equivalence types, then $nA \leqslant nB$ is equivalent to $A \leqslant B$.

This generalizes the analogous theorem 40 a of [1] for A and B isols. The generalization will be obtained by a method resembling J. MYHILL's proof in [3] that all creative sets are isomorphic. Since $X = Y$ is equivalent to $X \leqslant Y$ & $X \geqslant Y$ our theorem has as a corollary R. FRIEDBERG's result in [2], which can be formulated as follows: if n is a positive integer and A and B are recursive equivalence types, then $nA = nB$ is equivalent to $A = B$.

We use notations and concepts of [1]. So $x, y, z, \ldots$ will denote natural numbers, $\alpha, \beta, \gamma, \ldots$ sets of natural numbers, o (omicron) the empty set of natural numbers and ε the set of all natural numbers (included zero). $\alpha\beta$ will denote the intersection of α and β, $\alpha+\beta$ the union and $\alpha\times\beta$ the cartesian product of α and β.

The cardinal number of a set α is denoted by $Nc(\alpha)$. $\alpha \simeq \beta$ stands for: α is recursively equivalent to β. A recursive equivalence type is in effect an equivalence class under the relation of recursive equivalence.

We will use the abbreviations "r.e." for "recursively enumerable", "p.i." for "partial isomorfism" which is by definition a partial recursive one-one function of one variable, and "R.E.T." for "recursive equivalence type".

P, Q, R will often denote binary relations on the set of natural numbers ε, and $\emptyset$ the empty relation. R^{-1} will denote the converse of R, defined by $yR^{-1}x \leftrightarrow xRy$. For xRy we will write sometimes $(x, y) \in R$; we will write sometimes $R(x)$ for $\{y : xRy\}$, $R(\alpha)$ for $\bigcup_{x\in\alpha} R(x)$, ϱR for $R(\varepsilon)$, δR

[1]) This result was obtained in spring 1965 during a seminar under direction of Prof. B. van Rootselaar on Recursive Equivalence Types at Amsterdam. At the Tenth Logic Colloquium (Leicester 1965) the author learned that the result was known to Prof. A. Nerode, who is able to derive the result by slightly adapting certain proofs of a paper of his, to appear in the "Mathematische Annalen". His proof, however, uses the priority method, in contrast to the present proof.

for $R^{-1}(\varepsilon)$. For $R^{-1}(R(x)) \times R(x)$ or more explicitly: $\{(s, t): (\exists u)(sRu \ \& \ \& \ xRu) \ \& \ xRt\}$ we will write sometimes R_x.

Now we define R to be "semitransitive" if $(\forall x)(\forall y)(\forall z)(\forall u)(xRy \ \& \ \& \ zRy \ \& \ xRu \to zRu)$.

We define R to be "balanced" if R is semitransitive and in addition $(\forall x)(\forall y)(xRy \to NcR^{-1}(y) = NcR(x))$. The last equality can be formulated also as $Nc\{z: zRy\} = Nc\{u: xRu\}$.

For a balanced relation R the equality $NcR(a) = NcR^{-1}R(a)$ holds and analogously $NcR^{-1}(a) = NcRR^{-1}(a)$. The proof of the first equality is as follows. Let $b \in R(a)$. Then $R^{-1}R(a) = R^{-1}(b)$ holds: for if $c \in R^{-1}R(a)$ then $aRb \ \& \ (\exists d)(cRd \ \& \ aRd)$, hence by the semitransitivity of R, cRb, i.e. $c \in R^{-1}(b)$; and if $c \in R^{-1}(b)$ then a fortiori $c \in R^{-1}R(a)$. From $R^{-1}R(a) = = R^{-1}(b)$ it follows by the balancedness of R that $NcR(a) = NcR^{-1}(b) = = NcR^{-1}R(a)$. If no $b \in R(a)$ exist then both $R(a)$ and $R^{-1}R(a)$ are empty.

The following lemma may claim some interest of its own.

Lemma. If there exists a balanced recursively enumerable relation R such that $R(\alpha) = \beta$ and $R^{-1}(\beta) = \alpha$ then α is recursively equivalent to β.

Proof. The sets R, δR and ϱR are r.e. Let $R = \{(r_1, r_2), (r_3, r_4), (r_5, r_6), \ldots\}$, $\delta R = \{a_1, a_3, a_5, \ldots\}$ and $\varrho R = \{a_2, a_4, a_6, \ldots\}$.

Remark. One may put $a_i = r_i$ for all i but that is not necessary.

Now we construct a $1-1$ r.e. subrelation $Q \subset R$ such that the associated function q, defined by $q(x) = y$ if and only if $(x, y) \in Q$ is a partial isomorfism between α and β.

We define Q by induction: $Q = \bigvee_k Q_k$, $Q_0 = \emptyset$ and $Q_k = Q_{k-1} + C_k$ where C_k is either empty or consists of one element of R.

Definition of C_k.

Case 1. k odd. If $a_k \in \delta Q_{k-1}$ then $C_k = \emptyset$.

If $a_k \notin \delta Q_{k-1}$ then $C_k = \{c_k\}$,

where $c_k =$ the first pair (a_k, y) in R (in the enumeration $(r_1, r_2), (r_3, r_4), (r_5, r_6), \ldots$) such that $y \notin \varrho Q_{k-1}$, if such a pair exists, otherwise $C_k = \emptyset$.

Case 2. k even. If $a_k \in \varrho Q_{k-1}$ then $C_k = \emptyset$.

If $a_k \notin \varrho Q_{k-1}$ then $C_k = \{c_k\}$,

where $c_k =$ the first pair (x, a_k) in R (in the enumeration) such that $x \notin \delta Q_{k-1}$, if such a pair exists, otherwise $C_k = \emptyset$.

It follows from the definition that indeed $Q = C_1 + C_2 + C_3 + \ldots \subset R$.

An easy consequence is that Q is $1-1$. For $Q_0 = \emptyset$, therefore Q_0 is a fortiori $1-1$. Suppose Q_{k-1} is $1-1$. Now $Q_k = Q_{k-1} + C_k$. If $C_k = \emptyset$ then $Q_k = Q_{k-1}$ and thus Q_k $1-1$. If $C_k = \{(x, y)\}$ then either k is odd, so $x = a_k$ and therefore $x \notin \delta Q_{k-1}$ and $y \notin \varrho Q_{k-1}$, or k is even, so $y = a_k$ and therefore as well $y \notin \varrho Q_{k-1}$ and $x \notin \delta Q_{k-1}$, from which follows that Q_k is $1-1$. As a consequence also Q is $1-1$.

Now we are going to prove for every non-zero natural number k the following statements. If $a \in \delta R$, but $a \notin \delta Q_{k-1}$ then there is an $y \notin \varrho Q_{k-1}$ such that aRy holds. And likewise, if $a \in \varrho R$ but $a \notin \varrho Q_{k-1}$ then there is an $x \notin \delta Q_{k-1}$ such that xRa holds.

Proof. Q_k is finite. Let $Q_{k-1}R_a = \{(x_1, y_1), \ldots, (x_d, y_d)\}$ with $x_1, \ldots, x_d$ all different. Since $a \notin \delta Q_{k-1}$, a is different from $x_1, \ldots, x_d$. It follows that $x_1, \ldots, x_d, a \in R^{-1}R(a)$ and all different, so $NcR^{-1}R(a) \geqslant d+1$. R is balanced so $NcR^{-1}R(a) = NcR(a)$, and therefore $NcR(a) \geqslant d+1$. Hence there is an $y \in R(a)$ which is different from $y_1, \ldots, y_d$. But then $y \notin \varrho Q_{k-1}$ and aRy. The proof of the second statement is analogous to that of the first one.

A consequence is that $\delta Q = \delta R$ and also $\varrho Q = \varrho R$. For let $a \in \delta R$. Let k be a number such that $a = a_k$. If $a \in \delta Q_{k-1}$ then a fortiori $a \in \delta Q$. If $a \notin \delta Q_{k-1}$ then we know that there is an $y \notin \varrho Q_{k-1}$ such that aRy holds. So one will find, by going along the enumeration (r_1, r_2), (r_3, r_4), (r_5, r_6), ... of R, a first element y_1 such that $y_1 \notin \varrho Q_{k-1}$ and aRy_1 holds. Therefore by the construction of Q, $(a, y_1) \in Q_k$, i.e. $a \in \delta Q_k$, hence $a \in \delta Q$. Proof of $\varrho Q = \varrho R$ analogously.

Q is r.e. For by the preceding proof if k is odd and $a_k \notin \delta Q_{k-1}$ then $C_k \neq \emptyset$ and if k is even and $a_k \notin \varrho Q_{k-1}$ then also $C_k \neq \emptyset$, so the clauses "otherwise $C_k = \emptyset$" in the definition of C_k don't actually occur, whence it follows from the construction of Q_k that Q is r.e.

Furthermore $\alpha \subset \delta Q$ and $\beta \subset \varrho Q$. The first inclusion follows from $\alpha \subset \delta R$ and $\delta R = \delta Q$, and the second one from $\beta \subset \varrho R$ and $\varrho R = \varrho Q$.

Also $Q(\alpha) \subset \beta$, since $Q(\alpha) \subset R(\alpha)$ and $R(\alpha) \subset \beta$. Likewise $Q^{-1}(\beta) \subset \alpha$.

From the last four statements it follows that $Q(\alpha) = \beta$. For let $y \in \beta$. Then $y \in \varrho Q$, so $Q^{-1}(y)$ is not empty and therefore $y \in QQ^{-1}(y)$. Hence $\beta \subset QQ^{-1}(\beta)$. Also $QQ^{-1}(\beta) \subset Q(\alpha)$. As a consequence $\beta \subset Q(\alpha)$. $Q(\alpha) \subset \beta$ is also valid and so $Q(\alpha) = \beta$.

Since Q is $1-1$ and *r.e.*, $\alpha \subset \delta Q$, $\beta \subset \varrho Q$ and $Q(\alpha) = \beta$ it follows that q is a *p.i.* between α and β.

THEOREM. Let n be a positive integer, and let A and B be recursive equivalence types. Then $nA \leqslant nB$ is equivalent to $A \leqslant B$.

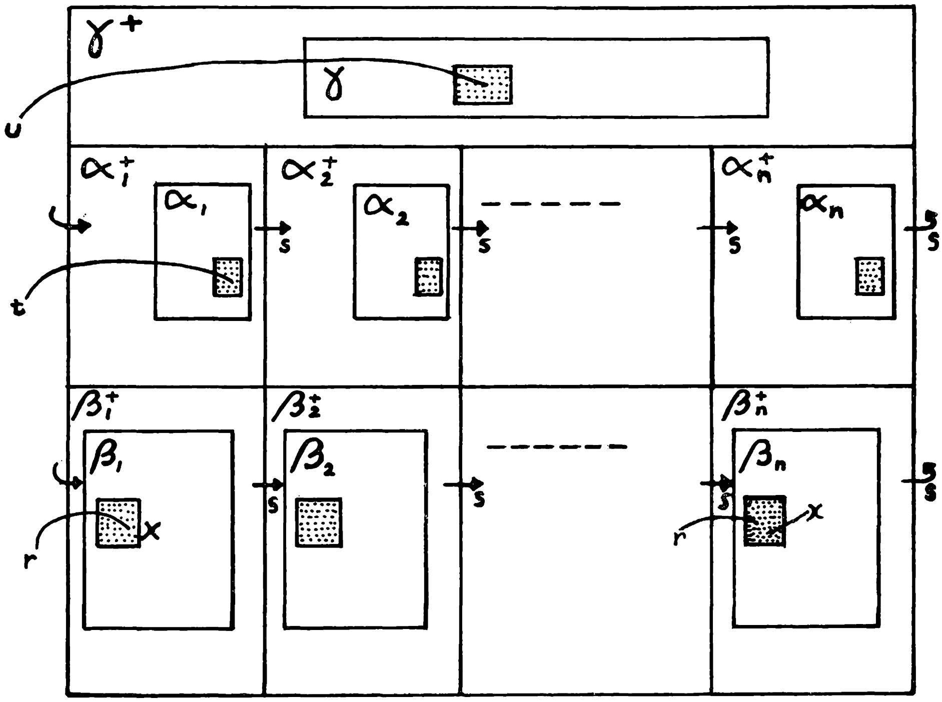
Proof. We must show: $nA + C = nB \rightarrow (\exists D) A + D = B$. Let $\xi \in A$, $\eta \in B$, $\zeta \in C$.

First we introduce some number theoretical functions f_i and some sets. For $i = 1, \ldots, 2n$ let $f_i(x) = 4nx + 2i - 1$; let $f_0(x) = 2x$. For $i = 1, \ldots, n$ let $\alpha_i = f_i(\xi)$ and $\beta_i = f_{n+i}(\eta)$; let $\gamma = f_0(\zeta)$, $\alpha = \alpha_1 + \ldots + \alpha_n$ and $\beta = \beta_1 + \ldots + \beta_n$.

Then $\alpha \in nA$, $\beta \in nB$ and $\gamma \in C$, so $\alpha + \gamma \simeq \beta$, say by p_1. All ϱf_i are disjoint and recursive, and therefore also separable. In addition all f_i are $1-1$.

Therefore one can define a function s by the following.

Let $y \in \varepsilon$. There are exactly one i and one x such that $y = f_i(x)$. If $i = 1, \ldots, n-1, n+1, \ldots$ or $2n-1$, then put $s(y) = f_{i+1}(x)$. If $i = n$ or $2n$, then put $s(y) = f_{i+1-n}(x)$. In all other cases (in fact only the case $i = 0$) $s(y)$ will be undefined.



Notice that s is a p.i. and that s^n is the identity on the set of odd numbers.

For $i = 1, \ldots, n-1$: $s(\alpha_i) = s(f_i(\xi)) = f_{i+1}(\xi) = \alpha_{i+1}$ and $s(\alpha_n) = s(f_n(\xi)) = f_1(\xi) = \alpha_1$. From this follows $s(\alpha) = \alpha$, since $s(\alpha) = s(\alpha_1 + \ldots + \alpha_n) = s(\alpha_1) + \ldots + s(\alpha_n) = \alpha_2 + \ldots + \alpha_n + \alpha_1 = \alpha$.

Analogously for β instead of α, i.e. for $i = 1, \ldots, n-1$: $s(\beta_i) = \beta_{i+1}$, $s(\beta_n) = \beta_1$ and $s(\beta) = \beta$.

Let $\delta_2 = \{x \colon s^1(x), \ldots, s^n(x) \in \delta p_1(\varrho f_1 + \ldots + \varrho f_n)\}$ and

$$\varrho = \{x \colon s^1(x), \ldots, s^n(x) \in \varrho p_1 \cdot (\varrho f_{n+1} + \ldots + \varrho f_{2n})\}.$$

Then δ_2 and ϱ are r.e. and $\alpha \subset \delta_2$, $\beta \subset \varrho$.

E.g. $\alpha \subset \delta_2$ can be verified as follows. Let $x \in \alpha$. Then $s(x) \in \alpha$, $s^2(x) \in \alpha, \ldots, s^n(x) \in \alpha$. Combined with $\alpha \subset \delta p_1 \cdot (\varrho f_1 + \ldots + \varrho f_n)$ one obtains $x \in \delta_2$.

We introduce one more function p and some more sets. Let $p = p_1 \cdot [(\delta_2 + \varrho f_0) \times \varrho]$ i.e. p is the restriction of p_1 to the set $\delta_2 + \varrho f_0$ in the domain and to the set ϱ in the range. Then p is, like p_1, a p.i. between α and β. Let for $i = 1, \ldots, n$: $\alpha_i^+ = \varrho f_i \cdot \delta p$ and $\beta_i^+ = \varrho f_{n+i} \cdot \varrho p$. Let $\gamma^+ = \varrho f_0 \cdot \delta p$. Let $\alpha^+ = \alpha_1^+ + \ldots + \alpha_n^+$, $\beta^+ = \beta_1^+ + \ldots + \beta_n^+$, $\delta = \alpha + \beta + \gamma$ and $\delta^+ = \alpha^+ + \beta^+ + \gamma^+$.

Then $\alpha_1^+, \ldots, \alpha_n^+, \beta_1^+, \ldots, \beta_n^+, \gamma^+$ are r.e. and mutually disjoint (and therefore separable). Furthermore $\alpha_i \subset \alpha_i^+$, $\beta_i \subset \beta_i^+$, $\gamma \subset \gamma^+$, and hence also $\alpha \subset \alpha^+$, $\beta \subset \beta^+$ and $\delta \subset \delta^+$; $\alpha_i \subset \alpha_i^+$ holds because $\alpha_i \subset \alpha \subset \delta_2 \cdot \delta p_1 \subset \delta p$ and $\alpha_i \subset \varrho f_i$ so $\alpha_i \subset \delta p \cdot \varrho f_i$, which by definition equals α_i^+; $\beta_i \subset \beta_i^+$ holds likewise because $\beta_i \subset \beta \subset \varrho \cdot \varrho p_1 = \varrho p$ and $\beta_i \subset \varrho f_{n+i}$ so $\beta_i \subset \varrho p \cdot \varrho f_{n+i} = \beta_i^+$ and $\gamma \subset \gamma^+$ holds because $\gamma \subset \varrho f_0 \cdot \delta p = \gamma^+$.

Also hold $\delta p = \alpha^+ + \gamma^+$ and $\varrho p = \beta^+$, since $\delta p = (\varrho f_1 + \ldots + \varrho f_n + \varrho f_0) \cdot \delta p = (\varrho f_1 + \ldots + \varrho f_n) \cdot \delta p + \varrho f_0 \cdot \delta p = \alpha^+ + \gamma^+$ and $\varrho p = (\varrho f_{n+1} + \ldots + \varrho f_{2n}) \cdot \varrho p = \beta^+$.

We have also the following equalities: for $i = 1, \ldots, n-1$ is $s(\alpha_i^+) = \alpha_{i+1}^+$ and $s(\beta_i^+) = \beta_{i+1}^+$, $s(\alpha_n^+) = \alpha_1$, $s(\beta_n^+) = \beta_1$, $s(\alpha) = \alpha$ and $s(\beta^+) = \beta$. For example one may verify the first equality as follows.

$\alpha_i^+ = \varrho f_i \cdot \delta p = \varrho f_i \cdot \delta p_1(\delta_2 + \varrho f_0) = \varrho f_i \cdot \delta p_1 \delta_2 = \varrho f_i \cdot \delta_2$ and in the same way $\alpha_{i+1}^+ = \varrho f_{i+1} \cdot \delta_2$. Suppose $x \in \alpha_i^+$. Then $x \in \delta_2$ so for all k, $s^{k+1}(x) \in \delta p_1 \cdot (\varrho f_1 + \ldots + \varrho f_n)$, so $s(x) \in \delta_2$. And $s(x) \in \varrho f_{i+1}$, so $s(x) \in \delta_2 \cdot \varrho f_{i+1} = \alpha_{i+1}^+$.

For $i = 1, \ldots, n-1$: s is a p.i. between α_i and α_{i+1}, s is a p.i. between α_n and α_1 and s is a p.i. between α and α. The same for β, α^+ and β^+ instead of α.

Summarizing we have obtained by the above "cleaning" $2n+1$ mutually disjoint *r.e.* sets $\alpha_1^+, \ldots, \alpha_n^+$, $\beta_1^+, \ldots, \beta_n^+$ and γ^+, and $2n+1$ sets $\alpha_1, \ldots, \alpha_n \in A$, $\beta_1, \ldots, \beta_n \in B$ and $\gamma \in C$ such that $\alpha \subset \alpha_1^+, \ldots, \alpha_n \subset \alpha_n^+$, $\beta_1 \subset \beta_1^+, \ldots, \beta_n \subset \beta_n^+$ and $\gamma \subset \gamma^+$ and two p.i.'s p and s ("shift") such that, with the six notations $\alpha = \alpha_1 + \ldots + \alpha_n$, $\beta = \beta_1 + \ldots + \beta_n$, $\alpha^+ = \alpha_1^+ + \ldots + \alpha_n^+$, $\beta^+ = \beta_1^+ + \ldots + \beta_n^+$, $\delta = \alpha + \beta + \gamma$ and $\delta^+ = \alpha^+ + \beta^+ + \gamma^+$: $\delta p = \alpha^+ + \gamma^+$, $\varrho p = \beta^+$, $p(\alpha + \gamma) = \beta$, $\delta s = \varrho s = \alpha^+ + \beta^+$, $s(\alpha_1) = \alpha_2, \ldots, s(\alpha_{n-1}) = \alpha_n$, $s(\alpha_n) = \alpha_1$, $s(\beta_1) = \beta_2, \ldots, s(\beta_{n-1}) = \beta_n$, $s(\beta_n) = \beta_1$ and $s^n =$ the identity on $\alpha^+ + \beta^+$.

Now we want to obtain a $\gamma_1 \in D$ with $\gamma_1 | \alpha_1$ (i.e. γ_1 "separated" from α_1) and a p.i. between $\alpha_1 + \gamma_1$ and β_1 (it is clear that p does not satisfy, for in general $p(\alpha_1) \not\subset \beta_1^+$; also does not satisfy $s^k p$ where k is always chosen so large that $s^k p(x) \in \beta_1^+$, for in general this function will not be one-one). By the lemma it is sufficient to construct a balanced r.e. relation between $\alpha_1 + \gamma_1$ and β_1.

We begin with the introduction of some notations.

Let $x \in \delta^+$. Then put $\nu(x) = \{p^k s^i(x) : i = 0, \ldots, n-1; k = -1, 0, 1\}$ ("the first half round of the unity of x"). Then $x \in \nu(x)$.

If $\sigma \subset \delta^+$ then put $\nu(\sigma) = \cup_{z \in \sigma} \nu(z)$. Then $\sigma \subset \nu(\sigma)$. Put $\nu^m(x) = \nu(\nu^{m-1}(x))$.

For $x \in \delta^+$ put $\pi(x) = \nu(x) + \nu^2(x) + \nu^3(x) + \ldots$ ("the unity of x").

The set $\pi(x)$ is r.e. and consists of all elements which can be obtained from x by a finite number of applications of p, p^{-1} and s. We will call σ "closed" (under p, p^{-1} and s) iff $\nu(\sigma) = \sigma$. If σ is closed then also $\pi(\sigma) = \sigma$ holds.

The binary relation on δ^+, defined by $y \in \pi(x)$, is an equivalence relation. More precisely: if $y \in \pi(x)$ then $x \in \pi(y)$, if $z \in \pi(y)$ and $y \in \pi(x)$ then $z \in \pi(x)$ and if $\pi(x) \cdot \pi(y) \neq o$ then $\pi(x) = \pi(y)$.

For $x \in \delta^+$, $\pi(x)$ is closed under p, p^{-1} and s. Also δ is closed under p, p^{-1} and s, so $\pi(\delta)=\delta$.

Let $x \in \delta^+$. We define $\pi_3(x)$ ("the additional unity of x") by the following process.

Calculate $\nu(x)$, $\nu^2(x)$, ... Stop the calculation as soon as $\nu^m(x)=\nu^{m-1}(x)$. If the calculation never stops then put $\pi_3(x)=o$. This occurs when $\pi(x)$ is infinite. If the above calculation stops, then order the obtained elements of $\pi(x)\cdot\gamma^+$ according to their magnitude: $c_1<c_2<\ldots<c_u$, and put $\pi_3(x)= =\{c_n, c_{2n}, \ldots, c_{vn}\}$, where $v=[u/n]$, i.e. v is the largest integer smaller than or equal to u/n.

Then $\pi_3(x)$ is r.e.

If $x_1 \in \pi(x)$ then $\pi_3(x_1)=\pi_3(x)$, i.e. $\pi_3(x_1)$ is independent of x_1 as long as $x_1 \in \pi(x)$.

Put $\gamma_1{}^+=\cup_x\pi_3(x)$, $\gamma_1=\gamma\gamma_1{}^+$, $\delta_1=\alpha_1+\beta_1+\gamma_1$ and $\delta_1{}^+=\alpha_1{}^++\beta_1{}^++\gamma_1{}^+$.

Then also $\gamma_1{}^+$ is r.e.

We now define a relation R by $xRy \leftrightarrow y \in \pi(x)$ & $x \in \alpha_1{}^++\gamma_1{}^+$ & $y \in \beta_1{}^+$. It will turn out that this R is a balanced r.e. relation between $\alpha_1+\gamma_1$ and β_1.

First we state and prove a crucial property.

Property 1. If $\pi(x)\beta_1{}^+$ is infinite then also $\pi(x)\alpha^+$ is infinite.

Proof. Let $x \in \beta^+$ and suppose in addition that $\nu(x)\alpha^+=o$. Then holds $\nu\nu(x)=\nu(x)$. For let $y \in \nu\nu(x)$. Then $y=p^ks^ip^ls^j(x)$ for certain integers i, j, k and l which satisfy $0\leqslant i\leqslant n-1$, $0\leqslant j\leqslant n-1$, $-1\leqslant k\leqslant 1$ and $-1\leqslant l\leqslant 1$. Since $x \in \beta^+ \subset \varrho R$ l must be 0 or -1. If $l=0$ then $y=p^ks^{i+j}(x)$, so $y \in \nu(x)$. Suppose next $l=-1$. By assumption $\nu(x)\alpha^+=o$, so $p^{-1}s^j(x) \in \gamma^+$. Since s is not defined on γ^+, i must be zero, whence $y=p^kp^{-1}s^j(x)=p^{k-1}s^j(x)$. Since $p^{-1}s^j(x) \in \delta R$, k must be 0 or 1, so $k-1=-1$ or $k-1=0$ hence $y \in \nu(x)$.

If $x \in \beta_1{}^+$ and $\pi(x)$ is infinite then $\nu(x)\alpha^+\neq o$. For suppose $\nu(x)\alpha^+=o$. Then $\nu\nu(x)=\nu(x)$ so $\pi(x)=\nu(x)$ so that $\pi(x)$ is finite.

If x and y are different members of $\beta_1{}^+$ then $\nu(x)\alpha^+$ and $\nu(y)\alpha^+$ are disjoint. For suppose $z \in \nu(x)\alpha^+$ and $z \in \nu(y)\alpha^+$ i.e. $z=p^{-1}s^k(x)=p^{-1}s^l(y)$ for certain integers k and l. Then $y=s^{k-1}(x) \in s^{k-1}(\beta_1{}^+)=\beta^+_{1+k-1}$. Also $y \in \beta_1{}^+$ so $1+k-l=1$ hence $k=l$ and $y=x$.

Now suppose $\pi(x)\beta_1{}^+$ is infinite. Let $\pi(x)\beta_1{}^+=\{x_1, x_2, \ldots\}$. Then $\nu(x_1)\alpha^++ +\nu(x_2)\alpha^++\ldots \subset \pi(x)\alpha^+$. By the above considerations we know that $\nu(x_i)\alpha^+$ and $\nu(x_j)\alpha^+$ are nonempty and disjoint for all i and j such that $x_i\neq x_j$, in other words $\nu(x_1)\alpha^++\nu(x_2)\alpha^++\ldots$ is infinite and therefore also $\pi(x)\alpha^+$ is infinite, which was to be proved.

Next we state some equalities between the cardinals of our sets. $Nc\pi(x)\alpha_1{}^+=Nc\pi(x)\alpha_k{}^+$ since $\pi(x)\alpha_1{}^+\simeq\pi(x)\alpha_k{}^+$ by s^{k-1}. Likewise $Nc\pi(x)\beta_1{}^+=Nc\pi(x)\beta_k{}^+$. $Nc\pi(x)(\alpha^++\gamma^+)=Nc\pi(x)\beta^+$ since $\pi(x)(\alpha^++\gamma^+)\simeq \simeq\pi(x)\beta^+$ by p. $Nc\pi(x)\alpha^+=nNc\pi(x)\alpha_1{}^+$ since both members equal

$Nc\pi(x)(\alpha_1^+ + \ldots + \alpha_n^+)$. Likewise $Nc\pi(x)\beta^+ = nNc\pi(x)\beta_1^+$. From the definition of $\pi_3(x)$ it follows that if we put $u = Nc\pi(x)\gamma^+$ and $v = Nc\pi(x)\gamma_1^+$ then either $\pi(x)$ is finite and $v = [u/n]$ or $\pi(x)$ is infinite and $v = 0$. Less trivial is the following statement.

Property 2. $Nc\pi(x)(\alpha_1^+ + \gamma_1^+) = Nc\pi(x)\beta_1^+$.

Proof. Suppose first that $\pi(x)$ is finite. Put $t = Nc\pi(x)\alpha_1^+$, $r = Nc\pi(x)\beta_1^+$ and $u = Nc\pi(x)\gamma^+$ and $v = Nc\pi(x)\gamma_1^+$. Then $v = [u/n]$.

With the above notations we can write down: $Nc\pi(x)\alpha^+ = Nc\pi(x)(\alpha_1^+ + \ldots + \alpha_n^+) = nt$, $Nc\pi(x)\beta^+ = Nc\pi(x)(\beta_1^+ + \ldots + \beta_n^+) = nr$ and $Nc\pi(x)(\alpha^+ + \gamma^+) = nt + u$.

From $Nc\pi(x)(\alpha^+ + \gamma^+) = Nc\pi(x)\beta^+$ it follows that $nt + u = nr$, so $u = n(r-t)$ so $v = r - t$. Hence $Nc\pi(x)(\alpha_1^+ + \gamma_1^+) = t + v = t + (r-t) = r = Nc\pi(x)\beta_1^+$.

Suppose next that $\pi(x)$ is infinite. Then also $\pi(x)\beta^+$ is infinite, and therefore $\pi(x)\beta_1^+$ infinite so by property 1 $\pi(x)\alpha^+$ infinite, therefore also $\pi(x)\alpha_1^+$ infinite so a fortiori $\pi(x)(\alpha_1^+ + \gamma_1^+)$ infinite, which was to be proved.

From the presented definition of R and the fact that $\pi(x)$ is *r.e.* for all x it follows that R is also r.e.

The relation R is semitransitive. For suppose xRy, uRy and xRv hold. Then $u \in \alpha_1^+ + \gamma_1^+$, $v \in \beta_1^+$, $y \in \pi(u)$, $v \in \pi(x)$ and $y \in \pi(x)$. By combination one obtains: $v \in \pi(u)$ and hence uRv.

R is also balanced. For suppose xRy holds, so $x \in \alpha_1^+ + \gamma_1^+$, $y \in \beta_1^+$ and $y \in \pi(x)$; also $\pi(y) = \pi(x)$. Then $R(x) = \pi(x)\beta_1^+$ and $R^{-1}(y) = \pi(x)(\alpha_1^+ + \gamma_1^+)$ which are equal by property 2.

Furthermore $\delta R = \alpha_1^+ + \gamma_1^+$. For suppose $x \in \alpha_1^+ + \gamma_1^+$. Then (since $x \in \pi(x)$) $Nc\pi(x)(\alpha_1^+ + \gamma_1^+) \geqslant 1$ so also $Nc\pi(x)\beta_1^+ \geqslant 1$ i.e. there is an $y \in \pi(x)\beta_1^+$. But then xRy holds, so $x \in \delta R$.

In the same way $\varrho R = \beta_1^+$.

Also $R(\alpha_1 + \gamma_1) \subset \beta_1$. For suppose $y \in R(\alpha_1 + \gamma_1)$ i.e. there is an x such that $x \in \alpha_1 + \gamma_1 \subset \delta$ and xRy. Then $y \in \pi(x) \subset \delta$. Hence $y \in \delta \cdot \varrho R = \delta\beta_1^+ = \beta_1$.

Likewise $R(\beta_1) \subset \alpha_1 + \gamma_1$.

From $\beta_1 \subset \varrho R$ and $R^{-1}(\beta_1) \subset \alpha_1 + \gamma_1$ it follows that $\beta_1 \subset R(R^{-1}(\beta_1)) \subset R(\alpha_1 + \gamma_1)$. Likewise it follows from $\alpha_1 + \gamma_1 \subset \delta R$ and $R(\alpha_1 + \gamma_1) \subset \beta_1$ that $\alpha_1 + \gamma_1 \subset R(\beta_1)$.

By combination one obtains the equalities $R(\alpha_1 + \gamma_1) = \beta_1$ and $R(\beta_1) = \alpha_1 + \gamma_1$. Since in addition we know that R is r.e. and balanced, we can apply the Lemma, and obtain $\alpha_1 + \gamma_1 \simeq \beta_1$. Because $\gamma_1 \subset \gamma$ and $\gamma|\alpha$ (i.e. γ separable from α) it follows that $A + \text{Req}\gamma_1 = B$ ($\text{Req}\gamma_1$ stands for the R.E.T. of γ_1).

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