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Nonexistence for mixed-type equations with critical exponent nonlinearity in a ball

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ABSTRACT

In this work, we consider the following isotropic mixed-type equations:

$$y|y|^{\alpha-1}u_{xx} + x|x|^{\alpha-1}u_{yy} = f(x, y, u) \tag{0.1}$$

in $B_r(0) \subset R^2$ with $r > 0$. By proving some Pohozaev-type identities for (0.1) and dividing $B_r(0)$ naturally into six regions Ω_i $(i = 1, 2, 3, 4, 5, 6)$, we can show that the equation

$$yu_{xx} + xu_{yy} = \text{sign}(x + y)|u|^2 u \tag{0.2}$$

with Dirichlet boundary conditions on each natural domain Ω_i has no nontrivial regular solution in $B_r(0)$.

1. Introduction

In this work, we consider the following isotropic mixed-type equations:

$$y|y|^{\alpha-1}u_{xx} + x|x|^{\alpha-1}u_{yy} = f(x, y, u) \tag{1.1}$$

in $B_r(0)$ for any given $r > 0$.

Tricomi problems and mixed-type equations like (1.1) with $f(u) = 0$ or $f(u) = \lambda u$ and other types have been widely considered (see [1–14]). The existence and uniqueness were obtained in these earlier papers under suitable conditions on different domains. For example, the mixed-type equation

$$Lu = \text{sign}\, t \cdot |t|^m u_{xx} + u_{tt} - b^2 \text{sign}\, t \cdot |t|^m u = 0$$

with $m = \text{const} > 0$ and $b = \text{const} \geq 0$ was considered in [4,13] in the rectangular domain $D = \{(x, t)|0 < x < 1,\ -\alpha < t < \beta\}$ and a criterion for the uniqueness and existence of a solution to this equation with certain conditions was established by applying a method of spectral analysis for boundary value problems.

In the present work, we will consider the mixed-type Eq. (1.1) in a usual domain, that is, in a ball $B_r(0)$ with $r > 0$. It is well known that the semilinear elliptic equation

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

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has no positive solution in $H_0^1(\Omega)$ if the bounded domain $\Omega \subset R^N$ ($N \geq 3$) is star-shaped with respect to some interior point and $f(u) = |u|^{p-1}u$ with $p \geq 2^* - 1 = \frac{N+2}{N-2}$ (see [10]), especially for $\Omega = B_r(0)$. It is interesting to ask whether a mixed-type equation like (1.1) leads to a similar result on a ball. We will show that the answer is positive.

To give our results, we divided $B_r(0)$ naturally into six domains Ω_i ($i = 1, 2, 3, 4, 5, 6$) using the x-axis, y-axis and characteristic line $y = -x$, where $\Omega_1 = \left\{(t\cos\theta, t\sin\theta)|0 \leq t \leq r, -\frac{\pi}{4} \leq \theta \leq 0\right\}$, $\Omega_2 = \left\{(t\cos\theta, t\sin\theta)|0 \leq t \leq r, 0 \leq \theta \leq \frac{\pi}{2}\right\}$, $\Omega_3 = \left\{(t\cos\theta, t\sin\theta)|0 \leq t \leq r, \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{4}\right\}$. By the symmetry of $B_r(0)$, we can set $\Omega_4 = -\Omega_1 = \{(x, y) : (-x, -y) \in \Omega_1\}$, $\Omega_5 = -\Omega_2$ and $\Omega_6 = -\Omega_3$ and $OA = \Omega_1 \bigcap \Omega_6$ with $O = (0, 0)$ and $A = \left(\frac{\sqrt{2}}{2}r, -\frac{\sqrt{2}}{2}r\right)$, $OB = \Omega_1 \bigcap \Omega_2$ with $B = (r, 0)$, $OC = \Omega_2 \bigcap \Omega_3$ with $C = (0, r)$ and $OD = \Omega_3 \bigcap \Omega_4$ with $D = \left(-\frac{\sqrt{2}}{2}r, \frac{\sqrt{2}}{2}r\right)$. Then by assuming that $u = 0$ on each boundary of the domains Ω_i, we can show that $u \equiv 0$ in $B_r(0)$.

Since the shape of the domain that we consider is different from that of [4,13] and there are nonlinearities in our case, we use a method different from those of [4,13]. We follow the approach of [7–9,15] to get some identities with conservation laws; then we will show that Eq. (0.2) has no nontrivial solution. The difference is that in [6–9], the Tricomi problem was considered for a set-up not isotropic in x, y and with a domain different from ours. Problems (1.1) make it more feasible to consider a natural ball.

In Section 2, we get some identities for Eq. (1.1) in $B_r(0)$. In Section 3, we prove the nonexistence for Eq. (0.2) in each domain Ω_i ($i = 1, 2, 3, 4, 5, 6$) with the help of the identities that we obtained in Section 2.

2. Conservation laws and identities

In this section, we consider Eq. (1.1) on a star-shaped domain $B_r(0)$ with power-type nonlinearities $f(x, y, u) = \mu|u|^{p-1}u$. We use conservation laws inspired by [7,8] to prove some identities for Eq. (1.1).

For any given $\gamma > 0$, we consider the one-parameter family of homogeneous dilations Φ_λ and the scaled functions u_λ defined by

$$u_\lambda(x, y) = \Phi_\lambda u(x, y) = \lambda^\gamma u(x/\lambda, y/\lambda). \tag{2.1}$$

Following direct calculations, it is easy to see that if u is a solution of (1.1) with power-type nonlinearity $f(x, y, u) = \mu|u|^{p-1}u$ (where $p \geq 1$), so is u_λ for any $\lambda > 0$ whenever $\gamma = \frac{\alpha-2}{p-1}$. Hence we have a multiplicative group R^+ of dilations as a symmetry group and an infinitesimal generator

$$Mu = \frac{\mathrm{d}}{\mathrm{d}\lambda}\bigg|_{\lambda=1} u_\lambda = \frac{\alpha - 2}{p - 1}u - xu_x - yu_y \tag{2.2}$$

as a multiplier (see [7,8,15] for more details); then we have the following results:

Theorem 2.1. *Suppose that $u \in C^2(B_r(0))$ is a solution of the equation*

$$y|y|^{\alpha-1}u_{xx} + x|x|^{\alpha-1}u_{yy} = f(u), \quad (x, y) \in B_r(0) \subset \mathbb{R}^2 \tag{2.3}$$

where $\alpha, r > 0$ and $f(t) = \mu|t|^{p-1}t$ with $p \geq 1$; we have the identities

$$\operatorname{div}\{yu_xMu + x[F(u) + L_0u], xu_yMu + y[F(u) + L_0u]\} = 2F(u) - \frac{\alpha}{2}uf(u), \tag{2.4}$$

where

$$L_0u = \frac{y|y|^{\alpha-1}u_x^2 + x|x|^{\alpha-1}u_y^2}{2}, \tag{2.5}$$

Mu is as given in (2.2) and $F(t) = \int_0^t f(s)\mathrm{d}s$ is a primary function of $f(t)$.

Proof. By multiplication of (2.3) with u followed by direct calculations, we get

$$\begin{aligned} \operatorname{div}\{y|y|^{\alpha-1}uu_x, x|x|^{\alpha-1}uu_y\} &= y|y|^{\alpha-1}u_x^2 + x|x|^{\alpha-1}u_y^2 + uf(u) \\ &= 2L_0u + uf(u). \end{aligned} \tag{2.6}$$

Also, multiplying (2.3) with xu_x and yu_y separately, we find

$$\operatorname{div}\{(xu_x)(y|y|^{\alpha-1}u_x, x|x|^{\alpha-1}u_y)\} = y|y|^{\alpha-1}u_x^2 + 2xy|y|^{\alpha-1}u_xu_{xx} + |x|^{\alpha+1}u_yu_{xy} + |x|^{\alpha+1}u_xu_{yy} \tag{2.7}$$

and

$$\operatorname{div}\{(yu_y)(y|y|^{\alpha-1}u_x, x|x|^{\alpha-1}u_y)\} = |y|^{\alpha+1}u_yu_{xx} + |y|^{\alpha+1}u_xu_{xy} + x|x|^{\alpha-1}u_y^2 + 2yx|x|^{\alpha-1}u_yu_{yy}. \tag{2.8}$$

Combining (2.6)–(2.8), we derive that

$$\begin{aligned}
\operatorname{div}[(y|y|^{\alpha-1}u_x, x|x|^{\alpha-1}u_y)Mu] &= \frac{\alpha-2}{1-p}uf(u) + \frac{\alpha-2}{p-1}(y|y|^{\alpha-1}u_x^2 + x|x|^{\alpha-1}u_y^2) \\
&\quad - (xu_x)[y|y|^{\alpha-1}u_{xx} + x|x|^{\alpha-1}u_{yy}] - (y|y|^{\alpha-1}u_x^2 + xy|y|^{\alpha-1}u_xu_{xx} + |x|^{\alpha+1}u_yu_{xy}) \\
&\quad - (yu_y)[x|x|^{\alpha-1}u_{yy} + y|y|^{\alpha-1}u_{xx}] - (x|x|^{\alpha-1}u_y^2 + yx|x|^{\alpha-1}u_yu_{yy} + |y|^{\alpha+1}u_xu_{xy}) \\
&= f(u)Mu + \frac{\alpha-2}{p-1}(y|y|^{\alpha-1}u_x^2 + x|x|^{\alpha-1}u_y^2) \\
&\quad - (y|y|^{\alpha-1}u_x^2 + xy|y|^{\alpha-1}u_xu_{xx} + |x|^{\alpha+1}u_yu_{xy}) \\
&\quad - (x|x|^{\alpha-1}u_y^2 + yx|x|^{\alpha-1}u_yu_{yy} + |y|^{\alpha+1}u_xu_{xy}). \qquad (2.9)
\end{aligned}$$

In addition we have that

$$\begin{aligned}
\operatorname{div}\{(x, y)F(u)\} &= 2F(u) + (xu_x + yu_y)f(u) \\
&= 2F(u) + \left(\frac{\alpha-2}{p-1}u - Mu\right)f(u) \qquad (2.10)
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{div}\{(x, y)L_0u\} &= \operatorname{div}\left\{(x, y)\left(y|y|^{\alpha-1}u_x^2 + x|x|^{\alpha-1}u_y^2\right)/2\right\} \\
&= \frac{1}{2}y|y|^{\alpha-1}u_x^2 + xy|y|^{\alpha-1}u_xu_{xx} + \frac{\alpha+1}{2}x|x|^{\alpha-1}u_y^2 + |x|^{\alpha+1}u_yu_{xy} \\
&\quad + \frac{\alpha+1}{2}y|y|^{\alpha-1}u_x^2 + |y|^{\alpha+1}u_xu_{xy} + \frac{1}{2}x|x|^{\alpha-1}u_y^2 + yx|x|^{\alpha-1}u_yu_{yy}. \qquad (2.11)
\end{aligned}$$

Combining (2.9)–(2.11), we have that

$$\begin{aligned}
&\operatorname{div}\{y|y|^{\alpha-1}u_xMu + x[F(u) + L_0u], x|x|^{\alpha-1}u_yMu + y[F(u) + L_0u]\} \\
&\quad = 2F(u) + \frac{\alpha-2}{p-1}uf(u) + \left(\frac{2(\alpha-2)}{p-1} + \alpha\right)L_0u. \qquad (2.12)
\end{aligned}$$

Then combining (2.12) with (2.6), we finally get that

$$\begin{aligned}
&\operatorname{div}\left\{yu_xMu - \left(\frac{\alpha-2}{p-1} + \frac{\alpha}{2}\right)y|y|^{\alpha-1}uu_x + x[F(u) + L_0u], xu_yMu - \left(\frac{\alpha-2}{p-1} + \frac{\alpha}{2}\right)x|x|^{\alpha-1}uu_y + y[F(u) + L_0u]\right\} \\
&\quad = 2F(u) - \frac{\alpha}{2}uf(u) \qquad (2.13)
\end{aligned}$$

and Theorem 2.1 is proved. □

Suppose that f is a power-type nonlinearity $f(t) = \mu|t|^{p-1}t$ where $p = \frac{4}{\alpha} - 1$ for any $1 \le \alpha \le 2$; it is obvious that $2F(t) - \frac{\alpha}{2}tf(t) = 0$. So, $p = \frac{4}{\alpha} - 1$ is called the *critical exponent* for (1.1). Similarly, multiplying (1.2) with $x \cdot \nabla u$, we have

$$\operatorname{div}\left(\nabla ux \cdot \nabla u - x\frac{|\nabla u|^2}{2} + xF(u)\right) = NF(u) - \frac{N-2}{2}uf(u)$$

and the critical exponent for (1.2) with $f(u) = |u|^{p-1}u$ is $p = \frac{N+2}{N-2}$.

A directly corollary of Theorem 2.1for mixed-type Eq. (1.1) with critical exponent nonlinearity reads as follows:

Corollary 2.1. *Suppose that $u \in C^2(B_r(0))$ is a solution of the equation*

$$y|y|^{\alpha-1}u_{xx} + x|x|^{\alpha-1}u_{yy} = \mu|u|^{\frac{4}{\alpha}-1}, \quad (x, y) \in B_r(0) \subset \mathbb{R}^2 \qquad (2.14)$$

for any $\alpha \in [1, 2]$; then we have the conservation law

$$\operatorname{div}\{y|y|^{\alpha-1}u_xMu + x(F(u) + L_0u), x|x|^{\alpha-1}u_yMu + y(F(u) + L_0u)\} = 0. \qquad (2.15)$$

3. Nonexistence

In [1–5,10–14] and other papers, the existence and uniqueness of solutions for equations like (1.1) with linearities $f(x, y, u)$ were obtained on kinds of domains with different boundary conditions. In these papers, the uniqueness was proved directly. In the present work, we will prove the uniqueness of Eq. (1.1) with a power-type critical nonlinearity $f(x, y, u)$ in a different way. We will give a proof of the uniqueness of (1.1) with $\alpha = 1$ below. One can see from the proof of Theorem 3.1 that with $\alpha = 1$, it is natural to divide $B_r(0)$ into domains Ω_i $(i = 1, 2, 3, 4, 5, 6)$ by using the x-axis, the y-axis and the characteristic line $\{(x, y) : x + y = 0\}$; the proof will be clearer.

In this section, we consider Eq. (1.1) on $B_r(0)$ with $\alpha = 1$ and $f(x, u) = \text{sign}(x + y)|u|^{p-1}u$, where $p = 3$ is the critical exponent (see Corollary 2.1). That is, we consider the following equation:

$$yu_{xx} + xu_{yy} = \text{sign}(x + y)|u|^2u, \quad (x, y) \in B_r(0) \subset \mathbb{R}^2. \tag{3.1}$$

We will use the identities that we got in Section 2 to prove our results in this section. To prove our results, we set the following boundary conditions:

$$u|_{\bigcup_{i=1}^{6} \partial\Omega_i} = 0. \tag{3.2}$$

Note that one can get the same results for the linearity $f(x, y)$ in the same way; we omit this here.

Theorem 3.1. *Let $u \in C^2(B_r(0))$ be a solution of Eq. (3.1) satisfying the boundary condition (3.2); then $u \equiv 0$ in $\Omega_1 \bigcup \Omega_4$.*

Proof. We will give the proof for $u \equiv 0$ in Ω_1 only. Since $x + y > 0$ in Ω_1 except for the points on segment OA, we have that $f(u) = |u|^2u$ is of power type with a critical exponent and Ω_1 is simply connected and star-shaped with respect to the origin $O = (0, 0)$. By Corollary 2.1, we have the conservation law $\text{div}(U_1, U_2) = 0$ where

$$\begin{aligned} U_1(x, y) &= 2xF(u) - yuu_x - xyu_x^2 - 2y^2u_xu_y + x^2u_y^2, \\ U_2(x, y) &= 2yF(u) - xuu_y - xyu_y^2 - 2x^2u_xu_y + y^2u_x^2. \end{aligned} \tag{3.3}$$

Since Ω_1 is simply connected, the conservative vector field $V = (V_1, V_2) = (U_2, -U_1)$ admits a potential function φ; that is, we have

$$\begin{aligned} \varphi_x &= V_1 = U_2, \\ \varphi_y &= V_2 = -U_1. \end{aligned} \tag{3.4}$$

In fact, we can define

$$\varphi(P) = \int_{\Gamma_P} V_1\text{d}x + V_2\text{d}y, \quad P \in \overline{\Omega}_1 \tag{3.5}$$

where Γ_P is a segment from $O = (0, 0)$ to the point $P \in \overline{\Omega}_1$.

Without loss of generality, we take $r = 1$ for $B_r(0)$. Then, for each $P = (x, 0) \in OB$, we can parameterize $\Gamma_P(t) = (tx, 0)$ with $t \in [0, 1]$ to find

$$\varphi(x, 0) = \int_0^x V_1(t, 0)\text{d}t$$

and so

$$\varphi_x(x, 0) = V_1(x, 0) = -xuu_y - 2x^2u_xu_y.$$

Since $u(x, 0) \equiv 0$ for each $x \in [0, 1]$, φ is constant on OB and vanishes at $O(0, 0)$, so it vanishes identically, which implies that

$$\varphi(B) = \varphi(O) = 0. \tag{3.6}$$

On $\widehat{AB}$, we define

$$v(\theta) = \varphi(\cos\theta, \sin\theta), \quad \theta \in \left[-\frac{\pi}{4}, 0\right]. \tag{3.7}$$

Since $u \equiv 0$ along $\widehat{AB}$, by (3.3) we have that

$$\begin{aligned} v'(\theta) &= -yV_1 + xV_2 \\ &= -x[2xF(u) - yuu_x - xyu_x^2 - 2y^2u_xu_y + x^2u_y^2] - y[2yF(u) - xuu_y - xyu_y^2 - 2x^2u_xu_y + y^2u_x^2] \\ &= -2(x^2 + y^2)F(u) + xyu(u_x + u_y) + 2xy(x + y)u_xu_y + (x^2y - y^3)u_x^2 + (xy^2 - x^3)u_y^2 \\ &= (x + y)[2xyu_xu_y + (x - y)(yu_x^2 - xu_y^2)]. \end{aligned} \tag{3.8}$$

Note that $u_\theta = -yu_x + xu_y$ on $\widehat{AB}$. Then, it follows from $u \equiv 0$ on $\widehat{AB}$ that

$$-yu_x + xu_y = 0 \tag{3.9}$$

on $\widehat{AB}$. Inserting (3.9) into (3.8) gives the expression

$$\begin{aligned} v'(\theta) &= (x+y)[2x^2u_y^2 + (x-y)(x^2u_y^2 - xyu_y^2)/y] \\ &= u_y^2[(x^2+y^2)x(x+y)/y] \\ &= u_y^2(\cos\theta + \sin\theta)\cos\theta/\sin\theta \\ &\le 0 \end{aligned} \tag{3.10}$$

for $-\frac{\pi}{4} < \theta < 0$. This implies that for any $P \in \widehat{AB}$

$$\varphi(B) \le \varphi(P) \le \varphi(A). \tag{3.11}$$

Next we examine φ along characteristic segments. For each $P = (x, -x) \in OA$ we use the parameterization

$$\Gamma(t) = (t, -t), \quad t \in [0, x]. \tag{3.12}$$

Setting

$$\begin{aligned} w(x) &= \varphi(\Gamma(t)) \\ &= \int_0^x V_1(t,-t)\mathrm{d}t - \int_0^x V_2(t,-t)\mathrm{d}t \end{aligned}$$

and $\psi(x) = u(\Gamma(x))$, for $0 < x < \frac{\sqrt{2}}{2}$ we have that

$$\begin{aligned} w'(x) &= V_1(x,-x) - V_2(x,-x) \\ &= xu(u_x - u_y) - 4x^2u_xu_y + 2x^2(u_x^2 + u_y^2) \\ &= xu(u_x - u_y) - 2x^2(u_x - u_y)^2 \\ &= x\psi(x)\psi'(x) - 2x^2[\psi'(x)]^2. \end{aligned} \tag{3.13}$$

Since $\psi(x) = u(\Gamma(x)) \equiv 0$ implies that $\psi'(x) \equiv 0$ on $\left(0, \frac{\sqrt{2}}{2}\right)$, from (3.13) we have that

$$w'(x) \equiv 0, \quad \text{on } \left(0, \frac{\sqrt{2}}{2}\right). \tag{3.14}$$

(3.14) implies that

$$\varphi(A) = w\left(\frac{\sqrt{2}}{2}\right) = w(0) = 0. \tag{3.15}$$

Consequently, combining (3.6) and (3.11) with (3.15) we get that for any $P \in \widehat{AB}$, $0 = \varphi(B) \le \varphi(P) \le \varphi(A) = 0$. Hence $\varphi|_{\widehat{AB}} = 0$.

Finally, we show that $u \equiv 0$ in Ω_1. To prove that, we consider u on the arc $\widehat{QP} = \{\Gamma(\theta) = (\tau\cos\theta, \tau\sin\theta); \theta \in [-\pi/4, 0]\}$ for some $0 < \tau < 1$ with P on segment OB and Q on segment OA. Then

$$\begin{aligned} 0 &= \varphi(Q) - \varphi(P) \\ &= \int_{-\frac{\pi}{4}}^0 (xV_2 - yV_1)\mathrm{d}\theta \\ &= \int_{-\frac{\pi}{4}}^0 [-2\tau^2F(u) + xyu(u_x + u_y) + 2xy(x+y)u_xu_y + y(x^2 - y^2)u_x^2 + x(y^2 - x^2)u_y^2]\mathrm{d}\theta \\ &= \int_{-\frac{\pi}{4}}^0 [-2\tau^2F(u)]\mathrm{d}\theta + \mathrm{I} + \mathrm{II} \end{aligned} \tag{3.16}$$

where

$$\mathrm{I} = \int_{-\frac{\pi}{4}}^0 xyu(u_x + u_y)\mathrm{d}\theta \tag{3.17}$$

and

$$\mathrm{II} = \int_{-\frac{\pi}{4}}^{0} 2xy(x+y)u_x u_y + y(x^2-y^2)u_x^2 + x(y^2-x^2)u_y^2 \mathrm{d}\theta. \tag{3.18}$$

Note that $u_\theta = xu_y - yu_x$; we have that

$$\begin{aligned}\mathrm{I} &= \int_{-\frac{\pi}{4}}^{0} xyu(u_x+u_y)\mathrm{d}\theta = \int_{-\frac{\pi}{4}}^{0} [xyuu_x + yuxu_y]\mathrm{d}\theta \\ &= \int_{-\frac{\pi}{4}}^{0} [yu(xu_y - yu_x)] + [y^2uu_x + xyuu_x]\mathrm{d}\theta \\ &= \int_{-\frac{\pi}{4}}^{0} [yuu_\theta + y(x+y)uu_x]\mathrm{d}\theta. \end{aligned} \tag{3.19}$$

On one hand, on $\widehat{QP}$, we have that

$$\int_{-\frac{\pi}{4}}^{0} yuu_\theta \mathrm{d}\theta = \int_{-\frac{\pi}{4}}^{0} y\mathrm{d}\left(\frac{u^2}{2}\right) = y\left(\frac{u^2}{2}\right)\Bigg|_{-\pi/4}^{0} - \int_{-\frac{\pi}{4}}^{0} x\left(\frac{u^2}{2}\right)\mathrm{d}\theta, \tag{3.20}$$

$$|y(x+y)uu_x| = (x+y)|uyu_x| \le (x+y)u^2/2 + (x+y)y^2u_x^2/2. \tag{3.21}$$

So, from (3.19)–(3.21), we have that

$$\mathrm{I} \le \int_{-\frac{\pi}{4}}^{0} [yu^2/2 + (x+y)y^2u_x^2/2]\mathrm{d}\theta. \tag{3.22}$$

On the other hand,

$$\mathrm{II} = \int_{-\frac{\pi}{4}}^{0} [xy(x+y)(u_x+u_y)^2 - (x+y)(x^2u_y^2 + y^2u_x^2)]\mathrm{d}\theta. \tag{3.23}$$

Hence, from (3.16), (3.22) and (3.23) we get that

$$0 \le \int_{-\frac{\pi}{4}}^{0} \left[-2\tau^2F(u) + y\frac{u^2}{2} + xy(x+y)(u_x+u_y)^2 - (x+y)\left(x^2u_y^2 + y^2\frac{u_x^2}{2}\right)\right]\mathrm{d}\theta \le 0.$$

Note that the integral in (3.19) is strictly negative unless $u \equiv u_x \equiv u_y \equiv 0$; hence $u \equiv 0$ on $\widehat{QP} = \{(\tau\cos\theta, \tau\sin\theta), \theta \in [-\pi/4, 0]\}$. By the arbitrariness of τ, we have that $u \equiv 0$ in Ω_1 and Theorem 3.1 is proved. □

Then we give the uniqueness on Ω_3 stated as follows:

Theorem 3.2. *Let $u \in C^2(B_r(0))$ be a solution of Eq. (3.1) satisfying the boundary condition (3.2); then $u \equiv 0$ in $\Omega_3 \bigcup \Omega_6$.*

Proof. Since in Ω_3, $x+y > 1$ except at the points on the characteristic line OD, we have $f(x, y, u) = |u|^2u$ which is of power type with a critical exponent. Note that Ω_3 is star-shaped too; by Corollary 2.1 we have the conservation law $\mathrm{div}(U_1, U_2) = 0$ as in Theorem 3.1. Then we have equations which are similar to (3.3) and (3.5) for any $(x, y) \in \Omega_3$, to (3.8)–(3.10) for any (x, y) on $\widehat{CD}$, and to (3.12), (3.14) and (3.15) for any $(x, y) \in OD$. Finally, we get that $\varphi|_{\partial\Omega_3} = 0$.

Next we will show that $u \equiv 0$ in Ω_3. In the same way as in Theorem 3.1, we consider the arc $\{\Gamma(\theta) = (\tau\cos\theta, \tau\sin\theta);\ \theta \in [\pi/2, 3\pi/4]\}$ for any given $0 < \tau < r$; then we have

$$\begin{aligned} 0 &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} (x\varphi_y - y\varphi_x)\mathrm{d}\theta \\ &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} [-2\tau^2F(u) + xyu(u_x+u_y) + 2xy(x+y)u_xu_y + y(x^2-y^2)u_x^2 + x(y^2-x^2)u_y^2]\mathrm{d}\theta \\ &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} [-2\tau^2F(u)]\mathrm{d}\theta + \mathrm{I} + \mathrm{II} \end{aligned} \tag{3.24}$$

where

$$\mathrm{I} = \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} xyu(u_x+u_y)\mathrm{d}\theta$$

and

$$\mathrm{II} = \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} 2xy(x+y)u_x u_y + y(x^2-y^2)u_x^2 + x(y^2-x^2)u_y^2 \mathrm{d}\theta.$$

Note that $u_\theta = xu_y - yu_x$; we have that

$$\begin{aligned} \mathrm{I} &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} xyu(u_x+u_y)\mathrm{d}\theta = \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} [xuyu_x + xyuu_y]\mathrm{d}\theta \\ &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} [xu(yu_x - xu_y)] + [x^2uu_y + xyuu_y]\mathrm{d}\theta \\ &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} [-xuu_\theta + x(x+y)uu_y]\mathrm{d}\theta. \end{aligned} \tag{3.25}$$

In fact, on the arc $\{\Gamma(\theta) = (\tau\cos\theta, \tau\sin\theta);\ \theta \in [\pi/2, 3\pi/4]\}$ we have that

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} xuu_\theta \mathrm{d}\theta = \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} x\mathrm{d}\left(\frac{u^2}{2}\right) = x\left(\frac{u^2}{2}\right)\Big|_{\pi/2}^{3\pi/4} + \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} y\left(\frac{u^2}{2}\right)\mathrm{d}\theta, \tag{3.26}$$

$$|x(x+y)uu_y| = (x+y)|xu_y u| \le (x+y)u^2/2 + (x+y)x^2u_y^2/2. \tag{3.27}$$

So, from (3.25)–(3.27), we have that

$$\mathrm{I} \le \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} [xu^2/2 + (x+y)x^2u_y^2/2]\mathrm{d}\theta. \tag{3.28}$$

Also, it follows from (3.24), (3.23) and (3.28) that

$$0 \le \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \left[-\tau^2|u|^4/2 + x\frac{u^2}{2} + xy(x+y)(u_x+u_y)^2 - (x+y)\left(x^2\frac{u_y^2}{2} + y^2u_x^2\right)\right]\mathrm{d}\theta \le 0. \tag{3.29}$$

Note that in Ω_6, we have $f(x, y, u) = -|u|^2u$. In the same way as above, we finally get

$$0 \ge \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \left[\tau^2|u|^4/2 + x\frac{u^2}{2} + xy(x+y)(u_x+u_y)^2 - (x+y)\left(x^2\frac{u_y^2}{2} + y^2u_x^2\right)\right]\mathrm{d}\theta \ge 0 \tag{3.30}$$

where we use

$$|x(x+y)uu_y| = (x+y)|u \cdot xu_y| \ge (x+y)u^2/2 + (x+y)x^2u_y^2/2,$$

instead of (3.27). By (3.29) and (3.30) and the arbitrariness of τ, we get that $u \equiv 0$ in $\Omega_3 \bigcup \Omega_6$ and Theorem 3.3 is proved. □

Then we give the uniqueness on Ω_2 and Ω_5, that is, we have:

Theorem 3.3. *Let $u \in C^2(B_r(0))$ be a solution of Eq. (3.1) satisfying the boundary condition (3.2); then $u \equiv 0$ in $\Omega_2 \bigcup \Omega_5$.*

Proof. Note that both Ω_2 and Ω_5 are elliptic domains; by the Hopf maximum principle, one can show that u cannot reach its positive maximum or negative minimum in Ω_2, so it does this in Ω_5. That is, $u \equiv 0$ in $\Omega_2 \bigcup \Omega_5$ if (3.2) is satisfied. □

From Theorems 3.2–3.4 we have:

Theorem 3.4. *Let $u \in C^2(B_r(0))$ be a solution of Eq. (3.1) satisfying the boundary condition (3.2); then $u \equiv 0$ in $B_r(0)$.*

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