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# Categories with finite limits and stable binary coproducts can be subdirectly decomposed

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*Abstract*

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Categories in which the binary coproduct is preserved by pulling back are of particular relevance to computer science. An important subclass of such categories are those which are finitely complete and have disjoint coproducts, *distributive categories*, as they are a natural setting for the study of data structures.

Unfortunately, stability of binary coproducts does not imply disjointness of coproducts. The simplest counter-example to this is provided by a nontrivial distributive lattice. However, a finitely complete category with stable coproducts may always be subdirectly decomposed into a distributive poset and a distributive category. Furthermore, the distributive component occurs as a reflexive subcategory.

## 1. Introduction

With the increased interest in using 'term models' (initial categories) to provide semantics for programming languages, one would dearly like these settings to have disjoint coproducts which are preserved by pulling back. It often turns out, however, even when they are finitely complete, that this is not the case. The main problem concerns the initial object which is rarely present explicitly because no computation is of that type. Despite this, it is often natural to have binary coproducts and to demand, if it is not already the case, that they are preserved by pulling back. With this condition (and, in fact, the more general *weak stability* property) while there may be no initial object there are certainly many preinitial objects.

The property of being preserved by pullbacks is often called *stability*. Accordingly, categories with stable binary coproducts shall be called *stable* categories,

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and shall have both pullbacks and stable binary coproducts. *Disjoint* stable categories then have stable binary coproducts, an initial object, and disjoint coproducts. Coproducts are disjoint in case the embeddings of all coproducts intersect at the initial object. In developing the properties of these categories we shall start by considering *weakly stable* categories, that is, categories with binary coproducts but in which pullbacks and stability of coproducts are only guaranteed over coproduct embeddings.

In weakly stable categories coproducts may not be disjoint but their embeddings intersect at preinitial objects. Furthermore, the full subcategory of preinitial objects forms a coreflexive subcategory which is equivalent to a distributive poset. If there is a largest preinitial object, which certainly happens in the presence of a final object, then there is a full subcategory of the weakly stable category which has disjoint coproducts and a reflecting functor. These functors, the coreflector to the full subcategory of preinitials and the reflector to the full subcategory with disjoint unions, provide for stable categories a full and faithful stable subdirect decomposition. This means that all stable categories with a final object are (equivalent to) a stable full subcategory of a categorical product of a distributive poset and a distributive category.

Distributive categories were introduced by Bill Lawvere and Stephen Schanuel, who has studied isomorphism classes of various distributive categories through the properties of the Burnside 'rig'. Bob Walters has considered distributive categories for the study of data structures [4], in this regard they are of considerable interest as they permit the use of the coproduct in specification which cannot be accomplished employing ordinary algebraic specification methods. The author has also used distributive categories as the basis for studying list arithmetic [1, 2] and the observations of this paper allow a small generalization to stable categories for those results.

## 2. Notation

The binary coproduct of two objects $A$ and $B$ is denoted by $A + B$. The coproduct embeddings are denoted by

$$b_0 : A \to A + B , \qquad b_1 : B \to A + B .$$

A category with binary coproducts is *weakly stable* if

(i) pullbacks along coproduct embeddings always exist, that is, if $h$ is a coproduct embedding and $g$ is arbitrary, then the pullback

$$([h \wedge g], p_0^{h \wedge g}, p_1^{h \wedge g})$$

exists allowing the pullback square

$$\begin{array}{ccc} [h \wedge g] & \xrightarrow{p_0^{h \wedge g}} & B \\ {\scriptstyle p_1^{h \wedge g}} \downarrow & & \downarrow {\scriptstyle g} \\ X & \xrightarrow{h} & Y \end{array}$$

to be formed;

(ii) these pullbacks preserve coproducts, that is,

$$\begin{array}{ccc} [h \wedge f] + [h \wedge g] & \xrightarrow{\langle p_0^{h \wedge f} + p_0^{h \wedge g} \rangle} & A + B \\ {\scriptstyle \langle p_1^{h \wedge f};\, p_1^{h \wedge g} \rangle} \downarrow & & \downarrow {\scriptstyle \langle f;\, g \rangle} \\ X & \xrightarrow[h]{} & Y \end{array}$$

when $h$ is a coproduct embedding the square is a pullback.

A functor which preserves the coproducts and pullbacks along embeddings is a *weakly stable functor*.

An important source of weakly stable categories are categories with pullbacks and *stable* coproducts, that is, coproducts which satisfy the second property above, without the restriction that $h$ be a coproduct embedding. These we shall refer to as *stable categories*. Functors which preserve pullbacks (in so far as they exist) and coproducts shall be called *stable functors*.

A weakly stable category has *disjoint coproducts* if it has an initial object and the pullback of the embeddings into any coproduct is always initial. A weakly stable category is said to be a *disjoint weakly stable category* if it has disjoint coproducts.

A source of examples of weakly stable categories important to this paper is posets. It is easy to see that posets with a binary distributive join and meet is exactly what both a *weakly stable* and a *stable poset* must be, this we call a *distributive poset*. The join is given by the coproduct while the meet is given by the pullback of the embeddings into the coproduct. An observation, which is central to this paper, is that disjoint distributive posets are trivial: the only possible element is the initial object. This means that any nontrivial disjoint weakly stable category cannot have any poset component to it. The orthogonality of distributive posets and disjoint stable categories is the subject of this paper.

A disjoint stable category with a final object is clearly finitely complete and is called a *distributive category*. These categories, as mentioned in the Introduction, are a useful setting for modeling data structures and for discussing simple computations. They also crop up naturally in many other areas of mathematics.

## 3. Preinitial objects

While a weakly stable category need not have an initial object it turns out that it does have *preinitial objects*. These objects and the properties of coproducts in weakly stable and stable categories are discussed in this section.

**Definition 3.1.** (i) A *preinitial object* is an object which has at most one map to any given object.

(ii) A preinitial object is *strict* if every object with a map to it is preinitial.

An initial object is preinitial but not conversely as there may be some object to which a preinitial has no map. We first develop some elementary properties of coproducts in weakly stable categories.

**Proposition 3.2.** *Let* **X** *be a weakly stable category.*

(i) *If a coproduct embedding is a retract (split epic), then it is an isomorphism.*

(ii) *Coproduct embeddings are monic.*

(iii) *An object $P$ is preinitial if and only if there is some object $X$ such that $b_1 : X \to P + X$ is an isomorphism.*

(iv) *If $P$ is preinitial and there is a map $P \to X$, then $b_0 : X \to P + X$ is an isomorphism.*

(v) *The pullback of the embeddings of all binary coproducts is preinitial.*

The first and fourth part of this lemma is true in any category. The remaining parts of the lemma require the weak stability property.

**Proof.** (i) Suppose $a : A + B \to A$ is a section which splits $b_0$, then as $a.b_0.\langle i_A; b_1.a\rangle = a$ we conclude that $\langle i_A; b_1.a\rangle = a$ and that $a$ is an isomorphism.

(ii) Consider the pullback of the coproduct along an embedding:

$$\begin{array}{ccc} [b_0 \wedge b_0] + [b_0 \wedge b_1] & \xrightarrow{\langle p_0^{b_0 \wedge b_0} + p_0^{b_0 \wedge b_1}\rangle} & A + B \\ {\scriptstyle \langle p_1^{b_0 \wedge b_0};\, p_1^{b_0 \wedge b_1}\rangle}\downarrow & & \downarrow{\scriptstyle \langle b_0; b_1\rangle} \\ A & \xrightarrow[b_0]{} & A + B \end{array}$$

The coproduct embedding $p_1^{b_0 \wedge b_0}$ is then a retract.

(iii) If $P$ is preinitial, then $b_0 : P \to P + P$ is a retract with section $\langle i_P; i_P\rangle$ and so an isomorphism. Conversely if $b_1 : X \to P + X$ is an isomorphism and $x, y : P \to Y$ then $x + i_X$, $y + i_X : P + X \to Y + X$ are equal so $b_0.\langle x + i_X\rangle = x.b_0$, $b_0.\langle y + i_X\rangle = y.b_0 : P \to Y + X$ are equal. However, as coproduct embeddings are monic this forces $x = y$.

(iv) Obvious.

(v) In the above pullback square $b_0 \wedge b_1$ satisfies part (iii) as $p_1^{b_0 \wedge b_0}$ is a retract. □

In a weakly stable category it is not necessarily the case that preinitals are strict. However, in the presence of stability this is indeed the case.

**Corollary 3.3.** *In a stable category preinitials are strict.*

**Proof.** Suppose $g : R \to P$, where $P$ is preinitial, then

$$\begin{array}{ccc} R + R & \xrightarrow{\langle i_R;\, i_R \rangle} & R \\ {\scriptstyle g+g}\downarrow & & \downarrow{\scriptstyle g} \\ P + P & \xrightarrow[\langle i_P;\, i_P \rangle]{} & P \end{array}$$

is a pullback. As $\langle i_P; i_P \rangle$ is an isomorphism, $R$ satisfies (iii). □

The full subcategory of preinitial objects of a weakly stable category $\mathbf{X}$ shall be denoted $\mathbf{X}_\emptyset$. Clearly there is at most one map between any two objects and so it is equivalent to a poset. Furthermore, it is immediate that it is a weakly stable category and so equivalent to a distributive poset. In fact, it is a coreflexive subcategory whose coreflector is a stable functor.

**Proposition 3.4.** *The full subcategory of preinitial objects $\mathbf{X}_\emptyset$ of a weakly stable category $\mathbf{X}$ is a coreflexive weakly stable category whose coreflector*

$$\emptyset : \mathbf{X} \to \mathbf{X}_\emptyset$$

*is a stable functor.*

**Proof.** Define $\emptyset$ to take an object to the pullback

$$\begin{array}{ccc} \emptyset(X) & \xrightarrow{\eta_X} & X \\ {\scriptstyle \eta_X}\downarrow & & \downarrow{\scriptstyle b_0} \\ X & \xrightarrow[b_1]{} & X + X \end{array}$$

This induces an obvious assignment for maps which is clearly functional.

$\emptyset$ is a coreflector as if $P$ is preinitial with $q : P \to X$, then certainly

$$\begin{array}{ccc} P & \xrightarrow{q} & X \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle b_0} \\ X & \xrightarrow[b_1]{} & X+X \end{array}$$

commutes. This gives a unique map of $P$ to $\emptyset(X)$ as desired.

As $\emptyset$ is a right adjoint it certainly preserves what pullbacks are present. To show $\emptyset$ is stable it suffices to show that coproducts are preserved. For this we observe that for preinitials the coproduct absorbs, that is, if $P' \to P$, then $P + P' \equiv P$. Next by breaking down the pullback $\emptyset(X + Y)$ and observing that the cross-terms are absorbed, we obtain an isomorphism with $\emptyset(X) + \emptyset(Y)$. □

In a disjoint weakly stable category the coreflector $\emptyset$ is to the trivial category. Notice that both $\emptyset$ and the inclusion of $\mathbf{X}_\emptyset$ are stable functors. However, the latter will not preserve the final object usually.

**Corollary 3.5.** *If a weakly stable category* $\mathbf{X}$ *has a final object, then* $\emptyset(1)$ *is final in* $\mathbf{X}_\emptyset$. □

If there is a final object in $\mathbf{X}_\emptyset$, then we shall denote it 0. The next results will show how it can be turned into an initial object.

An object has *cosupport* $P$ if there is a map from $P$ to that object. If $P$ is preinitial, then $\mathbf{X}\backslash P$, the coslice category, is equivalent to the full subcategory with objects having cosupport $P$. This makes the functor $P + -$ a reflecting functor for the full subcategory.

**Proposition 3.6.** *For any weakly stable category* $\mathbf{X}$:

(i) *if $P$ is a preinitial object, the full subcategory of objects with cosupport $P$ is a reflexive subcategory with reflector $P + (-)$ and initial object $P$,*

(ii) *if* $\mathbf{X}$ *has a final preinitial object* 0, *the full subcategory* $\mathbf{X}\backslash 0$ *is a disjoint weakly stable category.* □

The second part is immediate as 0 is both the initial and final preinitial in $(\mathbf{X}\backslash 0)_\emptyset$.

## 4. Subdirect decomposition of stable categories

The properties of a weakly stable category are not quite sufficient to give the decomposition results we seek. The stability of coproducts along arbitrary maps is a vital ingredient. In particular, this ingredient allows the strengthening of the above result described below.

A functor $F$ is *nearly full* if, whenever $\mathrm{Hom}(A, B)$ is nonempty, $F : \mathrm{Hom}(A, B) \to \mathrm{Hom}(F(A), F(B))$ is surjective.

**Proposition 4.1.** *If $P$ is a preinitial in a stable category $\mathbf{X}$, $P + (-)$ is a faithful, nearly full, stable reflecting functor for the full subcategory $\mathbf{X}\backslash P$.*

**Proof.** It remains to show that $P + (-)$ is nearly full and preserves pullbacks.

For the former suppose there is a map $x : A \to B$ and $g : A + P \to B + P$; then we wish to show that $g$ is of the form $g_0 + g_1$. It is clear that $g$ is determined by its effect on the first component $b_0.g$. But $A$ can be split into a coproduct by pulling back $B + P$ along $b_0.g$. The preimage of $P$ will be preinitial and as there is a map to $B$ can be itself uniquely mapped to $B$. This shows that $b_0.g = g_0.b_0$ which gives the desired decomposition of $g$.

For the latter consider

$$\begin{array}{ccc} V & \xrightarrow{\quad f \quad} & X + P \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle x + i_P} \\ Y + P & \xrightarrow[\quad y + i_P \quad]{} & Z + P \end{array}$$

We wish to show that $V$ can be decomposed as $V' + P'$ where $p' : P' \to P$, $f = f' + p'$ and $g = g' + p'$ with

$$\begin{array}{ccc} V' & \xrightarrow{f'} & X \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle x} \\ Y & \xrightarrow[y]{} & Z \end{array}$$

commuting. This will show that the pullback is of the form $[x \wedge y] + P$.

However, $V$ can be split along $f$ and $g$, clearly the intersection of the nonpreinitial parts is the desired $V'$ and the sum of the remaining components is preinitial and the desired $P'$. Whence pullbacks are preserved by $P + (-)$. $\square$

This leads to the following subdirect decomposition:

**Theorem 4.2.** *If $\mathbf{X}$ is stable and $P$ preinitial, then*

$$\langle \emptyset, P + (-) \rangle : \mathbf{X} \to \mathbf{X}_\emptyset \times \mathbf{X}\backslash P$$

*is a full and faithful stable embedding.*

**Proof.** It remains only to show that the functor is full. Suppose

$$\langle h, k\rangle : \langle \emptyset(X), X + P\rangle \rightarrow \langle \emptyset(Y), Y + P\rangle \ ,$$

then $b_0.k$ can be used to split $X$ into a preinitial and a component with a map to $Y$. However, this preinitial is contained in $\emptyset(X)$ which in turn is contained in $\emptyset(Y)$. Whence $k$ can be split into $k' + p'$ which proves fullness. □

It is clear that the final object will always be preserved by $P + (-)$ and this allows the following corollary, which is also the main observation of the paper.

**Corollary 4.3.** *Any finitely complete stable category can be subdirectly decomposed into a distributive poset and a distributive category. The decompostion is explicitly*:

$$\langle \emptyset, \emptyset(1) + - \rangle : \mathbf{X} \rightarrow \mathbf{X}_\emptyset \times \mathbf{X} \backslash \emptyset(1) \ .$$

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