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The Gauss–Green theorem and removable sets for PDEs in divergence form

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Abstract

Applying a very general Gauss–Green theorem established for the generalized Riemann integral, we obtain simple proofs of new results about removable sets of singularities for the Laplace and minimal surface equations. We treat simultaneously singularities with respect to differentiability and continuity.

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0. Introduction

In an open set $U \subset \mathbb{R}^m$ we consider the second order partial differential equation

$$\operatorname{div}(h \circ \nabla u)(x) = f[x, u(x)], \qquad (*)$$

where $h : \mathbb{R}^m \to \mathbb{R}^m$ and $f : U \times \mathbb{R}^m \to \mathbb{R}$ are given. Note that depending on the maps h and f, Eq. $(*)$ need not be linear; e.g., the minimal surface equation (4.2) below. A set

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$E \subset U$ is called *removable* if u satisfies $(*)$ in U whenever it satisfies $(*)$ in $U - E$. The goal is to find a useful class of removable sets.

While this is a classical problem studied by many mathematicians, e.g., [3,9–11,20] to mention a few, it appears that our approach has not been used previously. Applying a very general Gauss–Green theorem established for the *generalized Riemann integral* [17], we obtain simple proofs of new results about removable sets for the *Laplace* and *minimal surface equations*. We deal simultaneously with removable sets where the solution u of $(*)$, or $h \circ \nabla u$, or both, lack differentiability, denoted by E_{d}, as well as those where they lack continuity, denoted by E_{c}. Typically, we assume the Hausdorff measure $\mathscr{H}^{m-1}$ of E_{d} is σ-finite, and that of E_{c} is zero. Notwithstanding, we also study the case where the size of removable sets is measured by the *integral-geometric measure* $\mathscr{I}_1^{m-1}$. In the main result (Theorem 4.2 below), no topological restrictions are imposed on removable sets.

We illustrate our idea by proving a known fact for the Laplace equation $\Delta u = 0$, i.e., for Eq. $(*)$ where h is the identity map and $f = 0$. Let E be a relatively closed subset of U whose Hausdorff measure $\mathscr{H}^{m-1}$ is σ-finite. Assuming a function $u \in C^1(U)$ is harmonic in $U - E$, we wish to prove that it is harmonic in U. Since each weak solution (in the sense of distributions) of the Laplace equation is harmonic [18, Corollary of Theorem 8.12], it suffices to show $\int_U u\Delta\varphi = 0$ for every $\varphi \in C_c^\infty(U)$. In view of our assumptions, this is true whenever the following integrations by parts are valid:

$$\int_U u\Delta\varphi = -\int_U \nabla u \cdot \nabla\varphi = \int_U \varphi\Delta u.$$

While the first equality is standard, the second depends on the Gauss–Green theorem for the vector field $\varphi\nabla u$ that is continuous in U but differentiable only in $U - E$. Since such a theorem holds [17, Proposition 5.1.2, Corollary 5.1.13], the removability of E follows.

In Theorem 4.1 below, we show that the same technique provides a simple proof of a slightly improved classical result, due to Besicovitch [2], about removable sets for holomorphic functions.

As all functions involved in the previous example, as well as in the proof of Besicovitch's theorem, are Lebesgue integrable, the above mentioned generalized Riemann integral, called the *R-integral* in [17, Chapter 5], is used only indirectly: we merely apply the Gauss–Green theorem established for the R-integral to the Lebesgue integral. However, the Lebesgue integral cannot be used when $f \neq 0$ and it is not a priori clear, or actually not true, that the function $x \mapsto f[x, u(x)]$ belongs to $L^1_{\mathrm{loc}}(U)$; cf. Theorem 4.2 below.

To deal simultaneously with the sets E_{d} and E_{c}, we need a Gauss–Green theorem for discontinuous vector fields, which has not been available previously even in the context of Lebesgue integration—cf. [1,19]. The necessary result is obtained in Section 3 by extending the R-integral to a larger class of integrable functions; the definition and basic properties of the R-integral are stated in this section without proofs. The R-integral, its extension, and the associated Gauss–Green theorem depend on the concepts of *charge* and *weak charge*. These are linear functionals, continuous with respect to suitable topologies, on the linear space of all bounded BV functions with compact support. Under the name of *continuous additive functions*, charges were defined in [16, Section 4]. Their properties are summarized in Section 2; the details can be found in [17, Chapter 2]. In addition, Section 2 contains the main results concerning weak charges, which are more restrictive than *bounded additive functions* introduced in [16, Definition 10.1]. Applications to removable sets for Eq. $(*)$ are given in Section 4.

1. The setting

The ambient space of this paper is $\mathbb{R}^m$ where $m \geqslant 1$ is a fixed integer. In $\mathbb{R}^m$ we shall use exclusively the Euclidean norm $|\cdot|$ induced by the usual inner product $x \cdot y$. The diameter of a set $E \subset \mathbb{R}^m$ is denoted by $d(E)$. We denote by $B(x,r)$ and $B[x,r]$, respectively, the open and closed ball of radius $r > 0$ centered at $x \in \mathbb{R}^m$. The origin of $\mathbb{R}^m$ is denoted by 0, and we write $B(r)$ and $B[r]$ instead of $B(0,r)$ and $B[0,r]$, respectively.

By a measure we always mean an *outer measure*. Lebesgue measure in $\mathbb{R}^m$ is denoted by $\mathscr{L}^m$; however, for $E \subset \mathbb{R}^m$, we usually write $|E|$ instead of $\mathscr{L}^m(E)$. If $0 \leqslant s \leqslant m$, we denote by $\mathscr{H}^s$ the s-dimensional Hausdorff measure in $\mathbb{R}^m$. In addition, we shall use the *integral-geometric measure* $\mathscr{I}_1^{m-1}$ defined in [7, Section 2.10.5]. Unless specified otherwise, the words "measure," "measurable," and "negligible" as well as the expressions "almost all" and "almost everywhere" always refer to Lebesgue measure $\mathscr{L}^m$; similarly, the symbols $\int_E f$ and $L^p(E)$ refer to $\mathscr{L}^m$.

Let $E \subset \mathbb{R}^m$. We denote by $\operatorname{cl} E, \operatorname{int} E$, and ∂E the closure, interior, and boundary of E, respectively. If E is measurable, we denote by $\operatorname{cl}_* E$ and $\partial_* E$, the essential closure and essential boundary of E, respectively. A measurable set E is called *essentially closed* whenever $\operatorname{cl}_* E$ is closed.

Let $U \subset \mathbb{R}^m$ be an open set. The collections of all BV subsets of U and all locally BV subsets of U are denoted by $\boldsymbol{BV}(U)$ and $\boldsymbol{BV}_{\mathrm{loc}}(U)$, respectively. We denote by $\boldsymbol{BV}_c(U)$ the collection of all bounded BV subsets of U whose closure is also contained in U. In the absence of additional attributes, a BV set or a locally BV set is always a BV subset or a locally BV subset of $\mathbb{R}^m$, respectively. We write, respectively, $\boldsymbol{BV}, \boldsymbol{BV}_{\mathrm{loc}}$, and $\boldsymbol{BV}_c$ instead of $\boldsymbol{BV}(\mathbb{R}^m), \boldsymbol{BV}_{\mathrm{loc}}(\mathbb{R}^m)$, and $\boldsymbol{BV}_c(\mathbb{R}^m)$. The perimeter and unit exterior normal of a BV set A are denoted by $\|A\|$ and ν_A, respectively. The *regularity* of a BV set A is the number

$$r(A) := \begin{cases} \frac{|A|}{d(A)\|A\|} & \text{if } |A| > 0, \\ 0 & \text{if } |A| = 0. \end{cases}$$

Throughout, by a function we mean a real-valued function. When considered individually, functions are generally not identified with the equivalence classes they determine. On the other hand, by spaces of functions, we usually mean the spaces of the equivalence classes determined by these functions. Often we denote by f both a function defined on a set A and its restriction $f \restriction B$ to $B \subset A$.

Let $U \subset \mathbb{R}^m$ be an open set. We denote by $BV(U)$ the family of all BV functions in U, and give the symbols $BV_{\text{loc}}(U)$ and $BV_c(U)$ the obvious meaning. We let

$$BV^\infty(U) := BV(U) \cap L^\infty(U),$$

and define $BV^\infty_{\text{loc}}(U)$ and $BV^\infty_c(U)$ similarly. We write BV instead of $BV(\mathbb{R}^m)$, and use the same convention for the other spaces introduced in this paragraph. If $g \in BV_{\text{loc}}(U)$, we denote by Dg the distributional gradient of g, and by $||Dg||$ the variational measure of g. We let $||g|| := ||Dg||(U)$.

Observation 1.1. *If $g \in BV_c(\mathbb{R})$, then $|g|_\infty \leqslant ||g||$.*

Proof. If $\{g \neq 0\} \subset (a,b)$ and g^* is the precise value of g, denote by $V(g^*)$ the classical variation of g^* in the interval $[a,b]$. In view of [7, Theorem 4.5.9, (23)],

$$|g|_\infty = |g^*|_\infty \leqslant \sup_{t \in [a,b]} |g^*(t)| \leqslant V(g^*) \leqslant ||g||. \qquad \square$$

Proposition 1.2. *If E is a bounded measurable subset of $\mathbb{R}^m$, then $||\cdot||$ is a Banach norm in the linear space*

$$BV_E := \{g \in BV : \{g \neq 0\} \subset E\}.$$

Proof. Clearly $||\cdot||$ is a norm in BV_E. If $\{g_i\}$ is a Cauchy sequence in $(BV_E, ||\cdot||)$, then it is a Cauchy sequence in $L^1(\mathbb{R}^m)$. Indeed, this follows from Observation 1.1 if $m = 1$, and from the Hölder and Sobolev inequalities if $m \geqslant 2$. Thus $\{g_i\}$ converges to a $g \in L^1(\mathbb{R}^m)$, and we may assume $\{g \neq 0\} \subset E$. As $\{||g_i||\}$ is a Cauchy sequence of real numbers, $||g|| \leqslant \lim ||g_i|| < \infty$. Consequently $g \in BV_E$.

Now given $\varepsilon > 0$, there is an integer $k \geqslant 1$ such that $||g_i - g_j|| < \varepsilon$ for all $i, j \geqslant k$. If $i \geqslant k$ is an integer, then the sequence $\{g_i - g_j\}_j$ converges to $g_i - g$ in $L^1(\mathbb{R}^m)$. Therefore

$$||g_i - g|| \leqslant \liminf_j ||g_i - g_j|| \leqslant \varepsilon,$$

and the proposition follows. $\square$

Let $E \subset C \subset \mathbb{R}^m$ and let $v : C \to \mathbb{R}^n$. We say v is *pointwise Lipschitz* in E if given $x \in E$, we can find $c_x > 0$ and $\delta_x > 0$ so that

$$|v(x) - v(y)| \leqslant c_x |x - y|$$

for all $y \in C \cap B(x, \delta_x)$. Recall v is called *continuous* in E if it is continuous at each $x \in E$. An easy variant of Whitney's extension theorem [21, Chapter 6, Section 2] yields the following:

If C is a closed set and v is continuous or pointwise Lipschitz in E, then v has an extension $w : \mathbb{R}^m \to \mathbb{R}^n$ that is C^∞ in $\mathbb{R}^m - C$ and continuous or pointwise Lipschitz in E, respectively.

In particular, if v is pointwise Lipschitz in E, then by Stepanoff's theorem, w is differentiable at almost all $x \in E$; moreover, for almost all $x \in E$, the derivative $Dw(x)$ depends only on v and not on the extension w [17, Lemma 1.6.3].

2. Charges

For $n = 1, 2, \ldots$, we topologize the convex set

$$BV_n := \{g \in BV_c^\infty : \ \operatorname{supp} g \subset B[n] \text{ and } \|g\| + |g|_\infty \leqslant n + 1\}$$

by two different metrics:

$$\tau : (f, g) \mapsto |f - g|_1 \quad \text{and} \quad \varpi : (f, g) \mapsto \|f - g\|.$$

The space (BV_n, τ) is compact by [5, Theorem 4, Section 5.2.3], and it follows from Proposition 1.2 that the space (BV_n, ϖ) is complete. In BV_c^∞ we consider the largest topology $\mathcal{T}$ for which all inclusion maps

$$(BV_n, \tau) \hookrightarrow (BV_c^\infty, \mathcal{T})$$

are continuous, and the largest topology $\mathcal{W}$ for which all inclusion maps

$$(BV_n, \varpi) \hookrightarrow (BV_c^\infty, \mathcal{W})$$

are continuous. Both topologies $\mathcal{T}$ and $\mathcal{W}$ are Hausdorff, sequential and sequentially complete, but not metrizable. Moreover, the topology $\mathcal{T}$ is locally convex; whether the same is true about the larger topology $\mathcal{W}$ is unclear. Identifying each set B in $\boldsymbol{BV}_c$ with its indicator $\chi_A \in BV_c^\infty$, we view $\boldsymbol{BV}_c$ as a closed subspace of $(BV_c^\infty, \mathcal{T})$ and $(BV_c^\infty, \mathcal{W})$.

Let $\{g_i\}$ be a sequence in BV_c^∞ and $g \in BV_c^\infty$. We write $\{g_i\} \to g$ or $\{g_i\} \rightrightarrows g$ according to whether $\{g_i\}$ $\mathcal{T}$-converges or $\mathcal{W}$-converges to g, respectively. Observe

- $\{g_i\} \to g$ if and only if each g_i vanishes outside a fixed compact set $K \subset \mathbb{R}^m$, and

$$\sup(\|g_i\| + |g_i|_\infty) < \infty \quad \text{and} \quad \lim |g_i - g|_1 = 0;$$

- $\{g_i\} \rightrightarrows g$ if and only if each g_i vanishes outside a fixed compact set $K \subset \mathbb{R}^m$, and

$$\sup |g_i|_\infty < \infty \quad \text{and} \quad \lim ||g_i - g|| = 0.$$

Using Observation 1.1 if $m = 1$, and the Hölder and Sobolev inequalities if $m \geqslant 2$, it is easy to show that $\{g_i\} \rightrightarrows g$ implies $\{g_i\} \to g$. For a sequence $\{B_i\}$ in $\boldsymbol{BV}_c$ and a set $B \in \boldsymbol{BV}_c$, the meaning of the symbols $\{B_i\} \to B$ and $\{B_i\} \rightrightarrows B$ is obvious.

Note $\{g_i\} \to g$ and $\{B_i\} \to B$ can be defined for $g \in L^1(\mathbb{R}^m)$ and a bounded measurable set $B \subset \mathbb{R}^m$. However, in this case [5, Section 5.2.1] implies $g \in BV_c^\infty$ and $B \in \boldsymbol{BV}$.

Definition 2.1. A linear functional $F : BV_c^\infty \to \mathbb{R}$ is called a *charge* or a *weak charge* (abbreviated as *w-charge*) according to whether F is $\mathscr{T}$-continuous of $\mathscr{W}$-continuous, respectively.

It is easy to see a linear functional $F : BV_c^\infty \to \mathbb{R}$ is, respectively, a charge or w-charge whenever $\lim \langle F, g_i \rangle = 0$ for each sequence $\{g_i\}$ in BV_c^∞ for which $\{g_i\} \to 0$ or $\{g_i\} \rightrightarrows 0$.

Remark 2.2. It follows from [17, Section 4.1] that a charge F is uniquely determined by the restriction $F \restriction \boldsymbol{BV}_c$, and that each additive $\mathscr{T}$-continuous function F on $\boldsymbol{BV}_c$ defines a charge by the formula

$$\langle F, g \rangle := \int_0^\infty F(\{g^+ > t\})\, dt - \int_0^\infty F(\{g^- > t\})\, dt$$

for each $g \in BV_c^\infty$. In general, neither is true for w-charges.

Clearly, each charge is a w-charge, and the next example shows the converse is false.

Example 2.3 (The flux of a vector field). If $v : \mathbb{R}^m \to \mathbb{R}^m$ is a locally bounded Borel vector field, we define a linear functional F_v on BV_c^∞ by the formula

$$\langle F_v, g \rangle := \int_{\mathbb{R}^m} v \cdot d(Dg)$$

for each $g \in BV_c^\infty$. If $|v(x)| \leqslant \theta$ for every $x \in \operatorname{supp} g$, then

$$|\langle F_v, g \rangle| \leqslant \theta ||Dg||(\mathbb{R}^m) = \theta ||g||,$$

and it follows F_v is a w-charge. Since

$$\langle F_v, \chi_B \rangle = \int_{\partial_* B} v \cdot \nu_B \, d\mathscr{H}^{m-1},$$

for each bounded BV set B, we call F_v the *flux* of v. Choosing a vector field v with a suitable discontinuity, it is easy to see that the flux of v need not be a charge [17, Example 2.1.11].

Proposition 2.4. *If F is a linear functional on BV_c^∞, then*

- *F is a charge if and only if given $\varepsilon > 0$, there is a $\theta > 0$ such that*

$$|\langle F, g \rangle| < \theta |g|_1 + \varepsilon(||g|| + |g|_\infty)$$

for each $g \in BV_c^\infty$ with $\{g \neq 0\} \subset B(1/\varepsilon)$;

- *F is a w-charge if and only if given $\varepsilon > 0$, there is a $\theta > 0$ such that*

$$|\langle F, g \rangle| < \theta ||g|| + \varepsilon |g|_\infty$$

for each $g \in BV_c^\infty$ with $\{g \neq 0\} \subset B(1/\varepsilon)$.

Proof. The statement about charges follows immediately from Remark 2.2 and [17, Proposition 2.2.6].

As the converse is obvious, suppose F is a w-charge, and choose an $\varepsilon > 0$. Observe there is an $\eta > 0$ such that $|\langle F, g \rangle| < \varepsilon/2$ for each $g \in BV_c^\infty$ with $||g|| < \eta$, $|g|_\infty < 1$, and $\{g \neq 0\} \subset B(1/\varepsilon)$. Let $\theta := \varepsilon/(2\eta)$ and select a $g \in BV_c^\infty$ for which $\{g \neq 0\} \subset B(1/\varepsilon)$. With no loss of generality, we may assume $g \geqslant 0$.

Let p and q be the smallest positive integers for which $||g||/p < \eta$ and $|g|_\infty / q < 1$. Note $p \leqslant ||g||/\eta + 1$ and $q \leqslant |g|_\infty + 1$. Since

$$s \mapsto \int_0^s ||\{g > t\}||\, dt : [0, |g|_\infty] \to [0, ||g||]$$

is an increasing continuous function, the coarea theorem implies there are $0 = a_0 < \cdots < a_p = |g|_\infty$ such that

$$\int_{a_{i-1}}^{a_i} ||\{g > t\}||\, dt = \frac{1}{p} ||g|| < \eta, \quad i = 1, \ldots, p.$$

For $i = 0, \ldots, q$, let $b_i = (i/q)|g|_\infty$, and order the set $\{a_0, \ldots, a_p;\ b_0, \ldots, b_q\}$ into a sequence $0 = c_0 < \cdots < c_r = |g|_\infty$. Clearly $r \leqslant p + q - 1$. Now it is easy to verify

$$g_i := \max\{\min\{g, c_i\}, c_{i-1}\} - c_{i-1}, \quad i = 1, \ldots, r,$$

are BV functions with $|g_i|_\infty < 1$, $\{g_i \neq 0\} \subset B(1/\varepsilon)$, and $\sum_{i=1}^r g_i = g$. As each $[c_{i-1}, c_i]$ is contained in some $[a_{j-1}, a_j]$,

$$||g_i|| = \int_0^1 ||\{g_i > t\}|| \, dt = \int_{c_{i-1}}^{c_i} ||\{g > t\}|| \, dt$$
$$\leqslant \int_{a_{j-1}}^{a_j} ||\{g > t\}|| \, dt < \eta,$$

by the coarea theorem. We conclude

$$|\langle F, g\rangle| \leqslant \sum_{i=1}^r |\langle F, g_i\rangle| < \frac{\varepsilon r}{2} \leqslant \frac{\varepsilon}{2}(p+q-1)$$
$$\leqslant \frac{\varepsilon}{2}\left(\frac{||g||}{\eta} + |g|_\infty + 1\right) = \theta||g|| + \frac{\varepsilon}{2}(|g|_\infty + 1),$$

from which the desired inequality follows whenever $|g|_\infty \geqslant 1$. If $0 < |g|_\infty < 1$, we apply the previous result to $h := g/|g|_\infty$:

$$|\langle F, g\rangle| = |g|_\infty |\langle F, h\rangle| < |g|_\infty(\theta||h|| + \varepsilon|h|_\infty) = \theta||g|| + \varepsilon|g|_\infty.$$

As the case $|g|_\infty = 0$ is trivial, the proposition is established. □

If F is a charge and $f \in BV_{\mathrm{loc}}^\infty$, then it is easy to see we can define a charge $F \llcorner f$ by the formula

$$\langle F \llcorner f, g\rangle := \langle F, fg\rangle$$

for all $g \in BV_c^\infty$. Showing that the same construction is possible for w-charges requires some work.

Observation 2.5. *The variational measure $||Df||$ of a BV function f is absolutely continuous with respect to the Hausdorff measure $\mathscr{H}^{m-1}$.*

Proof. By [17, Theorem 1.8.2, (3)], this is true if f is the indicator of a locally BV set. For an arbitrary BV function, the observation follows from the coarea theorem [17, Proposition 1.8.10]. □

Lemma 2.6. *Let $\{g_i\}$ be a sequence in $C_c^1(\mathbb{R}^m)$. If $\lim ||g_i|| = 0$, then $\{g_i\}$ has a subsequence that converges to zero $\mathscr{H}^{m-1}$-almost everywhere.*

Proof. For $m = 1$, Observation 1.1 implies $\{g_i\}$ converges uniformly to zero everywhere.

If $m \geqslant 2$, select a subsequence of $\{g_i\}$, still denoted by $\{g_i\}$, so that $||g_i|| \leqslant 2^{-i}$ for $i = 1, 2, \ldots$, and let

$$B_{i,k} := \left\{ x \in \mathbb{R}^m : \ g_i(x) > \frac{1}{k} \right\} \quad \text{and} \quad B_k := \bigcap_{j=1}^{\infty} \bigcup_{i=j}^{\infty} B_{i,k}.$$

It follows directly from the definition of capacity Cap_1 [5, Section 4.7.1] that $\mathrm{Cap}_1(B_{i,k}) \leqslant ||kg_i|| \leqslant k2^{-i}$. Since Cap_1 is a measure in $\mathbb{R}^m$,

$$\mathrm{Cap}_1(B_k) \leqslant \mathrm{Cap}_1\left(\bigcup_{i=j}^{\infty} B_{i,k} \right) \leqslant k2^{1-j}$$

for $j = 1, 2, \ldots$. We infer $\mathrm{Cap}_1(B_k) = 0$, and if $B = \bigcup_{k=1}^{\infty} B_k$ then $\mathrm{Cap}_1(B) = 0$. According to [5, Section 5.6.3, Theorem 3], we have $\mathscr{H}^{m-1}(C) = 0$ for each compact subset of B. As B is a Borel set, it follows from [6, Theorems 1.6 and 5.6] that $\mathscr{H}^{m-1}(B) = 0$. A direct verification reveals

$$B = \{ x \in \mathbb{R}^m : \ \limsup g_i(x) > 0 \},$$

which means $\limsup g_i \leqslant 0$ $\mathscr{H}^{m-1}$-almost everywhere. Applying this result to the sequence $\{-g_i\}$, we obtain $\liminf g_i \geqslant 0$ $\mathscr{H}^{m-1}$-almost everywhere, and the lemma follows. $\square$

Lemma 2.7. *Let $\{g_i\}$ be a sequence in BV_c^{∞} such that $\{g_i\} \rightrightarrows 0$, and let $f \in BV_{\mathrm{loc}}^{\infty}$. Then $\{fg_i\} \rightrightarrows 0$.*

Proof. Select a compact set $K \subset \mathbb{R}^m$ with $\mathrm{supp}\, g_i \subset \mathrm{int}\, K$, and let $c = \sup |g_i|_{\infty}$. We may assume $\{f \neq 0\} \subset K$, in which case

$$\sup |fg_i|_{\infty} \leqslant c|f|_{\infty} < \infty.$$

Thus we only need to prove $\lim ||fg_i|| = 0$. Clearly, it suffices to show that this is true for a subsequence of $\{g_i\}$.

Assume first that $\{g_i\}$ is a sequence in $C_c^1(\mathbb{R}^m)$. Using Lemma 2.6, find a subsequence of $\{g_i\}$, still denoted by $\{g_i\}$, that converges to zero $\mathscr{H}^{m-1}$-almost everywhere, and in view of Observation 2.5, also $||Df||$-almost everywhere. Choose a sequence $\{f_j\}$ in $C_c^1(\mathbb{R}^m)$ so that $\lim |f_j - f|_1 = 0$ and $\lim ||f_j|| = ||f||$. If $U \subset \mathbb{R}^m$ is an open set, then

$$||Df||(U) \leqslant \liminf ||Df_j||(U)$$

by [5, Section 5.2, Theorem 1]. As $||Df||$ and $||Df_j||$ are finite measures and $\lim ||Df_j||(\mathbb{R}^m) = ||Df||(\mathbb{R}^m)$, the previous inequality implies

$$\limsup ||Df_j||(C) \leqslant ||Df||(C)$$

for each closed set $C \subset \mathbb{R}^m$. According to [5, Section 1.9, Theorem 1], the measures $||Df_j||$ converge weakly to $||Df||$. Since

$$\begin{aligned}||fg_i|| &\leqslant \liminf_{j\to\infty} ||f_j g_i|| = \liminf_{j\to\infty} \int_{\mathbb{R}^m} |D(f_j g_i)|\, d\mathscr{L}^m \\ &\leqslant \lim_{j\to\infty} \int_{\mathbb{R}^m} |Df_j| \cdot |g_i|\, d\mathscr{L}^m + \lim_{j\to\infty} \int_{\mathbb{R}^m} |f_j| \cdot |Dg_i|\, d\mathscr{L}^m \\ &= \int_{\mathbb{R}^m} |g_i|\, d||Df|| + \int_{\mathbb{R}^m} |f| \cdot |Dg_i|\, d\mathscr{L}^m \\ &\leqslant \int_{\mathbb{R}^m} |g_i|\, d||Df|| + |f|_\infty ||g_i||,\end{aligned}$$

the dominated convergence theorem implies $\lim ||fg_i|| = 0$.

In the general case, find functions $g_{i,k},\ i,k = 1,2,\ldots,$ in $C_c^1(\mathbb{R}^m)$ so that each $\{g_{i,k} \neq 0\}$ is contained in K, all $|g_{i,k}|_\infty$ are bounded by a fixed constant, and for $i = 1,2,\ldots,$

$$\lim_{k\to\infty} |g_{i,k} - g_i|_1 = 0 \quad \text{and} \quad \lim_{k\to\infty} ||g_{i,k}|| = ||g_i||.$$

Since $\{fg_{i,k}\}$ converges to fg_i in L^1, we have $||fg_i|| \leqslant \liminf ||fg_{i,k}||$. Thus for $i = 1,2,\ldots,$ there is an integer $k_i \geqslant 1$ such that

$$||g_{i,k_i}|| < ||g_i|| + \frac{1}{i} \quad \text{and} \quad ||fg_i|| - \frac{1}{i} \leqslant ||fg_{i,k_i}||;$$

in particular, $\lim ||g_{i,k_i}|| = 0$. Now $\lim ||fg_{i,k_i}|| = 0$ by the first part of the proof, and consequently $\lim ||fg_i|| = 0$. $\square$

Corollary 2.8. *If F is a w-charge and $f \in BV_{\mathrm{loc}}^\infty$, then*

$$F \llcorner f : g \mapsto \langle F, fg \rangle : BV_c^\infty \to \mathbb{R}$$

is a w-charge.

Let F be a charge or a w-charge, and let $A \in \boldsymbol{BV}$. We define

$$F(A) := \langle F, \chi_A \rangle \quad \text{and} \quad F \llcorner A := F \llcorner \chi_A,$$

and say that F is, respectively, a charge or w-charge *in* A whenever $F = F \llcorner A$.

Note. Independently of the previous paragraph, when μ is a measure in $\mathbb{R}^m$ and $E \subset \mathbb{R}^m$, we define the measure $\mu \llcorner E$ in the customary way:

$$(\mu \llcorner E)(A) := \mu(A \cap E)$$

for each set $A \subset \mathbb{R}^m$.

3. Integrals

A set $E \subset \mathbb{R}^m$ is called *thin* whenever $\mathscr{H}^{m-1} \llcorner E$ is a σ-finite measure. A *gage* on a set $E \subset \mathbb{R}^m$ is a function $\delta : E \to [0, \infty)$ such that $\{\delta \neq 0\}$ is a thin set. A *partition* is a finite collection

$$P = \{(A_1, x_1), \dots, (A_p, x_p)\},$$

where $A_1, \dots, A_p$ are disjoint bounded BV sets and $x_1, \dots, x_p$ are points of $\mathbb{R}^m$.

Definition 3.1. A function f defined almost everywhere in a bounded BV set A is called *R-integrable* in A if there is a charge F in A and an extension of f to $\mathrm{cl}_* A$, still denoted by f, such that the following condition is satisfied: given $\varepsilon > 0$, we can find a gage δ on $\mathrm{cl}_* A$ so that

$$\sum_{i=1}^{p} |f(x_i)|A_i| - F(A_i)| < \varepsilon$$

for every partition $\{(A_1, x_1), \dots, (A_p, x_p)\}$ with $x_i \in \mathrm{cl}_* A, A_i \subset A$,

$$r(A_i \cup \{x_i\}) > \varepsilon \quad \text{and} \quad d(A_i \cup \{x_i\}) < \delta(x_i)$$

for $i = 1, \dots, p$.

The charge F of Definition 3.1, which is uniquely determined by f, is called the *R-primitive* of f, denoted by $(R)\int f$. For a BV set $B \subset A$, the charge $F \llcorner B$ is the R-primitive of $f \restriction B$; in particular, $f \restriction B$ is R-integrable in B. The *R-integral* of f over A is the number $(R)\int_A f := F(A)$. By [17, Proposition 5.1.3], the R-integral is a nonnegative linear functional on the linear space $R(A)$ of all R-integrable functions in A.

Without proofs, we summarize the main properties of the R-integral established in [17, Chapter 5].

Theorem 3.2. *Let A be bounded BV set.*

(1) $L^1(A) \subset R(A)$ *and* $(R)\int_A f = \int_A f$ *for each* $f \in L^1(A)$. *The inclusion* $L^1(A) \subset R(A)$ *is proper whenever* $\mathrm{int}\, A \neq \emptyset$.
(2) *Each function* $f \in R(A)$ *is measurable; moreover, if* $f \in R(A)$ *is nonnegative, then* $f \in L^1(A)$.
(3) *If the R-primitive of* $f \in R(A)$ *equals zero, then* $f(x) = 0$ *for almost all* $x \in A$.
(4) *Suppose B is a bounded BV set disjoint from A. Let f be a function defined on $A \cup B$ that is R-integrable in A and B, and let F_A and F_B be the R-primitives of $f \restriction A$ and $f \restriction B$, respectively. If both A and B are essentially closed, then $f \in R(A \cup B)$ and $F_A + F_B$ is the R-primitive of f.*

(5) *Let F be the R-primitive of $f \in R(A)$, and let $g \in BV^{\infty}_{\mathrm{loc}}$. Then $fg \in R(A)$ and $F \llcorner g$ is the R-primitive of fg. In particular,*

$$(R)\int_A f(x)g(x)\,dx = \int_0^{\infty}\left[(R)\int_{\{g>t\}} f(x)\,dx\right]dt$$

whenever g is nonnegative.

(6) *Let $v \in C(\mathrm{cl}\,A;\mathbb{R}^m)$, let T be a thin set, and let v be pointwise Lipschitz at each $x \in \mathrm{cl}_* A - T$. Then* $\mathrm{div}\,v$ *belongs to $R(A)$ and the flux of v is the R-primitive of* $\mathrm{div}\,v$.

(7) *If $\phi : A \to \mathbb{R}^m$ is a lipeomorphism and $f \in R[\phi(A)]$, then $(f \circ \phi)J_\phi$ belongs to $R(A)$ and*

$$(R)\int_A f[\phi(x)]J_\phi(x)\,dx = (R)\int_{\phi(A)} f(y)\,dy;$$

here $J_\phi \coloneqq |\det D\phi|$ is the Jacobian of ϕ.

Note. Four comments concerning Theorem 3.2 are in order.

(i) An easy consequence of part (1) is the following observation: if the pair (f, F) satisfies the conditions of Definition 3.1 for a particular extension of f to $\mathrm{cl}_* A$, then it satisfies these conditions for an arbitrary extension of f to $\mathrm{cl}_* A$.

(ii) Part (4) is false without assuming the sets A and B are essentially closed [17, Proposition 6.1.1, Remark 6.1.2, (4)]. We shall improve on this situation in Theorem 3.5 below.

(iii) As part (6) implies that Fubini's theorem is generally false for the R-integral [17, Example 5.1.14], part (5) asserts a nontrivial fact; cf. [17, Remark 5.2.3].

(iv) Part (7) can be generalized to a geometrically intuitive transformation formula for local lipeomorphisms [17, Section 5.3].

If f is a function defined almost everywhere in $A \in \boldsymbol{BV}$ and F is a w-charge, we denote by $\boldsymbol{R}(f, F; A)$ the family of all BV sets $B \subset A$ for which $F \llcorner B$ is the R-primitive of $f \restriction B$.

Definition 3.3. A function f defined almost everywhere in $A \in \boldsymbol{BV}$ is called *W-integrable* in A if there is a w-charge F in A and a sequence $\{A_i\}$ in $\boldsymbol{R}(f, F; A)$ with $\{A_i\} \rightrightarrows A$.

The family of all W-integrable functions in $A \in \boldsymbol{BV}$ is denoted by $W(A)$. Each charge F associated with $f \in W(A)$ according to Definition 3.3 is called a *W-primitive* of f.

Let F be a W-primitive of $f \in W(A)$, and choose a sequence $\{A_i\}$ in $\boldsymbol{R}(f, F; A)$ with $\{A_i\} \rightrightarrows A$. If $B \subset A$ is a BV set, then $\{B \cap A_i\}$ is a sequence in $\boldsymbol{R}(f \restriction B, F \llcorner B; B)$, and it follows from Lemma 2.7 that $\{B \cap A_i\} \rightrightarrows B$. Consequently, $F \llcorner B$ is a W-primitive of $f \restriction B$.

Observation 3.4. *Let A be a bounded BV set. Each $f \in W(A)$ has a unique W-primitive, denoted by $(W)\int f$.*

Proof. Suppose F and G are W-primitives of $f \in W(A)$, and find sequences $\{A_i\}$ in $\boldsymbol{R}(f, F; A)$ and $\{B_i\}$ in $\boldsymbol{R}(f, G; A)$ so that $\{A_i\} \rightrightarrows A$ and $\{B_i\} \rightrightarrows A$. Then $\{A_i \cap B_i\}$ is a sequence in $\boldsymbol{R}(f, F; A) \cap \boldsymbol{R}(f, G; A)$ and $\{A_i \cap B_i\} \rightrightarrows A$. As each R-integrable function has a unique R-primitive,

$$F(A) = \lim F(A_i \cap B_i) = \lim G(A_i \cap B_i) = G(A),$$

and the observation follows from the previous paragraph. □

Let A be a bounded BV set. If F is the W-primitive of $f \in W(A)$, we call the number $(W)\int_A f := F(A)$ the *W-integral* of f over A. Employing proofs similar to that of Observation 3.4, it is easy to show that the W-integral is a linear functional on $W(A)$, and that parts (1)–(3) and (7) of Theorem 3.2 hold for the W-integral.

Theorem 3.5 (Additivity). *Let A and B be bounded BV sets, and let f be a function defined almost everywhere on $A \cup B$. If f is W-integrable in A and B, then it is W-integrable in $A \cup B$, and*

$$(W)\int_{A \cup B} f = (W)\int_A f + (W)\int_B f$$

whenever A and B are disjoint.

Proof. It suffices to prove the theorem when A and B are disjoint. Denote by F_A and F_B the W-primitives of f in A and B, respectively, and find sequences $\{A_i\}$ in $\boldsymbol{R}(f, F_A; A)$ and $\{B_i\}$ in $\boldsymbol{R}(f, F_B; B)$ so that $\{A_i\} \rightrightarrows A$ and $\{B_i\} \rightrightarrows B$. According to [22], for $i, j = 1, 2, \ldots$, there are essentially closed BV sets $A_{i,j} \subset A_i$ and $B_{i,j} \subset B_i$ such that $\{A_{i,j}\}_j \rightrightarrows A_i$ and $\{B_{i,j}\}_j \rightrightarrows B_i$. For each i, find a j_i so that

$$\|A_i - A_{i,j_i}\| < 1/i \quad \text{and} \quad \|B_i - B_{i,j_i}\| < 1/i,$$

and let $C_i := A_{i,j_i} \cup B_{i,j_i}$. Part (4) of Theorem 3.2 shows that $\{C_i\}$ is a sequence in $\boldsymbol{R}(f, F_A + F_B; A \cup B)$. Since $\{C_i\} \rightrightarrows A \cup B$, the theorem follows. □

Theorem 3.6 (Multipliers). *Let A be a bounded BV set, and let F be the W-primitive of $f \in W(A)$. If $g \in BV^{\infty}_{\mathrm{loc}}$, then $fg \in W(A)$ and $F \llcorner g$ is the W-primitive of fg. If, in addition, $\{g_i\}$ is a sequence in $BV^{\infty}_{\mathrm{loc}}$ such that $\{g_i \chi_A\} \rightrightarrows g\chi_A$, then*

$$\lim (W)\int_A f g_i = (W)\int_A f g.$$

Proof. There is a sequence $\{A_i\}$ in $\boldsymbol{R}(f, F; A)$ with $\{A_i\} \rightrightarrows A$. As part (5) of Theorem 3.2 implies $\boldsymbol{R}(f, F; A) \subset \boldsymbol{R}(fg, F \llcorner g; A)$, it follows from Corollary 2.8 that $fg \in W(A)$ and $F \llcorner g$ is the W-primitive of fg. If $\{g_i\}$ is a sequence in $BV_{\mathrm{loc}}^{\infty}$ with $\{g_i \chi_A\} \rightrightarrows g\chi_A$, then

$$(W)\int_A fg = (F \llcorner g)(A) = \langle F, g\chi_A \rangle = \lim \langle F, g_i \chi_A \rangle$$
$$= \lim\,(F \llcorner g_i)(A) = \lim\,(W)\int_A fg_i. \qquad \square$$

If $U \subset \mathbb{R}^m$ is an open set, we denote by $W_{\mathrm{loc}}(U)$ the linear space of all functions $f : U \to \mathbb{R}$ such that $f \restriction A$ belongs to $W(A)$ for each $A \in \boldsymbol{BV}_c(U)$. The elements of $W_{\mathrm{loc}}(U)$ are called *locally W-integrable functions* in U. If $f \in W_{\mathrm{loc}}(U)$ and $g \in BV_c^{\infty}(U)$, find an $A \in \boldsymbol{BV}_c(U)$ with $\operatorname{supp} g \subset \operatorname{int} A$ and, using the multipliers theorem, let

$$(W)\int_U fg := (W)\int_A fg.$$

The W-integral $(W)\int_U fg$ is well defined, since by the additivity theorem, its value does not depend on the choice of A. In particular, it is easy to see that for each $f \in W_{\mathrm{loc}}(U)$, the linear map

$$\Lambda_f : \varphi \mapsto (W)\int_U f\varphi : C_c^{\infty}(U) \to \mathbb{R}$$

is a distribution in U [18, Definition 6.7]. Showing that $\Lambda_f = 0$ implies $f(x) = 0$ for almost all $x \in U$ requires some work.

Lemma 3.7. *Let F be a w-charge in $A \in \boldsymbol{BV}_c$, and suppose there is a sequence $\{A_i\}$ of BV subsets of A such that $\{A_i\} \rightrightarrows A$ and each $F \llcorner A_i$ is a charge. If $C_x := C + x$ is the translation of $C \in \boldsymbol{BV}_c$ by $x \in \mathbb{R}^m$, then the function $x \mapsto F(C_x)$ is uniformly continuous on $\mathbb{R}^m$ and has compact support.*

Proof. Choose an $\varepsilon > 0$, and using Proposition 2.4, find a $\theta > 0$ so that $|F(E)| < \theta\|E\| + \varepsilon$ for each $E \in \boldsymbol{BV}_c$. Let $\alpha := \varepsilon/\theta$, and for $i = 1, 2, \ldots,$ let $F_i := F \llcorner A_i$ and $B_i := A - A_i$.

Given a positive integer $j \leqslant m$, denote by Π_j the $(m-1)$-dimensional subspace of $\mathbb{R}^m$ perpendicular to the jth coordinate axis. For each $y \in \Pi_j$ denote by l_y the line passing through y and perpendicular to Π_j. According to [5, Section 5.10.2, Theorem 2], for $\mathscr{H}^{m-1}$-almost all $y \in \Pi_j$, the intersection $l_y \cap C$ is a one-dimensional BV set, and we denote by $\|l_y \cap C\|$ its perimeter. We employ the usual relationships between $\|l_y \cap C\|$ and $\|C\|$; see [17, Section 1.9]. As

$$\int_{\Pi_j} \|l_y \cap C\|\, d\mathscr{H}^{m-1}(y) \leqslant \|C\| < \infty,$$

there is a $\beta_j > 0$ such that $\int_E ||l_y \cap C|| \, d\mathscr{H}^{m-1}(y) < \alpha$ for each $\mathscr{H}^{m-1}$-measurable set $E \subset \Pi_j$, with $\mathscr{H}^{m-1}(E) < \beta_j$. The set

$$E_{i,j} := \{y \in \Pi_j: \, ||l_y \cap B_i|| > 0\} = \{y \in \Pi_j: \, ||l_y \cap B_i|| \geqslant 2\} \tag{3.1}$$

is $\mathscr{H}^{m-1}$-measurable by [5, Section 5.10.2, Lemma 1]. Denote by $x^{(j)}$ the orthogonal projection of $x \in \mathbb{R}^m$ to Π_j, and by $E_{i,j} - x^{(j)}$ the translation of $E_{i,j}$ by $x^{(j)}$. Observe

$$\begin{aligned}\int_{\Pi_j} ||l_y \cap (B_i \cap C_x)|| \, d\mathscr{H}^{m-1}(y) &= \int_{E_{i,j}} ||l_y \cap (B_i \cap C_x)|| \, d\mathscr{H}^{m-1}(y) \\ &\leqslant \int_{E_{i,j}} ||l_y \cap B_i|| \, d\mathscr{H}^{m-1}(y) + \int_{E_{i,j}} ||l_y \cap C_x|| \, d\mathscr{H}^{m-1}(y) \\ &\leqslant ||B_i|| + \int_{E_{i,j} - x^{(j)}} ||l_y \cap C|| \, d\mathscr{H}^{m-1}(y).\end{aligned}$$

Let $\beta = \min\{\beta_1, \dots, \beta_m\}$, and find an integer $k \geqslant 1$ with $||B_k|| < \min\{\alpha, \beta\}$. In view of (3.1),

$$\begin{aligned}\mathscr{H}^{m-1}(E_{k,j} - x^{(j)}) &= \mathscr{H}^{m-1}(E_{k,j}) \\ &\leqslant \int_{\Pi_j} ||l_y \cap B_k|| \, d\mathscr{H}^{m-1}(y) \leqslant ||B_k|| < \beta.\end{aligned}$$

Consequently,

$$||B_k \cap C_x|| \leqslant \sum_{j=1}^{m} \int_{\Pi_j} ||l_y \cap (B_k \cap C_x)|| \, d\mathscr{H}^{m-1}(y) < 2m\alpha,$$

and we conclude that for each $x \in \mathbb{R}^m$,

$$|F(C_x) - F_k(C_x)| = |F(B_k \cap C_x)| < 2m\alpha\theta + \varepsilon = \varepsilon(2m+1).$$

Since F_k is a charge, Proposition 2.4 and [17, Lemma 4.2.1] imply there is a $\delta > 0$ such that $|F_k(C_x) - F_k(C_z)| < \varepsilon$ for each $x, z \in \mathbb{R}^m$ with $|x - z| < \delta$. As $F(C_x) = 0$ whenever $|x|$ is sufficiently large, the lemma follows. □

Proposition 3.8. *Let $U \subset \mathbb{R}^m$ be an open set and let $f \in W_{\mathrm{loc}}(U)$. If $(W) \int_U f\varphi = 0$ for each $\varphi \in C_c^\infty(U)$, then $f(x) = 0$ for almost all $x \in U$.*

Proof. As U is Lindelöf, it suffices to show $f = 0$ almost everywhere in each open ball $V \subset U$. Choose an open ball $V \subset U$ and a diffeomorphism ϕ from $\mathbb{R}^m$ onto V. Observe $J_\phi(x) > 0$ for each $x \in \mathbb{R}^m$, and $g := (f \circ \phi)J_\phi$ belongs to $W_{\mathrm{loc}}(\mathbb{R}^m)$. Hence $(W) \int_{\mathbb{R}^m} g\phi = 0$ for each $\phi \in C_c^\infty(\mathbb{R}^m)$, and $g = 0$ almost everywhere yields $f(x) = 0$ for almost all $x \in V$. It follows we may assume $U = \mathbb{R}^m$ from the onset.

Select bounded BV sets A and B. Let $B_* := \{-x: x \in B\}$ and $B_x := B + x$ for each $x \in \mathbb{R}^m$. Denote by F the W-primitive of $f \restriction A$, and for $x \in \mathbb{R}^m$, let

$$\begin{aligned}(\chi_{B_*} * f)(x) &:= (W)\int_A \chi_{B_*}(x-y)f(y)\,dy \\ &= (W)\int_{A\cap B_x} f(y)\,dy = F(B_x).\end{aligned} \tag{3.2}$$

By Lemma 3.7, the "convolution" $\chi_B * f$ is a uniformly continuous function on $\mathbb{R}^m$ with compact support. Select a $\varphi \in C_c^\infty(\mathbb{R}^m)$, and observe that $\chi_B * \varphi$ is the usual convolution of χ_B and φ; in particular $\chi_B * \varphi$ belongs to $C_c^\infty(\mathbb{R}^m)$.

Claim. *$(W)\int_A f(\chi_B * \varphi) = \int_A (\chi_{B_*} * f)\varphi$ for each $\varphi \in C_c^\infty(\mathbb{R}^m)$.*

Proof. If H is a charge in A and $n \geqslant 1$ is an integer, let

$$||H||_n := \sup\{|H(C)|: ||C|| \leqslant n\}.$$

According to [17, Proposition 2.2.4], there is an integer $k \geqslant 1$, depending on A, such that $||\cdot||_k, ||\cdot||_{k+1}, \dots$ are equivalent Banach norms in the linear space of all charges in A. If H is the R-primitive of $h \in R(A)$, then $h \mapsto ||H||_n, n = k, k+1, \dots$ are equivalent norms in $R(A)$. The topology in $R(A)$ induced by any of these norms is denoted by $\boldsymbol{S}$. Given $\varphi \in C_c^\infty(\mathbb{R}^m)$, the linear functionals

$$R : h \mapsto (R)\int_A h(\chi_B * \varphi) \quad \text{and} \quad L : h \mapsto \int_A (\chi_{B_*} * h)\varphi$$

defined on $R(A)$ are $\boldsymbol{S}$-continuous. The $\boldsymbol{S}$-continuity of R follows from [17, Proposition 4.5.2]. The dominated convergence theorem implies the $\boldsymbol{S}$-continuity of L: since $||B_x|| = ||B||$ for each $x \in \mathbb{R}^m$, we infer from (3.2) the sequence $\{\chi_{B_*} * h_i\}$ converges uniformly to zero for every sequence $\{h_i\}$ in $R(A)$ that $\boldsymbol{S}$-converges to zero. The standard manipulation of convolutions by means of Fubini's theorem shows that $R(h) = L(h)$ for each $h \in L^1(A)$. This equality extends to every $h \in R(A)$, because $L^1(A)$ is a dense subspace of $(R(A), \boldsymbol{S})$; see [17, Corollary 4.2.3]. There is a sequence $\{A_i\}$ of BV subsets of A such that $\{A_i\} \rightrightarrows A$ and $f \in R(A_i)$ for $i = 1, 2, \dots$. Applying what we have already proved, and observing that $\lim |A - A_i| = 0$ establishes the claim:

$$\begin{aligned}(W)\int_A f(\chi_B * \varphi) &= \lim\,(R)\int_{A_i} f(\chi_B * \varphi) \\ &= \lim \int_{A_i} (\chi_{B_*} * f)\varphi = \int_A (\chi_{B_*} * f)\varphi.\end{aligned}$$

Now choose a $\varphi \in C_c^\infty(\mathbb{R}^m)$, and find an $A \in \boldsymbol{BV}_c$ whose interior contains the supports of both $\chi_B * \varphi$ and $\chi_{B_*} * f$. In view of the claim

$$\int_{\mathbb{R}^m} (\chi_{B_*} * f)\varphi = \int_A (\chi_{B_*} * f)\varphi$$
$$= (W) \int_A f(\chi_B * \varphi) = (W) \int_{\mathbb{R}^m} f(\chi_B * \varphi) = 0.$$

As $\chi_{B_*} * f$ is continuous, it is equal to zero everywhere. In particular $F(B) = \chi_{B_*} * f(0) = 0$, and the proposition follows from Theorem 3.2, part (3), and the arbitrariness of B. $\square$

An $\mathscr{H}^{m-1}$-negligible subset of $\mathbb{R}^m$ is called *slight*. Clearly, each slight set is thin but not vice versa.

Observation 3.9. *Given a bounded slight set S and $\varepsilon > 0$, there is an open set $U \in \boldsymbol{BV}$ such that $S \subset U$ and $\|U\| < \varepsilon$.*

Proof. Choose a bounded open set V containing S. Since S is negligible with respect to the $(m-1)$-dimensional spherical measure [13, Section 5.1], we can cover S by open balls $B_i \subset V$ so that $\sum_{i=1}^\infty \|B_i\| < \varepsilon$. It follows that $U = \bigcup_{i=1}^\infty B_i$ is the desired set. $\square$

Theorem 3.10 (Gauss–Green). *Let A, S, and T be, respectively, a bounded BV set, a slight set, and a thin set. Suppose $v : \operatorname{cl} A \to \mathbb{R}^m$ is a bounded vector field that is continuous in $\operatorname{cl} A - S$ and pointwise Lipschitz in $\operatorname{cl}_* A - T$. Then $\operatorname{div} v$ belongs to $W(A)$ and*

$$(W) \int_A \operatorname{div} v \, d\mathscr{L}^m = \int_{\partial_* A} v \cdot \nu_A \, d\mathscr{H}^{m-1}.$$

Proof. Extend v to a bounded vector field $w : \mathbb{R}^m \to \mathbb{R}^m$ that is continuous on $\mathbb{R}^m - (S \cap \operatorname{cl} A)$, and use Example 2.3 to define the flux F of w. By Observation 3.9, there is a sequences $\{U_i\}$ of open bounded BV sets such that $S \cap \operatorname{cl} A \subset U_i$ for $i = 1, 2, \dots$, and $\lim \|U_i\| = 0$. Let $A_i = A - U_i$ and observe $\{A_i\} \rightrightarrows A$. By part (6) of Theorem 3.2, each A_i belongs to $\boldsymbol{R}(\operatorname{div} v, F; A)$ and the theorem follows. $\square$

Theorem 3.11 (Integration by parts). *Let Ω, S, and T be, respectively, a Lipschitz domain, a slight set, and a thin set. Suppose $v :\ \operatorname{cl} \Omega \to \mathbb{R}^m$ is a bounded vector field that is continuous in $\operatorname{cl} \Omega - S$ and pointwise Lipschitz in $\Omega - T$. Then*

$$(W) \int_\Omega g \operatorname{div} v \, d\mathscr{L}^m = \int_{\partial\Omega} (\operatorname{Tr} g) v \cdot \nu_\Omega \, d\mathscr{H}^{m-1} - \int_\Omega v \cdot d(Dg)$$

for each $g \in BV^\infty(\Omega)$.

Proof. It suffices to prove the theorem for a nonnegative $g \in BV^{\infty}(\Omega)$. Extend v to a bounded vector field $w : \mathbb{R}^m \to \mathbb{R}^m$ that is continuous on $\mathbb{R}^m - (S \cap \operatorname{cl} \Omega)$. Using the standard mollifiers, find a sequence $\{v_k\}$ in $C^{\infty}(\mathbb{R}^m; \mathbb{R}^m)$ having the following properties

(i) $\sup |v_k|_{\infty} \leqslant |w|_{\infty}$;
(ii) $\lim v_k(x) = v(x)$ for each $x \in \operatorname{cl} \Omega - S$;
(iii) $\{v_k\}$ converges to w uniformly on each compact set K contained in $\mathbb{R}^m - (S \cap \operatorname{cl} \Omega)$.

According to [5, Section 5.3, Theorem 1],

$$\int_{\Omega} g \operatorname{div} v_k \, d\mathscr{L}^m = \int_{\partial\Omega} (\operatorname{Tr} g) v_k \cdot \nu_{\Omega} \, d\mathscr{H}^{m-1} - \int_{\Omega} v_k \cdot d(Dg)$$

for $k = 1, 2, \ldots$. Properties (i) and (ii), the dominated convergence theorem, and Observation 2.5 yield

$$\lim \int_{\partial\Omega} (\operatorname{Tr} g) v_k \cdot \nu_{\Omega} \, d\mathscr{H}^{m-1} = \int_{\partial\Omega} (\operatorname{Tr} g) v \cdot \nu_{\Omega} \, d\mathscr{H}^{m-1}$$

$$\lim \int_{\Omega} v_k \cdot d(Dg) = \int_{\Omega} v \cdot d(Dg),$$

and consequently

$$\lim \int_{\Omega} g \operatorname{div} v_k \, d\mathscr{L}^m = \int_{\partial\Omega} (\operatorname{Tr} g) v \cdot \nu_{\Omega} \, d\mathscr{H}^{m-1} - \int_{\Omega} v \cdot d(Dg).$$

In view of the multipliers and Gauss–Green theorems, $g \operatorname{div} v$ is W-integrable in Ω, and we only need to show

$$(W) \int_{\Omega} g \operatorname{div} v \, d\mathscr{L}^m = \lim \int_{\Omega} g \operatorname{div} v_k \, d\mathscr{L}^m.$$

By Observation 3.9, there is a sequences $\{U_i\}$ of open bounded BV sets such that $S \cap \operatorname{cl} \Omega \subset U_i$ for $i = 1, 2, \ldots$, and $\lim \|U_i\| = 0$. Letting $A_i := \Omega - U_i$, we see that $\{A_i\} \rightrightarrows \Omega$, and that $\operatorname{cl} A_i$ is a compact subset of $\mathbb{R}^m - (S \cap \operatorname{cl} \Omega)$. Part (6) of Theorem 3.2 implies

$$(W) \int_{A_i} g \operatorname{div} v \, d\mathscr{L}^m = (R) \int_{A_i} g \operatorname{div} v \, d\mathscr{L}^m$$

for $i = 1, 2, \ldots$, and as the W-primitive of $g \operatorname{div} v$ is a w-charge,

$$(W) \int_{\Omega} g \operatorname{div} v \, d\mathscr{L}^m = \lim_{i \to \infty} (R) \int_{A_i} g \operatorname{div} v \, d\mathscr{L}^m.$$

Choose an $\varepsilon > 0$, and find an integer $p \geqslant 1$ so that

$$\left| (W) \int_{\Omega} g \operatorname{div} v \, d\mathscr{L}^m - (R) \int_{A_p} g \operatorname{div} v \, d\mathscr{L}^m \right| < \varepsilon, \tag{3.3}$$

$\|A_p\| \leqslant \|\Omega\| + 1$ and $\|g\chi_{\Omega - A_p}\| < \varepsilon$; the last inequality follows from Lemma 2.7. Let $B_p := \Omega - A_p$ and $h := g\chi_{B_p}$. The Fubini, Gauss–Green, and coarea theorems, together with property (i), imply

$$\begin{aligned}
\left| \int_{B_p} g \operatorname{div} v_k \, d\mathscr{L}^m \right| &= \left| \int_0^{\infty} \left(\int_{\{h > t\}} \operatorname{div} v_k \, d\mathscr{L}^m \right) dt \right| \\
&= \left| \int_0^{\infty} \left(\int_{\partial_* \{h > t\}} v_k \cdot \nu_{\{h > t\}} \, d\mathscr{H}^{m-1} \right) dt \right| \\
&\leqslant |w|_{\infty} \int_0^{\infty} \mathscr{H}^{m-1}(\partial_* \{h > t\}) \, dt \\
&= |w|_{\infty} \|h\| < \varepsilon |w|_{\infty}
\end{aligned}$$

for $k = 1, 2, \ldots$, and hence

$$\left| \int_{A_p} g \operatorname{div} v_k \, d\mathscr{L}^m - \int_{\Omega} g \operatorname{div} v_k \, d\mathscr{L}^m \right| < \varepsilon |w|_{\infty}. \tag{3.4}$$

By property (iii), there is a q such that $|v_k(x) - v(x)| < \varepsilon$ for each $x \in \operatorname{cl} A_p$ and each $k \geqslant q$. Let $u := g\chi_{A_p}$, and select $k \geqslant q$. Part (5) of Theorem 3.2 yields

$$\begin{aligned}
&\left| (R) \int_{A_p} g \operatorname{div} v - \int_{A_p} g \operatorname{div} v_k \, d\mathscr{L}^m \right| \\
&\quad = \left| \int_0^{\infty} \left[(R) \int_{\{u > t\}} \operatorname{div}(v - v_k) \, d\mathscr{L}^m \right] dt \right| \\
&\quad = \left| \int_0^{\infty} \left(\int_{\partial_* \{u > t\}} (v - v_k) \cdot \nu_{\{u > t\}} \, d\mathscr{H}^{m-1} \right) dt \right| \\
&\quad \leqslant \varepsilon \int_0^{\infty} \mathscr{H}^{m-1}(\partial_* \{u > t\}) \, dt = \varepsilon \|u\| \\
&\quad \leqslant \varepsilon (\|g\| + |g|_{\infty} \|A_p\|) < \varepsilon (\|g\| + |g|_{\infty} + |g|_{\infty} \|\Omega\|).
\end{aligned}$$

Combining the previous inequality with inequalities (3.3) and (3.4), we conclude that for each $k \geqslant q$,

$$\left| (W) \int_{\Omega} g \operatorname{div} v \, d\mathscr{L}^m - \int_{\Omega} g \operatorname{div} v_k \, d\mathscr{L}^m \right| < \beta \varepsilon,$$

where $\beta = 1 + |w|_{\infty} + \|g\| + |g|_{\infty} + |g|_{\infty} \|\Omega\|$. $\square$

Remark 3.12. Given $g \in BV^{\infty}(\Omega)$, the definition of Dg implies

$$\int_{\Omega} g \operatorname{div} v \, d\mathscr{L}^m = -\int_{\Omega} v \cdot d(Dg)$$

for each $v \in C_c^1(\Omega; \mathbb{R}^m)$. Employing the W-integral, we substantially generalized this fact in Theorem 3.11. However, the W-integral is merely a tool, which can be replaced by the Lebesgue integral whenever $g \operatorname{div} v$ belongs to $L^1(\Omega)$ (part (1) of Theorem 3.2).

In conclusion we show that in the Gauss–Green theorem the exceptional sets can be defined by means of the integral-geometric measure $\mathscr{I}_1^{m-1}$ provided $|Dv|$ belongs to $L^1(A)$. Note $\mathscr{I}_1^{m-1}(E) \leqslant \mathscr{H}^{m-1}(E)$ for each set $E \subset \mathbb{R}^m$, and the equality holds whenever E is $(\mathscr{H}^{m-1}, m-1)$ rectifiable [7, Section 2.10.6, Theorem 3.2.26].

Denote by G the Grassmanian $G(m, m-1)$, and by γ the probability measure $\gamma_{m,m-1}$ on G defined in [13, Section 3.9]. For $\Pi \in G$, the unique orthogonal projection of $\mathbb{R}^m$ onto Π is denoted by π. If $B \subset \mathbb{R}^m$ is a Borel set, then

$$\mathscr{I}_1^{m-1}(B) = \kappa \int_G \left(\int_{\Pi} \mathscr{H}^0[B \cap \pi^{-1}(x)] \, d\mathscr{H}^{m-1}(x) \right) d\gamma(\Pi), \tag{3.5}$$

where $\kappa > 0$ is a constant depending only on the dimension [13, Section 5.14].

Observation 3.13. *If $N \subset G$ is a γ-negligible set, then $\mathbb{R}^m$ has an orthonormal base $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_m\}$ such that each $\Pi \in G$ perpendicular to some $\boldsymbol{e}_i$ belongs to $G - N$.*

Proof. Choose any orthonormal base $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m\}$ in $\mathbb{R}^m$, and denote by Π_i the elements of G orthogonal to $\boldsymbol{u}_i$. If θ_m is the Haar measure on the orthogonal group $\boldsymbol{O}(m)$ and

$$O = \bigcup_{i=1}^{m} \{g \in \boldsymbol{O}(m) \colon \, g(\Pi_i) \in N\},$$

then $\theta_m(O) \leqslant m\gamma(N) = 0$ by the definition of γ in [13, Section 3.9]. Thus there is a $g \in \boldsymbol{O}(n) - O$, and $\{g(\boldsymbol{u}_1), \ldots, g(\boldsymbol{u}_m)\}$ is the desired base in $\mathbb{R}^m$. $\square$

Theorem 3.14. *Let A be a bounded BV set, and let E_0 and E_σ be Borel subsets of $\mathbb{R}^m$ such that $\mathscr{I}_1^{m-1}(E_0) = 0$ and the measure $\mathscr{I}_1^{m-1} \llcorner E_\sigma$ is σ-finite. Suppose $v \colon \operatorname{cl} A \to \mathbb{R}^m$ belongs to $L^1(\partial_* A, \mathscr{H}^{m-1}; \mathbb{R}^m)$, is continuous in $\operatorname{cl} A - E_0$, and pointwise Lipschitz in $\operatorname{cl}_* A - E_\sigma$. Then*

$$\int_A \operatorname{div} v \, d\mathscr{L}^m = \int_{\partial_* A} v \cdot \nu_A \, d\mathscr{H}^{m-1} \tag{3.6}$$

whenever $|Dv| \in L^1(A, \mathscr{L}^m)$.

Proof. Denote by G_0 and G_σ the families consisting of all $\Pi \in G$ such that the sets $E_0 \cap \pi^{-1}(x)$ and $E_\sigma \cap \pi^{-1}(x)$ are, respectively, empty and countable for $\mathscr{H}^{m-1}$-almost all $x \in \Pi$. In view of (3.5), the set $N = G - G_0 \cup G_\sigma$ is γ-negligible. Observation 3.13 implies $\mathbb{R}^m$ has an orthonormal base $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_m\}$ such that each $\Pi \in G$ perpendicular to some $\boldsymbol{e}_i$ belongs to $G_0 \cup G_\sigma$. Since neither our assumptions nor the equalities (3.5) and (3.6) depend on the choice of an orthonormal base in $\mathbb{R}^m$, we may assume $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_m\}$ is the standard base of $\mathbb{R}^m$. We show that if $v = (v_1, \dots, v_m)$ and $\nu_A = (\nu_1, \dots, \nu_m)$, then

$$\int_A \frac{\partial v_i}{\partial \xi_i} \, d\mathscr{L}^m = \int_{\partial_* A} v_i \nu_i \, d\mathscr{H}^{m-1} \tag{3.7}$$

for $i = 1, \dots, m$. Our argument relies on $\partial v_i / \partial \xi_i \in L^1(A)$—a fact guaranteed by the assumption $|Dv| \in L^1(A)$ for any choice of a base in $\mathbb{R}^m$.

In view of symmetry, it suffices to verify equality (3.7) only for $i = m$. As $\Pi = \{(x, 0) \in \mathbb{R}^m \colon x \in \mathbb{R}^{m-1}\}$ is perpendicular to $\boldsymbol{e}_m$, our choice of $\boldsymbol{e}_m$ and [12, Section 2.2.1, Theorem 2] imply there is an $\mathscr{L}^{m-1}$-negligible set $E \subset \mathbb{R}^{m-1}$ such that for all x in $\mathbb{R}^{m-1} - E$, the section

$$A_x = \{t \in \mathbb{R} \colon (x, t) \in A\}$$

is a BV subset of $\mathbb{R}$, and the function $t \mapsto v_m(x, t)$ is continuous in $\operatorname{cl} A_x$ and Lipschitz at all but countably many $t \in \operatorname{cl} A_x$. According to Fubini's theorem, making E larger, we may assume that the function $t \mapsto (\partial v_m / \partial \xi_m)(x, t)$ belongs to $L^1(A_x, \mathscr{L}^1)$ for every x in $\mathbb{R}^{m-1} - E$. Parts (1) and (6) of Theorem 3.2 yield

$$\begin{aligned} \int_{A_x} \frac{\partial v_m}{\partial \xi_m}(x, t) \, d\mathscr{L}^1(t) &= \int_{\partial_* A_x} v_m(x, t) \cdot \nu_m(x, t) \, d\mathscr{H}^0(t) \\ &= \int_{(\partial_* A)_x} v_m(x, t) \cdot \nu_m(x, t) \, d\mathscr{H}^0(t) \end{aligned}$$

whenever x belongs to $\mathbb{R}^{m-1} - E$. Integrating over $\mathbb{R}^{m-1}$ and using Fubini's theorem, we obtain

$$\int_A \frac{\partial v_m}{\partial \xi_m} \, d\mathscr{L}^m = \int_{\partial_* A} v_m \nu_m \, d\mathscr{H}^{m-1}. \qquad \square$$

Remark 3.15. Let A be a bounded BV set, and let $E_0 \subset \mathbb{R}^m$ be $\mathscr{I}_1^{m-1}$-negligible. If $v \colon \operatorname{cl} A \to \mathbb{R}^m$ is *bounded* and continuous in $\partial_* A - E_0$, then $v \in L^1(\partial_* A, \mathscr{H}^{m-1}; \mathbb{R}^m)$. Indeed, as $E_0 \cap \partial_* A$ is an $(\mathscr{H}^{m-1}, m-1)$ rectifiable set, it is $\mathscr{H}^{m-1}$-negligible by [7, Theorem 3.2.26].

4. Removable singularities

To illustrate our technique, we present first a simple proof of a slightly improved classical result of Besicovitch [2] concerning the removable singularities of holomorphic functions defined on an open subset of the complex plane $\mathbb{C}$.

Theorem 4.1. *Let $U \subset \mathbb{C}$ be an open set, and let $f : U \to \mathbb{C}$ be locally bounded. Suppose f is continuous outside a slight set $E_c \subset U$ and pointwise Lipschitz outside a thin set $E_d \subset U$. If f has a complex derivative almost everywhere in U, then it can be redefined on E_c so that it is holomorphic in U.*

Proof. Let $\Re f$ and $\Im f$ denote, respectively, the real and imaginary part of f. The vector fields $u = (\Re f, -\Im f)$ and $v = (\Im f, \Re f)$ are locally bounded in U and continuous in $U - E_c$; in particular, they belong to $L^1_{\text{loc}}(U; \mathbb{R}^2)$. The *Cauchy–Riemann equations* yield

$$\operatorname{div} u = \operatorname{div} v = 0$$

almost everywhere in U. Using the integration by parts theorem, we infer

$$\int_U u \cdot \nabla\varphi = -\int_U \varphi \operatorname{div} u = 0,$$
$$\int_U v \cdot \nabla\varphi = -\int_U \varphi \operatorname{div} v = 0$$

for each $\varphi \in C^1_c(U)$. This means the vector field $(\Re f, \Im f)$ is a distributional solution of the Cauchy–Riemann equations. As these equations form an elliptic system, an application of the regularity theorem completes the argument; cf. [18, Example 8.14]. □

Throughout the remainder of this paper $U \subset \mathbb{R}^m$ is a nonempty open set, in which we consider the equation

$$\operatorname{div}(h \circ \nabla u)(x) = f[x, u(x)], \tag{4.1}$$

where

$$u : U \to \mathbb{R}, \quad h : \mathbb{R}^m \to \mathbb{R}^m, \quad \text{and} \quad f : U \times \mathbb{R} \to \mathbb{R}$$

are maps whose properties will be specified below. To avoid trivialities, we assume $m \geqslant 2$.

A *classical solution* of Eq. (4.1) is a function $u \in C^2(U)$ such that (4.1) holds for all $x \in U$. A *weak solution* of Eq. (4.1) is an almost everywhere differentiable function u such that

$$\int_U h[\nabla u(x)] \cdot \nabla\varphi(x)\, dx = -\int_U \varphi(x) f[x, u(x)]\, dx$$

for each $\varphi \in C_c^\infty(U)$. Under mild assumptions, we show that if u satisfies (4.1) at almost all $x \in U$, then u is a weak solution of (4.1).

Theorem 4.2. *Let $E_c \subset E_d$ be, respectively, a slight and a thin subset of U, and suppose the following conditions are satisfied*:

1. *u is differentiable almost everywhere in U*;
2. *$h \circ \nabla u$ has a locally bounded extension to U that is continuous in $U - E_c$ and pointwise Lipschitz in $U - E_d$*;
3. *u satisfies equation* (4.1) *for almost all $x \in U$.*

Then the equality

$$\int_U [h \circ \nabla u] \cdot d(Dg) = -(W) \int_U g(x) f[x, u(x)]\, dx$$

holds for each $g \in BV_c^\infty(U)$.

Proof. Select a $g \in BV_c^\infty(U)$. Since $\operatorname{supp} g$ is a compact subset of U, there is a Lipschitz domain Ω such that $\operatorname{supp} g \subset \Omega$ and $\operatorname{cl} \Omega \subset U$. By our assumptions, $h \circ \nabla u$ can be extended to a vector field $v : \operatorname{cl} \Omega \to \mathbb{R}^m$ that satisfies the assumptions of Theorem 3.11. Since $\operatorname{Tr} g = 0$ on $\partial\Omega$, the theorem follows. □

Under an additional assumption, Theorems 3.2, part (1), and 4.2 imply the aforementioned result.

Corollary 4.3. *If, in addition to the assumptions of Theorem* 4.2, *we assume the function $x \to f[x, u(x)]$ belongs to $L^1_{\mathrm{loc}}(U)$, then u is a weak solution of* (4.1).

We apply Corollary 4.3 to the Laplace equation $\Delta u = 0$ and to the minimal surface equation

$$\operatorname{div} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} = 0.$$

Proposition 4.4. *Let $E_c \subset E_d$ be, respectively, a slight and a thin subset of U, and suppose the following conditions are satisfied*:

1. *u is continuous in $U - E_c$, and pointwise Lipschitz in $U - E_d$*;
2. *∇u has a locally bounded extension to U that is continuous in $U - E_c$ and pointwise Lipschitz in $U - E_d$*;
3. *$\Delta u = 0$ almost everywhere in U.*

Then u can be redefined on E_c so that it is harmonic in U.

Proof. By Stepanoff's theorem, ∇u and Δu are defined almost everywhere in U. We show first that $u \in L^{\infty}_{\text{loc}}(U)$. To this end, choose a ball $B := B(z, r)$ whose closure is contained in U, and a $c > 0$ so that $|\nabla u(x)| \leqslant c$ for almost all $x \in B$. An *admissible segment* is the nonempty intersection l of B and a line in $\mathbb{R}^m$ such that $l \cap E_c = \emptyset$, and $l \cap E_d$ is countable. Since $\mathscr{I}_1^{m-1} \leqslant \mathscr{H}^{m-1}$, proceeding as in the proof of Theorem 3.14, we can find an orthonormal base $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_m$ in $\mathbb{R}^m$ so that the union S_i of all admissible segments parallel to $\boldsymbol{e}_i$ differs from B by a negligible set. If l is an admissible segment and $x, y \in l$, then the one-dimensional version of the Gauss–Green theorem yields $|u(y) - u(x)| \leqslant c|y - x|$. We infer u is Lipschitz on $S = \bigcap_{i=1}^m S_i$ with the Lipschitz constant not larger than mc. As $B - S$ is a negligible set, our assertion is proved.

Select a $\varphi \in C_c^{\infty}(U)$, and a bounded BV set A such that $\operatorname{cl} A \subset U$ and $\operatorname{supp} \varphi \subset \operatorname{int} A$. Corollary 4.3, applied to the function $h : x \mapsto x$, implies $\int_U \nabla u \cdot \nabla \varphi = 0$. As the vector $u \nabla \varphi$ satisfies the assumptions of the Gauss–Green theorem and $u \Delta \varphi \in L^1(A)$, we have

$$\begin{aligned} \int_U u\Delta\varphi &= \int_A u\Delta\varphi = \int_A (u\Delta\varphi + \nabla u \cdot \nabla\varphi) \\ &= \int_A \operatorname{div}(u\nabla\varphi) = \int_{\partial_* A} (u\nabla\varphi) \cdot \nu_A = 0; \end{aligned}$$

indeed, since $u\Delta\varphi$ belongs to $L^1(A)$ by the first part of the proof, part (1) of Theorem 3.2 shows the W-integrals, which would normally occur in the previous equality, can be replaced by the Lebesgue integrals. Thus u is a distributional solution of $\Delta u = 0$, and the proposition follows from [18, Corollary of Theorem 8.12]. □

Corollary 4.5. *Let $E_c \subset E_d$ be relatively closed subsets of U that are, respectively, slight and thin, and suppose $u \in C^1(U - E_c)$. If u is locally Lipschitz in U and harmonic in $U - E_d$, then it is harmonic in U.*

Proof. As E_c is relatively closed in U, the bounded vector field ∇u has a locally bounded extension $u_0 : U \to \mathbb{R}^m$ that is continuous in $U - E_c$. By our assumption, u is C^{∞} in $U - E_d$. Since E_d is relatively closed in U, both u and u_0 are differentiable in $U - E_d$. Now the corollary follows from Proposition 4.4. □

Remark 4.6. Let K be a compact subset of U, and suppose u is locally Lipschitz in U and harmonic in $U - K$. If $\mathscr{H}^{m-1}(K) = 0$, then letting $E_c = E_d = K$, Corollary 4.5 yields the classical result: *u can be redefined on K so that it is harmonic in U*. In special situations, the same conclusion holds under weaker assumptions. For $m = 2$, David and Mattila [3] have shown it suffices to assume $\mathscr{H}^1(K) < \infty$ and $\mathscr{I}_1^1(K) = 0$. For $m \geqslant 3$, it is not known whether the assumptions $\mathscr{H}^{m-1}(K) < \infty$ and $\mathscr{I}_1^{m-1}(K) = 0$ are sufficient; however, Mattila and Paramonov [14] proved that they are sufficient when K belongs to a class of self-similar sets.

Since closed $\mathscr{I}_1^{m-1}$-negligible sets are generally not removable for Lipschitz harmonic functions [14], the next proposition is interesting.

Proposition 4.7. *Let $E_0 \subset E_\sigma$ be relatively closed subsets of U such that $\mathscr{I}_1^{m-1}(E_0) = 0$ and the measure $\mathscr{I}_1^{m-1} \llcorner E_\sigma$ is σ-finite. Suppose $u \in C^1(U - E_0)$ is harmonic in $U - E_\sigma$. If u belongs to the Sobolev space $W^{2,1}_{\rm loc}(U)$, then it can be redefined on E_0 so that it is harmonic in U.*

Proof. Select a $\varphi \in C_c^\infty(U)$, and a bounded BV set A such that $\mathrm{cl}\, A \subset U$ and $\mathrm{supp}\, \varphi \subset \mathrm{int}\, A$. As $\varphi = \nabla\varphi = 0$ on $\partial_* A$, we can apply Theorem 3.14 to the vector fields $u\nabla\varphi$ and $\varphi\nabla u$. As in the proof of Proposition 4.4, we obtain $\int_U u\Delta\varphi = \int_A \varphi\Delta u = 0$, and the proposition follows from [18, Corollary of Theorem 8.12]. □

Remark 4.8. In view of the Hölder and Sobolev inequalities, in Proposition 4.7 the assumption $u \in W^{2,1}_{\rm loc}(U)$ is equivalent to assuming that all second partial derivatives of u belong to $L^1_{\rm loc}(U)$.

Prior to considering the minimal surface equation

$$\mathrm{div} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} = 0, \tag{4.2}$$

recall that according to the regularity result of De Giorgi [8], each weak solution of (4.2) in U which is locally Lipschitz is a real analytic function in U that solves (4.2) in the classical sense. In other words, among locally Lipschitz functions in U there is no difference between weak and classical solutions of (4.2).

Corollary 4.9. *Let $E_{\rm c} \subset E_{\rm d}$ be relatively closed subsets of U that are, respectively, slight and thin, and suppose $u \in C^1(U - E_{\rm c})$. If u solves equation* (4.2) *in $U - E_{\rm d}$, then u can be redefined on $E_{\rm c}$ so that it is locally Lipschitz in U and solves equation* (4.2) *in U.*

Proof. Since the function u is locally Lipschitz in the open set $U - E_{\rm c}$, it is real analytic in the open set $U - E_{\rm d}$. Letting

$$h(x) \coloneqq \frac{x}{\sqrt{1 + |x|^2}}$$

for each $x \in U$, the map $h \circ \nabla u$ is bounded and continuous in $U - E_{\rm c}$, and differentiable in $U - E_{\rm d}$. As $E_{\rm c}$ is a closed subset of U, the map $h \circ \nabla u$ has a bounded extension to U which is still continuous in $U - E_{\rm c}$ and differentiable in $U - E_{\rm d}$. Theorem 4.2 implies that u is a weak solution of (4.2) in U. Consequently u is a classical solution of (4.2) in $U - E_{\rm c}$, and the corollary follows from [20]; cf. Remark 4.10, (ii) below. □

Remark 4.10. Special cases of Corollary 4.9 were obtained previously by various authors. Specifically, the corollary was proved by

(i) De Giorgi and Stampacchia [9] if $E_d = E_c$ is a slight compact subset of U;
(ii) Simon [20] if $E_d = E_c$ is a slight relatively closed subset of U;
(iii) Harvey and Lawson [10] if $E_c = \emptyset$ and the measure $\mathscr{H}^{m-1} \llcorner E_d$ is locally finite; cf. [11].

5. Closing remarks

In U consider the equation

$$\operatorname{div}(h \circ \nabla u) = 0, \tag{5.1}$$

and define a distribution F by the formula $\langle F, \varphi \rangle \coloneqq \int_U (h \circ \nabla u) \cdot \nabla \varphi$ for each $\varphi \in C_c^\infty(U)$. Let E be a relatively closed subset of U. If (5.1) has a weak solution in $U - E$, then $\operatorname{supp} F \subset E$. Moreover, u is a weak solution of (5.1) in U whenever $F = 0$. Stated differently, E is removable whenever $\operatorname{supp} F \subset E$ implies $F = 0$. In Section 4 the conclusion $F = 0$ was inferred from the smallness of E (in the sense of measures $\mathscr{H}^{m-1}$ or $\mathscr{I}_1^{m-1}$) and the regularity results for weak solutions.

The previous paragraph suggest the following definition.

Definition 5.1. Let $\mathscr{F}$ be a family of distributions in an open set $U \subset \mathbb{R}^m$. A collection $\mathscr{E}$ of subsets of U is called *removable* with respect to $\mathscr{F}$ if $F = 0$ for each $F \in \mathscr{F}$ such that $\operatorname{supp} F \subset E$ for some $E \in \mathscr{E}$.

If $v : \mathbb{R}^m \to \mathbb{R}^m$ is a locally bounded Borel vector field, denote by F_v the flux of v (Example 2.3), viewed as a distribution.

Example 5.2. Let $\mathscr{F}_c$ consist of all F_v such that $v \in C^1(\mathbb{R}^m - \operatorname{supp} F_v)$. For instance,

$$u(x) \coloneqq \begin{cases} 0 & \text{if } \xi_1 = \xi_2 = 0, \\ \frac{1}{\sqrt{(\xi_1)^2 + (\xi_2)^2}}(-\xi_2, \xi_1, 0, \dots, 0) & \text{otherwise} \end{cases}$$

for each $x \coloneqq (\xi_1, \dots, \xi_m)$ in $\mathbb{R}^m$, defines $F_u \in \mathscr{F}_c$. We claim the collection of all slight sets is removable with respect to $\mathscr{F}_c$. Indeed, if $F_v \in \mathscr{F}_c$ and $S = \operatorname{supp} F_v$ is a negligible set, then by Theorem 3.11,

$$\int_{\mathbb{R}^m} \varphi \operatorname{div} v = -\int_{\mathbb{R}^m} v \cdot \nabla \varphi = -\langle F_v, \varphi \rangle = 0$$

for every $\varphi \in C_c^\infty(\mathbb{R}^m - S)$. Thus $\operatorname{div} v = 0$ almost everywhere outside S, and consequently almost everywhere in $\mathbb{R}^m$. Now if S is a slight set, then another application of Theorem 3.11 shows that $\langle F_v, \varphi \rangle = 0$ for each $\varphi \in C_c^\infty(\mathbb{R}^m)$. In particular $F_u = 0$, since $\operatorname{supp} F_u$ is a linear space of dimension $m-2$. On the other hand, if h is the indicator of $[0, \infty) \subset \mathbb{R}$ and $w(x) := (h(\xi_1), 0, \ldots, 0)$ for each $x = (\xi_1, \ldots, \xi_m)$ in $\mathbb{R}^m$, then $F_w \in \mathscr{F}_c, F_w \neq 0$, and $\operatorname{supp} F_w$ is not a slight set.

Example 5.3. If $\mathscr{F}_d$ consists of all $F_v \in \mathscr{F}_c$ for which v is continuous, then proceeding as Example 5.2, it is easy to see that the collection of all thin sets is removable with respect to $\mathscr{F}_d$. Let h be the Cantor-Vitali function extended to a continuous function on $\mathbb{R}$ by 0 and 1, and let $w(x) := (h(\xi_1), 0, \ldots, 0)$ for each $x = (\xi_1, \ldots, \xi_m)$ in $\mathbb{R}^m$. Then $F_w \in \mathscr{F}_d$, and as $F_w \neq 0$, we see that $\operatorname{supp} F$ is not a thin set.

It is legitimate to ask whether the removable collections for the families $\mathscr{F}_c$ and $\mathscr{F}_d$ indicated in the previous examples are the largest possible. The next example, based on the continuum hypothesis (CH), may bear on this question with regard to the family $\mathscr{F}_c$.

Example 5.4. Let $K \subset \mathbb{R}^m$ be a compact set with $\mathscr{H}^{m-1}(K) > 0$. According to Frostman's lemma [13, Theorem 8.8], there is a finite Radon measure μ in $\mathbb{R}^m$ such that $\mu(K) > 0, \operatorname{supp} \mu \subset K$, and $\mu[B(x,r)] \leqslant r^{m-1}$ for each $x \in \mathbb{R}^m$ and $r > 0$. In view of [15, Theorem 4.7], we can find a positive constant c so that $|\int_{\mathbb{R}^m} g \, d\mu| \leqslant c||g||$ for each $g \in BV$; in particular,

$$F : g \mapsto \int_{\mathbb{R}^m} g \, d\mu : BV_c^\infty \to \mathbb{R}$$

is a *nontrivial* w-charge with $\operatorname{supp} F \subset K$. Yet, following the argument of [4, Section 3], one can show that under CH there is an $\mathscr{H}^{m-1}$-measurable vector field $v : \mathbb{R}^m \to \mathbb{R}^m$ such that $F(g) = \int_{\mathbb{R}^m} v \cdot d(Dg)$ for every $g \in BV_c^\infty$. In particular, v is (Dg)-measurable for all $g \in BV_c^\infty$, and F is the "flux" of v.

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