A Modified Levenberg-Marquardt (L-M) Algorithm for Traffic Equilibrium Problem with Nonadditive Route Costs

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Abstract

The traditional traffic equilibrium problem (TEP) is mainly based on the additivity assumption that the route cost is simply the sum of the link costs on that route. However, there are many situations where this assumption on the route costs is inappropriate, and thus we have to explicitly formulate and solve the TEP in the route space instead of link space. In this paper, we firstly reformulate the TEP with nonadditive route cost function to a nonlinear complementarity problem (NCP), and then the NCP is converted to an equivalent least square problem (LSP) with a new NCP function; then we propose a modified Levenberg-Marquardt algorithm to solve the LSP, and also, the quadratic convergence and the equivalent condition of the proposed L-M algorithm are proved under some assumptions. Finally, a numerical example is presented in the paper. As the results shown, the proposed method has the capability to converge to a high level accuracy with reasonable computational efforts.

1. Introduction

Many researchers investigate the traffic equilibrium problem (TEP) on the assumption that the route cost is the sum of the link costs on that route, that is, the route cost is additive. Based on the additive assumption they can convert route flow variables into link flow variables without the need to enumerate routes or store routes, which is a

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significant benefit for analyzing user equilibrium (UE) condition or designing solution approaches to large-scale TEP.

However, there are a large number of cases in practice that the route costs are not additive (i.e. nonadditive), that is, the route cost is not the simply sum of the link costs. For example, the different pricing policies such as congestion pricing and the collection of emission fees add to the nonadditivity of travel costs. Moreover, different individuals have different valuations of time, which contributes to the nonadditivity of route costs. In these cases we have to explicitly formulate and solve the TEP in the route space instead of link space, which is quite difficult to realize in actual process. Therefore, it is necessary to find some efficient and feasible methods to reformulate the TEP with nonadditive route cost.

In fact, there are some important achievements for the nonadditive TEP being made at present. Gabriel and Bernstein (1997) studied the nonlinear complementarity problem (NCP) formulation of TEP with a general route cost function, and proposed an algorithm based on the non-smooth equations/sequential quadratic programming (NE/SQP) method. Lo and Chen (2000) studied the nonadditive TEP with a special route cost function, they defined a gap function to convert the NCP formulation to an equivalent mathematical program. Further, Lo and Chen (2000) used a new gap function, and converted the NCP formulation for the TEP to an equivalent unconstrained optimization. Meanwhile, they developed two solution approaches to solve the proposed formulation. Furthermore, Chen et al. (2012) proposed a self-adaptive projection and contraction method to solve the route-specific cost TEP formulated as a NCP. Larsson et al. (2002) studied TEP with a route cost function that combined time delay and monetary outlay with a nonlinear relation, and proposed two simplicial decomposition type methods for its solution. Besides, Agdeppa et al. (1979) presented a modified general route cost function by introducing a disutility function for each OD pair, and proposed a monotone mixed complementarity problem (MCP) formulation for the TEP. To sum up, the papers mentioned above have covered most of the existing studies on TEP with nonadditive route cost.

In this paper, we consider a solution algorithm of TEP with nonadditive route costs. We propose a new NCP function for solving the NCP formulation of the TEP, and we convert the NCP of TEP into an equivalent least square problem (LSP) by using the proposed NCP function. Then we present a modified Levenberg-Marquardt (L-M) algorithm with line search to solve the LSP; moreover, we also investigate the convergence of L-M algorithm and the equivalent condition under which the solution obtained by the L-M algorithm is necessarily the solution of original NCP.

The rest of the paper is organized as follows. In the next section, the NCP formulation of TEP with nonadditive route costs is presented; Section 3 proposes a modified Levenberg-Marquardt (L-M) algorithm with line search, and both the convergence and the equivalent condition of the proposed algorithm are also given in this section; Section 4 present a numerical example; Finally, we provide some concluding remarks in Section 5.

2. NCP formulation of Nonadditive TEP

2.1. User equilibrium condition

For a strongly connected network \([N, A]\), where \(N\) is the set of nodes and \(A\) is the set of links, the Wardrop’s user equilibrium (UE) principle, together with the condition imposed on the travel demand function, can be stated mathematically as

\[
f_k^{rs} (c_k^{rs} - u_k^{rs}) = 0, \quad \forall rs \in RS, \forall k \in K^{rs}
\]

(1)

\[
c_k^{rs} - u_k^{rs} \geq 0, \quad \forall rs \in RS, \forall k \in K^{rs}
\]

(2)

\[
\sum_{k \in K} f_k^{rs} - q^{rs} = 0, \quad \forall rs \in RS
\]

(3)

\[
f \geq 0, \quad u \geq 0
\]

(4)
Conditions (1) and (2) are the complementary slackness conditions. That is, for each OD pair \( rs \in RS \), if the travel cost on route \( k \) satisfies \( c^r_{rs} - u^r_{rs} > 0 \), the flow on that route is zero; if the travel cost on route \( k \) satisfies \( c^r_{rs} - u^r_{rs} = 0 \), its flow is equal to or greater than zero. These complementary slackness conditions are consistent with the Wardrop’s UE principle: All the used routes have equal and minimum travel times, and all the unused routes have equal or higher travel times.

The notations are as follows:

<table>
<thead>
<tr>
<th>Notation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>RS</td>
<td>the set of OD pairs for the whole network</td>
</tr>
<tr>
<td>Rs</td>
<td>an OD pair, ( rs \in RS )</td>
</tr>
<tr>
<td>K^rs</td>
<td>the set of routes connecting OD pair ( rs )</td>
</tr>
<tr>
<td>k</td>
<td>a route between an OD pair, ( p \in K^rs )</td>
</tr>
<tr>
<td>f^rs</td>
<td>the flow on route ( k ) between OD pair ( rs )</td>
</tr>
<tr>
<td>f</td>
<td>the vector of ((\cdots, f^r_{k}, \cdots)) with dimension ( n_1 = \sum_{rs}</td>
</tr>
<tr>
<td>e^r_{rs}(\cdot)</td>
<td>the route cost on route ( k ) between OD pair ( rs ), a function of ( f )</td>
</tr>
<tr>
<td>c</td>
<td>the vector of ((\cdots, e^r_{rs}, \cdots)) with dimension ( n_1 = \sum_{rs}</td>
</tr>
<tr>
<td>u^r_{rs}</td>
<td>the shortest travel cost (or disutility) between OD pair ( rs )</td>
</tr>
<tr>
<td>u</td>
<td>the vector of ((\cdots, u^r_{rs}, \cdots)) with dimension ( n_2 =</td>
</tr>
<tr>
<td>q^r_{rs}</td>
<td>the travel demand between OD pair ( rs )</td>
</tr>
<tr>
<td>q</td>
<td>the vector of ((\cdots, q^r_{rs}, \cdots)) with dimension ( n_2 =</td>
</tr>
<tr>
<td>n</td>
<td>the sum of ( n_1 ) and ( n_2 )</td>
</tr>
</tbody>
</table>

2.2. Nonadditive route cost function

Gabriel and Bernstein (1997) pointed out that route costs are not additive in situations with: (i) Nonlinear valuation of travel time - small amounts of time have relatively low value whereas large amounts of time are very valuable; (ii) Nonadditive tolls and fares - toll roads and transit systems have non-additive toll/fare structure; (iii) Emission fares - emissions of hydrocarbons and carbon monoxide are a nonlinear function of travel times (or speeds). Meanwhile, they presented a general route cost function for these scenarios:

\[
c^r_k(f) = \lambda^r_k + \sum_a \eta_i \delta^r_{k,a} t_a + g_k \left( \sum_a \delta^r_{s,a} t_a \right)
\]  

(5)

Where \( \lambda^r_k \) denotes the financial costs (such as tolls) specific to route \( k \), \( \eta_i \) the operating costs per travel-time (e.g., fuel consumption, vehicle rental), and \( g_k \) is a function describing the value of time for route \( k \), which could be nonlinear. In this paper, we define the route-specific costs function as follows:
2.3. NCP formulation of nonadditive TEP

With the nonadditive route cost function (6), we can reformulate the TEP as the following NCP:

\[ x^T \cdot F(x) = 0 \]
\[ F(x) \geq 0 \]
\[ x \geq 0 \]

by setting \( x \) and \( F(x) \) as follows

\[ x = \left\{ f \right\} \in R^n, \text{ where } n = n_1 + n_2 \]
\[ F(x) = \left( \begin{array}{c} (c_k^r - u^r, \forall rs \in RS, \forall k \in K^r) \\ \left( \sum_{k \in K} f_k^r - q^r, \forall rs \in RS \right) \end{array} \right) \in R^n \]

Aashtiani (1979) established that the above NCP is equivalent to the UE traffic assignment problem if and only if the travel cost function is positive and the demand function is nonnegative. These two assumptions are valid for virtually any problems and hence pose no restrictions. Furthermore, Aashtiani proved the existence of solutions of this NCP and the uniqueness of the solution in \( u \) (note: not \( f \)).

It is noteworthy that the travel demand between OD pair \( rs \) (i.e. \( q^r \)) is a constant for the fixed demand case and could be a function of \( u \) for the elastic demand case, \( q^r: R^n \rightarrow R^n; \) this paper focuses on the fixed demand case.

3. A Modified Levenberg-Marquardt(L-M) Method

3.1. A new NCP function

A function \( \phi(a, b): R^2 \rightarrow R \) is called a NCP function if the function satisfies \( \phi(a, b) = 0 \iff ab = 0, a \geq 0, b \geq 0 \). A NCP function can be used to convert a nonlinear complementarity problem into an equivalent unconstrained optimization problem (UOP). In this paper, we give a NCP function as follows:

\[ \phi_L(a, b) = \sqrt{(\sqrt{a^2 + b^2} - a - b)^2 + L^2} - L, \quad L \geq 0 \]

It is very easy to prove that the function \( \phi_L(a, b) \) has all the properties of a NCP function. Then we can propose a gap function \( \varphi(a, b): R^2 \rightarrow R \) based on \( \phi_L^2(a, b) \):

\[ \varphi(a, b) = (1/2)\phi_L^2(a, b) \]

This gap function has three properties, including:

(i) \( \varphi(a, b) \geq 0, \forall a, b \in R \);
(ii) \( \varphi(a, b) = 0 \iff a \geq 0, b \geq 0, ab = 0 \);
(iii) \( \varphi(a, b) \) is continuously differentiable on \( R^2 \); in particular, \( \nabla \varphi(0,0) = (0,0) \).
Then the NCP (7) for TEP with nonadditive path costs can be converted into the following equivalent unconstrained optimization problem (UOP).

\[
\min G(x) = \frac{1}{2} \| \Phi(x) \|^2 = \sum_{i=1}^{n} \varphi(x_i, F_i(x)) \quad (11)
\]

\[
\Phi(x) = \begin{bmatrix}
\phi_L(x_1, F_1(x)) \\
\vdots \\
\phi_L(x_n, F_n(x))
\end{bmatrix} \in \mathbb{R}^n
\]

where \( n = n_1 + n_2 \), and \( i \) represents the \( i \)th component in (8), and \( \| \cdot \| \) is the Euclidean norm.

### 3.2. A modified Levenberg-Marquardt (L-M) algorithm

In fact, the UOP (11) can be regarded as a least square problem (LSP); moreover, a large number of solution algorithms can solve the LSP at present. In this paper we apply the Levenberg-Marquardt (L-M) algorithm of LSP to solve the UOP (11).

Denote the function \( h(x) : \mathbb{R}^n \to \mathbb{R}^n \) as follows:

\[
h(x) = \Phi(x) = \begin{bmatrix}
\phi_L(x_1, F_1(x)) \\
\vdots \\
\phi_L(x_n, F_n(x))
\end{bmatrix}, \text{ where } h_i(x) = \phi_L(x_i, F_i(x)) \quad (12)
\]

Then the UOP (11) can be rewritten into a LSP (13) as follows:

\[
\min_{x \in \mathbb{R}^n} G(x) = \frac{1}{2} h(x)^T h(x) = \frac{1}{2} \sum_{i=1}^{\#} [h_i(x)]^2 
\]

Denote \( J(x) \) as the Jacobian matrix of \( h(x) \), that is

\[
J(x) = \begin{bmatrix}
\frac{\partial h_1}{\partial x_1} & \ldots & \frac{\partial h_1}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial h_n}{\partial x_1} & \ldots & \frac{\partial h_n}{\partial x_n}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial \phi_L(x_1, F_1)}{\partial a} + \frac{\partial \phi_L(x_1, F_1)}{\partial b} \frac{\partial F_1}{\partial x_1} & \ldots & \frac{\partial \phi_L(x_1, F_1)}{\partial a} + \frac{\partial \phi_L(x_1, F_1)}{\partial b} \frac{\partial F_1}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial \phi_L(x_n, F_n)}{\partial a} + \frac{\partial \phi_L(x_n, F_n)}{\partial b} \frac{\partial F_n}{\partial x_1} & \ldots & \frac{\partial \phi_L(x_n, F_n)}{\partial a} + \frac{\partial \phi_L(x_n, F_n)}{\partial b} \frac{\partial F_n}{\partial x_n}
\end{bmatrix}
\]

In the next, we will introduce a modified Levenberg-Marquardt (L-M) method with line search to solve the LSP (13), and the detailed procedure is as follows:

**Step 1:** Given \( x_0 \in \mathbb{R}^n \), \( \varepsilon > 0 \), \( \eta \in (0, 1) \), \( k := 0 \);

**Step 2:** Set \( \mu_k := \| h(x^k) \| \), if \( \mu_k \leq \varepsilon \), then stop; else, compute \( d_k \) by the following formula

\[
d_k = -(J(x^k)^T J(x^k) + \mu_k I)^{-1} J(x^k)^T h(x^k)
\]

**Step 3:** If \( d_k \) satisfies

\[
\| F(x_k + d_k) \| \leq \eta \| F(x_k) \|,
\]
then set $x^{k+1} = x^k + d_k$, otherwise $x^{k+1} = x^k + \alpha_k d_k$ where $\alpha_k$ is obtained by Wolfe or Armijo line search; 

**Step 4**: $k = k + 1$; go to **Step 2**.

Then the convergence and the equivalent condition of the above modified L-M algorithm are provided in the following paper.

**Assumption 1**: $F(x)$ is continuously differentiable, and the Jacobian $J(x)$ is Lipschitz continuous on some neighborhood of $x^* \in X^*$, i.e., there exist positive constants $L_1$ and $b_1 \in (0, 1)$ such that

$$
\|f(y) - J(x)\| \leq L_1 \|y - x\|, \quad \forall x, y \in N(x^*, b_1) = \{x : \|x - x^*\| \leq b_1\}
$$

**Assumption 2**: $\|F(x)\|$ provides a local error bound on $N(x^*, b_1)$ for the NCP (8), i.e., there exists a constant $c_1 > 0$ such that

$$
\|h(x)\| \geq c_1 \text{dist}(x, X^*), \quad \forall x \in N(x^*, b_1)
$$

**Theorem 1**: Suppose Assumption (1) and (2) holds and $F(x)$ is continuously differentiable. Let the sequence $\{x_k\}$ be generated by the foregoing Modified L-M algorithm. Then the sequence of $\{x_k\}$ converges to a stationary point of $G(x)$. Moreover, if the stationary point $x^*$ is a solution of NCP (7), then the whole sequence $\{x_k\}$ converges to $x^*$ quadratically.


In fact, the solution $x^*$ obtained by the abovementioned L-M algorithm is just a stationary point of $G(x)$ under normal circumstances, whereas it is not necessarily the solution of NCP (7). In the following paper, we give the condition under which the stationary point $x^*$ (i.e. the solution) of $G(x)$ obtained by the L-M algorithm is necessarily the solution of NCP (7).

**Theorem 2**: The stationary point $x^*$ of $G(x)$ obtained by the L-M algorithm is necessarily the solution of NCP (7) on condition that $\nabla F(x)$ is a positive semidefinite matrix.

**Proof**: Denote that

$$
\nabla G(x) = \left[ \begin{array}{c} 
\phi_L \frac{\partial \phi_L}{\partial a}(x, F) + \sum_{i=1}^{n} \phi_L(x_i, F_i) \frac{\partial \phi_L}{\partial b}(x_i, F_i) \\
\vdots \\
\phi_L(x_n, F_n) \frac{\partial \phi_L}{\partial a}(x_n, F_n) + \sum_{i=1}^{n} \phi_L(x_i, F_i) \frac{\partial \phi_L}{\partial b}(x_i, F_i) \frac{\partial F}{\partial x_i} \\
\vdots \\
\phi_L(x_n, F_n) \frac{\partial \phi_L}{\partial a}(x_n, F_n) + \sum_{i=1}^{n} \phi_L(x_i, F_i) \frac{\partial \phi_L}{\partial b}(x_i, F_i) \frac{\partial F}{\partial x_n} 
\end{array} \right]
$$

Then we give the proof by contradiction. Given that $x^*$ is a stationary point of $G(x)$ but not the solution of NCP (7), then it follows that

$$
\phi_L \frac{\partial \phi_L}{\partial a}(x^*, F(x^*)) \neq 0, \quad \phi_L \frac{\partial \phi_L}{\partial b}(x, F(x^*)) \neq 0
$$

Then we have the following equation
\[ \nabla G(x^*)^T \phi_L \frac{\partial \phi_L}{\partial b} (x, F(x^*)) \]

\[ = \left[ \phi_L \frac{\partial \phi_L}{\partial a} (x, F(x^*)) + \nabla F(x^*) \phi_L \frac{\partial \phi_L}{\partial b} (x, F(x^*)) \right]^T \phi_L \frac{\partial \phi_L}{\partial b} (x, F(x^*)) \]

\[ = \phi_L \frac{\partial \phi_L}{\partial a} (x, F(x^*))^T \phi_L \frac{\partial \phi_L}{\partial b} (x, F(x^*)) + \phi_L \frac{\partial \phi_L}{\partial b} (x, F(x^*))^T \nabla F(x^*) \phi_L \frac{\partial \phi_L}{\partial b} (x, F(x^*)) \]

It is noteworthy that the corresponding elements of the two nonzero vectors \( \phi_L \frac{\partial \phi_L}{\partial a} (x, F(x^*)) \) and \( \phi_L \frac{\partial \phi_L}{\partial b} (x, F(x^*)) \) have the same plus or minus, so

\[ \left[ \phi_L \frac{\partial \phi_L}{\partial a} (x, F(x^*)) \right]^T \phi_L \frac{\partial \phi_L}{\partial b} (x, F(x^*)) > 0 \]

Besides, considering that \( \nabla F(x) \) is a positive semidefinite matrix, we can have

\[ \left[ \phi_L \frac{\partial \phi_L}{\partial b} (x, F(x^*)) \right]^T \nabla F(x^*) \phi_L \frac{\partial \phi_L}{\partial b} (x, F(x^*)) \geq 0 \]

So it follows that

\[ \nabla G(x^*)^T \phi_L \frac{\partial \phi_L}{\partial b} (x, F(x^*)) > 0 \]

It is obviously contradictory to the foregoing assumption, so the statement is true. This completes the proof.

It is noteworthy that in the actual TEP the travel cost function is positive and the demand function is nonnegative; therefore, \( \nabla F(x) \) is a positive definite matrix for virtually any problems and hence Theorem 2 is necessarily valid in this paper.

4. Numerical example

To illustrate the proposed L-M algorithm, we apply it to a simple network with five nodes, five links, and two OD pairs, as given in Fig. 1. The travel demand between OD pair (1, 5) is 15 units, and (2, 5) is 18.75 units. The link performance function is the standard BPR function, and the link characteristics are provided in Table 1. Specially, we assume that a 6-unit cost was added to route 2 (link sequence: 1-3-5) and a 5-unit cost was added to route 4 (link sequence: 2-3-5) while no cost was added to the other routes, namely, \( \lambda_2 = 6, \lambda_4 = 5, \lambda_3 = 0 \).

\[ t_a = t^0_a [1 + 0.15 (\frac{v_a}{C_a})^4] \]

where \( t^0_a, v_a, \) and \( C_a \) are the travel time, free-flow travel time, flow, and capacity of link \( a \), respectively.

![Fig. 1. A simple network](image-url)
Table 1. Link characteristics of the example network

<table>
<thead>
<tr>
<th>Link</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Free-flow travel time ($t_{0i}$)</td>
<td>15</td>
<td>10</td>
<td>10</td>
<td>15</td>
<td>10</td>
</tr>
<tr>
<td>Capacity ($C_i$)</td>
<td>10</td>
<td>20</td>
<td>20</td>
<td>15</td>
<td>30</td>
</tr>
</tbody>
</table>

We solve this example two times. The first time is additive TEP, which corresponds to the classic TEP. The second time is nonadditive TEP, which is presented in this paper. The results for both cases are shown in Table 2 and Table 3.

Table 2. Link flows of additive and nonadditive TEP

| Link flows of additive TEP | 15.00 | 18.75 | 27.14 | 6.61 | 33.75 |
| Link flows of nonadditive TEP | 15.00 | 18.75 | 18.86 | 14.89 | 33.75 |

Table 3. Route flows and route costs of additive and nonadditive TEP

<table>
<thead>
<tr>
<th>OD</th>
<th>Route</th>
<th>Link sequence</th>
<th>Additive TEP</th>
<th>Nonadditive TEP</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Route flow</td>
<td>Route cost</td>
</tr>
<tr>
<td>(1, 5)</td>
<td>1</td>
<td>1-4-5</td>
<td>2.52</td>
<td>53.87</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1-3-5</td>
<td>12.48</td>
<td>53.87</td>
</tr>
<tr>
<td>(2, 5)</td>
<td>3</td>
<td>2-4-5</td>
<td>4.09</td>
<td>38.65</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>2-3-5</td>
<td>14.66</td>
<td>38.65</td>
</tr>
</tbody>
</table>

As shown in Table 2 and Table 3, the results satisfy the Wardrop’s UE principle, that is, the costs on all used routes between an OD pair are equal, and less than the costs on any unused routes for both models. It is noticeable that nearly all of the flow on route 2 shifts to route 1 when a 6-unit cost was added to route 2, which not only makes the travel time on link 4 increase but also makes the travel time on link 3 become much less. Therefore, even though a 5-unit cost was added to route 4, its total route cost is still lower than route 3 under UE, and that is the reason why the flow on route 3 is zero.

5. Conclusions

This paper presented a modified Levenberg-Marquardt algorithm for computing the TEP with nonadditive route cost function. This problem was first formulated as a NCP and then converted to a least square problem (LSP) with a new NCP function. Quadratic convergence and the equivalent condition of the proposed L-M algorithm were proved under some assumptions. Finally, a numerical example was given in the paper. As the results show the proposed method has the capability to converge to a high level accuracy with reasonable computational efforts.

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References


