Discrete Mathematics 14 (1976) 265–268.

ON DECOMPOSITION OF r-PARTITE GRAPHS INTO EDGE-DISJOINT HAMILTON CIRCUITS

Renu LASKAR and Bruce AUERBACH

Departments of Mathematical Sciences and Administrative Programming, Clemson University, Clemson, S.C. 29631, USA

Received 15 December 1973
Revised 18 June 1974
Final version received 25 April 1975

Let $K(n;r)$ denote the complete r-partite graph $K(n, n, ..., n)$. It is shown here that for all even $n(r-1) \geqslant 2$, $K(n;r)$ is the union of $n(r-1)/2$ of its Hamilton circuits which are mutually edge-disjoint, and for all odd $n(r-1) \geqslant 1$, $K(n;r)$ is the union of $(n(r-1)-1)/2$ of its Hamilton circuits and a 1-factor, all of which are mutually edge-disjoint.

1. Introduction

Let K_n and $K_{m,m}$ denote the complete graph on n vertices and the complete bipartite graph on $2m$ vertices with m vertices in each vertex set, respectively. The following properties of K_n and $K_{m,m}$ have long been known. For all odd $n \geqslant 3$, K_n is the union of $\frac{1}{2}(n-1)$ of its Hamilton circuits (which are mutually edge-disjoint). For all even $n \geqslant 2$, K_n is the union of $\frac{1}{2}n - 1$ of its Hamilton circuits and a 1-factor (which are mutually edge disjoint). For all even $m \geqslant 2$, $K_{m,m}$ is the union of $\frac{1}{2}m$ of its Hamilton circuits (which are mutually edge-disjoint). In addition, Dirac shows that for all odd $m \geqslant 1$, $K_{m,m}$ is the union of $\frac{1}{2}(m-1)$ of its Hamilton circuits and a 1-factor (which are mutually edge-disjoint).

Let $K(n;r)$ denote the complete r-partite graph, $K(n, n, ..., n)$. In this paper we show that for all even $n(r-1) \geqslant 2$, $K(n;r)$ is the union of $\frac{1}{2}n(r-1)$ of its Hamilton circuits (which are mutually edge-disjoint). For all odd $n(r-1) \geqslant 1$, $K(n;r)$ is the union of $\frac{1}{2}(n(r-1)-1)$ of its Hamilton circuits and a 1-factor (which are mutually edge-disjoint).

2. Known results

For definitions see [3]. The number of mutually edge-disjoint Hamilton circuits of a graph G whose union is G will be denoted by $h(G)$ and the complete r-partite graph with n vertices in each set will be denoted by $K(n;r)$.

The following decompositions will be needed.

(2.1) Lucas [1, p. 237] showed that if n is odd, it is possible to pair the numbers 1, 2, ..., n into n sets N_i, $1 \leqslant i \leqslant n$, with each set containing $\frac{1}{2}(n-1)$ pairs of elements, such that the element i is not in N_i. One such decomposition is given by

$$N_i = \{(i+s,\ i-s):\ s = 1, 2, ..., \tfrac{1}{2}(n-1)\},$$

where the numbers $i+s$ and $i-s$ are taken modulo n. The pairs (α, β) in N_i are taken such that $\alpha < \beta$.

(2.2) Lucas also showed that if r is even, it is possible to pair the numbers 1, 2, ..., r into $r-1$ sets R_i for $1 \leqslant i \leqslant r-1$ into $\frac{1}{2}r$ pairs with no elements in common. A decomposition is given by $R_i = N_i \cup \{(i, r)\}$, where N_i is as above. The pairs (α, β) in N_i and R_i are taken such that $\alpha < \beta$.

(2.3) For the graph $K(n; 2)$, let $V = \{v_1, v_2, ..., v_n\}$ and $W = \{w_1, w_2, ..., w_n\}$ be the two vertex sets. We define

$$\alpha_i = \{(v_1, w_i), (v_2, w_{i+1}), ..., (v_k, w_{i+k-1}), ..., (v_n, w_{i-1})\}, \quad \text{for } 1 \leqslant i \leqslant n\,,$$

where the indices are taken modulo n.

(2.4) For the graph $K(2;r)$, let $V_i = \{u_i, v_i\}$, $1 \leqslant i \leqslant r$, be the r vertex sets. We define

$$\gamma_j = \{(u_1, v_j), (u_2, v_{j+1}), ..., (u_k, v_{j+k}), ..., (u_n, v_{j-1})\}, \quad \text{for } 2 \leqslant j \leqslant r\,,$$

where the indices are taken modulo n.

(2.5) $h(K_{2n+1}) = n$ (see [3, p. 89]). To see this, label the vertices 1, 2, ..., $2n+1$, and define

$$P_i = \{i,\ i-1,\ i+1,\ i-2,\ ...,\ i-n,\ i+n\},$$

where all subscripts are taken as the integers 1, 2, ..., $2n$ (mod $2n$). Then $Z_i = P_i \cup 2n+1$ is a spanning cycle of K_{2n+1} for $1 \leqslant i \leqslant n$.

3. Main results

The following two lemmas can be proved easily. Proofs are constructive and are given in detail in [4].

Lemma 1. $h(K(2; 2k + 1)) = 2k$.

Lemma 2. $h(K(2; 2k)) = 2k - 1$.

Theorem. *For n odd and r even, $K(n; r)$ is the union of $\frac{1}{2}(n(r-1)-1)$ edge-disjoint Hamilton circuits and a 1-factor. Otherwise $h(K(n; r)) = \frac{1}{2}n(r-1)$.*

Proof. If n and r are both even, say $n = 2m$ and $r = 2k$, for each of the $2k - 1$ Hamilton circuits given in Lemma 2, m line-disjoint Hamilton circuits of $K(2m; 2k)$ can be constructed. Details of the construction are given in [4]. For the case n even and r odd, using Lemma 1 a similar construction gives the result. The details of the construction for the case in which both n and r odd are given in [4].

If n is odd and r even, let the n sets $N_1, N_2, \ldots, N_n$ and the $r - 1$ sets $R_1, R_2, \ldots, R_{r-1}$ be as described in (2.1) and (2.2). Let (N_i, R_j, γ_{j+1}) denote the subgraph of $K(n; r)$ where all the columns other than the ith one are paired as given in N_i and lines between each such pair of columns are taken as given by γ_{j+1} in (2.4), and the vertices of the ith column are joined as given in R_j. Clearly (N_i, R_j, γ_{j+1}) is a 1-factor and it can be easily checked that the triplets (N_i, R_j, γ_{j+1}), for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, r - 1$, decompose $K(n; r)$ into $n(r - 1)$ 1-factors. It can be proved that $(N_i, R_j, \gamma_{j+1}) \cup (N_k, R_l, \gamma_{l+1})$ with $i \neq k$, $j \neq l$ will give a Hamilton circuit of $K(n; r)$. It can be shown also that it is possible to form $\frac{1}{2}(n(r-1)-1)$ pairs of line-disjoint 1-factors $[(N_i, R_j, \gamma_{j+1}) \cup (N_k, R_l, \gamma_{l+1})]$ with $i \neq k$, $j \neq l$ and a 1-factor remains [4].

Example. K_5^4. $n = 5, r = 4, N_1 = (2,5), (3,4); N_2 = (1,3), (4,5); N_3 = (2,4), (1,5); N_4 = (3,5), (1,2); N_5 = (1,4), (2,3); R_1 = (1,4), (2,3); R_2 = (2,4), (1,3); R_3 = (3,4), (1,2)$. The Hamilton circuits are

$$H_1 = (N_2, R_1, \gamma_2) \cup (N_1, R_2, \gamma_3),$$
$$H_2 = (N_3, R_1, \gamma_2) \cup (N_1, R_3, \gamma_4),$$
$$H_3 = (N_3, R_2, \gamma_3) \cup (N_2, R_3, \gamma_4),$$
$$H_4 = (N_2, R_2, \gamma_3) \cup (N_3, R_3, \gamma_4),$$
$$H_5 = (N_4, R_1, \gamma_2) \cup (N_5, R_2, \gamma_3),$$
$$H_6 = (N_4, R_2, \gamma_3) \cup (N_5, R_3, \gamma_4),$$
$$H_7 = (N_5, R_1, \gamma_2) \cup (N_4, R_3, \gamma_4),$$

and

$$F_1 = (N_1, R_1, \gamma_2) \text{ is the 1-factor}.$$

References

[1] C. Berge, Graphes et hypergraphes (Dunod, Paris, 1970).
[2] G.A. Dirac, On Hamiltonian circuits and Hamiltonian paths, Math. Ann. 197 (1972) 57–70.
[3] F. Harary, Graph Theory (Addison–Wesley, Reading, Mass., 1969).
[4] R. Laskar and B. Auerbach, On Hamilton circuits of r-partite graphs, Clemson University, preprint, November 1973.
[5] E. Lucas, Récréations Mathematiques, I-IV, Vol. II (Paris, 1882–1894) 162–168.