Countable Fréchet topological groups under CH

A. Shibakov

Department of Mathematics, Auburn University, Auburn, AL 36849, USA

Received 9 October 1996

Abstract

We construct under CH a countable Fréchet topological group with sequential non-Fréchet square and a countable Fréchet $\alpha_2$-group which is not $\alpha_2$. We also give a simple proof that for Fréchet topological groups $\alpha_{1.5}$ is equivalent to $\alpha_1$. This answers some questions of D. Shakhmatov. © 1999 Elsevier Science B.V. All rights reserved.

Keywords: Fréchet space; Sequential space; $\alpha_i$-space; Topological group; Product operation

AMS classification: 54D55; 54G20; 54A35; 54H11; 54B15

1. Introduction and preliminary results

First-countable and Fréchet spaces are classical notions arising from an attempt to carry over the techniques of metric spaces to more general classes of topological spaces. In a Fréchet space $X$, any point $x \in X$ lying in the closure of some subset $A \subseteq X$ is a limit point of a convergent sequence in $A$. The next step in generalization is sequentiality. By definition, a space $X$ is sequential if any nonclosed subset $A \subseteq X$ contains a sequence converging to a point in the complement of $A$. Sequentiality proved to be a useful generalization of Fréchetness, as a product of two Fréchet spaces one of which is countably compact need not be Fréchet (see [20]) while it is always sequential [9].

A natural set of conditions that give the Fréchetness of the product was defined by A.V. Arhangel'skii in [3] and has been studied since then by many authors (see [6,12–14,17] and bibliography there). The classes of spaces defined by A. Arhangel'skii are now called $\alpha_i$-spaces where $i = 1, 1.5, 2, 3, 4$.

On the other hand Fréchetness and sequentiality play a very important role in functional analysis where many classical techniques deal with convergence of functions.

---

1 E-mail: shobaay@mail.auburn.edu, Web: http://www.auburn.edu/~shobaay.

0166-8641/99/$ – see front matter © 1999 Elsevier Science B.V. All rights reserved.
PII: S0166-8641(97)00195-8
operators, functionals, etc. One of the problems having its roots in functional analysis is metrization of topological groups. The classical Birkhoff–Kakutani theorem states that for Hausdorff topological groups first-countability is equivalent to metrizability. Thus it is natural to look for metrization conditions in the form of a strengthened Fréchet property.

P. Nyikos' paper [11] was an important step in this direction. He showed that Fréchet topological groups were \( \alpha_4 \) and applied this fact to the solution of a question by Arhangel'skii about metrizability of topological groups having a countable weak base. In the same paper he posed a question whether a similar looking property was satisfied by every Fréchet topological group.

The property was shown to be \( \alpha_2 \) and in [17] D. Shakhmatov answered the question of P. Nyikos negatively by constructing in a model of ZFC obtained by adding uncountably many Cohen reals an example of a Fréchet topological group which is not \( \alpha_3 \). [17] also contains a construction of a Fréchet \( \alpha_2 \) non \( \alpha_1 \) topological group in the same model. Such a group as well as a topological vector space were also constructed in [12] by P. Nyikos under \( p = b \).

In [17] D. Shakhmatov asked which implications besides the natural ones held for \( \alpha_i \)-properties in (countable) topological groups. His examples showed that neither \( \alpha_3 \Rightarrow \alpha_4 \) nor \( \alpha_{1.5} \Rightarrow \alpha_2 \) is reversible so the natural question is whether the same is true about \( \alpha_2 \Rightarrow \alpha_3 \) and \( \alpha_1 \Rightarrow \alpha_{1.5} \). In the final remark he pointed out a gap in Malykhin's construction [8] of a countable Fréchet topological group with non-Fréchet square and posed a question about existence of such a group.

Two countable Fréchet topological groups with sequential non-Fréchet square were constructed in [19]. In the case of general (even compact) topological spaces there is no difference between product and square for it is possible to take the union of the two spaces whose square will contain the initial product. Since no such tool is available in the case of topological groups other techniques are needed.

In the present paper we construct under CH a countable Fréchet topological group with sequential non-Fréchet square and a countable sequential topological \( \alpha_3 \)-group (which is Fréchet by a result in [11]) which is not \( \alpha_2 \) and give a simple proof that for Fréchet topological groups \( \alpha_{1.5} \) is equivalent to \( \alpha_1 \) thus answering the questions of D. Shakhmatov mentioned above.

The topology of each example is determined by a family of compact subsets of some first-countable group (either \( \mathbb{Q} \) or \( \mathbb{Q} \times \mathbb{Q} \)) which is constructed by transfinite induction so that the topology is also a common refinement of a family of first-countable group topologies finer than the initial topology of the group.

Let us now define the main notions. Let \( X \) be a topological space, \( S \) be a countable subset of \( X \). Let us say that \( S \) converges to a point \( x \in X \) if \( S \cup \{x\} \) is a compact subset of \( X \) with a unique nonisolated point \( x \). A family \( \{S_n\}_{n \in \omega} \) of countable subsets of \( X \) is called a normal sheaf with vertex \( x \in X \) if each \( S_n \) converges to \( x \) and \( S_n \cap S_{n'} = \emptyset \) if \( n \neq n' \). A point \( x \in X \) is called an \( \alpha_i \)-point where \( i = 1, 1.5, 2, 3, 4 \) if for any normal sheaf \( \{S_n\}_{n \in \omega} \) with vertex \( x \) there exists an \( S \subseteq X \) such that \( S \) converges to \( x \) and the corresponding property holds.
(α₁) $S \cap S_n$ is cofinite in $S_n$ for every $n \in \omega$.
(α₁,5) $S \cap S_n$ is cofinite in $S_n$ for infinitely many $n \in \omega$.
(α₂) $S \cap S_n$ is infinite for every $n \in \omega$.
(α₃) $S \cap S_n$ is infinite for infinitely many $n \in \omega$.
(α₄) $S \cap S_n$ is nonempty for infinitely many $n \in \omega$.

If for some $i = 1, 1.5, 2, 3, 4$ every point in $X$ is an $α_i$-point then $X$ is called an $α_i$-space. In [12] a slightly different definition of $α_i$-properties is given which is equivalent to the one above. Now

$$α_1 \Rightarrow α_{1.5} \Rightarrow α_2 \Rightarrow α_3 \Rightarrow α_4$$

and it is shown in [12] that these implications are not reversible for compact spaces.

All spaces are assumed to be Hausdorff. If $G$ is a group then $e$ denotes the unit element of $G$, if $G$ is commutative then the notation 0 for the unit element is used. Let $G$ denote either $Q$ or $Q \times Q$. Then for any $a \in Q$, $b \in G$ the element $a \cdot b \in G$ is naturally defined. Denote $\langle a, b \rangle = (a_1, \ldots, a_n) \in Q^n$, $\langle b_1, \ldots, b_n \rangle \in G^n$. Denote $\langle K \rangle = (K_1, \ldots, K_n)$ and each $K_i$ is a subset of $G$ then denote $\langle a, K \rangle = a_1 \cdot K_1 + \cdots + a_n \cdot K_n \subseteq G$. If $K \subseteq G$ denote $\langle a(K) = a_1 \cdot K + \cdots + a_n \cdot K \subseteq G$. Let $Q_\omega = Q \setminus \{0\}$ and $Q_M = \{a \in Q_\omega \mid |a| \leq M \}$ or $a_i = 0$, $k \leq M$. Put $Q^\omega = \bigcup_{M \in \omega} Q_M$.

Let us introduce some terminology for subsets of $\omega^2$. A subset $\subseteq \omega^2$ is called

1. thick if $|\{n \in \omega \mid |\sigma \cap \{n\} \times \omega| = \aleph_0\}| = \aleph_0$,
2. thin if for any $n \in \omega$ the set $\sigma \cap \{n\} \times \omega$ is finite,
3. small if $|\{n \in \omega \mid |\sigma \cap \{n\} \times \omega| = \aleph_0\}| < \aleph_0$,
4. short if $|\{n \in \omega \mid |\sigma \cap \{n\} \times \omega| \neq 0\}| < \aleph_0$.

If $f: \omega \to \omega$ is a map then denote $\omega^f = \{(m, n) \mid m \geq f(n)\}$. Then for any $f: \omega \to \omega$ the set $\omega^f$ is thick.

Let $π_\nu: \omega^2 \to \omega$ where $\nu \in \{1, 2\}$ be the natural projections. If $λ \subseteq \omega$ is infinite let $m_\lambda: \omega \to λ$ be the 1-1 map such that if $i < j$ then $m_\lambda(i) < m_\lambda(j)$. Let $σ \subseteq \omega^2$ be a thick subset and $λ = \{n \in \omega \mid |\sigma \cap \{n\} \times \omega| = \aleph_0\}$ and $λ_n = π_2(σ \cap \{m_\lambda(n)\} \times \omega)$. Define $ψ_\sigma: \omega^2 \to σ$ so that

$$ψ_\sigma(n, k) = (m_\lambda(n), m_\lambda(k)).$$

Then $ψ_\sigma$ is an injection such that for any $i \in \omega$ there is $m(i) \in \omega$ such that $ψ_\sigma([i] \times \omega) \subseteq \{m(i)\} \times \omega$ where $m(i) > m(j)$ if $i > j$ which implies that for any thick $δ \subseteq \omega^2$ the set $ψ_\sigma(δ)$ is thick and for any thin $θ \subseteq \omega^2$ the set $ψ_\sigma(θ)$ is thin.

We will use the following simple facts about topological groups:

5. a topological group $H$ is an $α_3$-space if for any injection $s: \omega^2 \to H$ such that $s(n, m) \to e \in H$ as $m \to \infty$ there is a convergent sequence $K \subseteq H$ such that $s^{-1}(K)$ is thick;
6. a sequential topological group $H$ is Fréchet if for any injection $s: \omega^2 \to H$ such that $s(n, m) \to s_n \in H$ as $m \to \infty$ and $s_n \to e \in H$ as $n \to \infty$ the set $s(\omega^2) \cup \{e\}$ contains a nontrivial convergent sequence with the limit point $e \in H$. 
A space \( X \) is called a \( k_\omega \)-space if \( X \) is a quotient space of a topological sum of countably many compact spaces. Every countable \( k_\omega \)-space is sequential and a product of two \( k_\omega \)-spaces is itself a \( k_\omega \)-space (see [10]).

Two classical spaces \( S_\omega \) and \( S_2 \) may be defined as follows. \( S_\omega \) denotes the set \( \omega^2 \cup \{\omega\} \) with the following topology: all points of \( \omega^2 \) are isolated, a basic neighborhood of \( \omega \) is of the form \( \{\omega\} \cup (\omega^2 \setminus \theta) \) for some thin \( \theta \subseteq \omega^2 \). \( S_2 \) is the set \( \omega^2 \cup \omega \cup \{\omega\} \) equipped with the topology in which every point of \( \omega^2 \) is isolated, a basic neighborhood of \( n \in \omega \) is of the form \( \{n\} \cup (\{n\} \times \lambda) \) where \( \lambda \subseteq \omega \) is cofinite and a basic neighborhood of \( \omega \) is of the form \( \{\omega\} \cup \lambda \cup (\lambda \times \omega \setminus \theta) \) where \( \lambda \subseteq \omega \) is cofinite and \( \theta \subseteq \omega^2 \) is thin.

The following simple lemma was proved in [18].

**Lemma 1.1.** A countable nondiscrete sequential topological group contains a closed copy of \( S_2 \) provided it is a \( k_\omega \)-space.

Lemmas 1.2–1.6 are essentially [18, Lemmas 2.1–2.5].

**Lemma 1.2.** Let \( \mathcal{K} \) be countable family of subsets of \( G \). Then there exists a countable family \( C(\mathcal{K}) \supseteq \mathcal{K} \) such that:

1. \( \{a\} \in C(\mathcal{K}) \) for any \( a \in G \),
2. if \( \vec{a} \in Q^n \), \( \vec{K} \in C(\mathcal{K})^n \) then \( \langle \vec{a}, \vec{K} \rangle \in C(\mathcal{K}) \),
3. if \( K^1 \in C(\mathcal{K}), \ldots, K^n \in C(\mathcal{K}) \) then \( \bigcup_{1 \leq i \leq n} K^i \in C(\mathcal{K}) \),
4. if \( \mathcal{K} \subseteq \mathcal{K}' \) and \( \mathcal{K}' \) has properties (7)–(9) then \( C(\mathcal{K}) \subseteq C(\mathcal{K}') \).

**Lemma 1.3.** If \( \mathcal{K} = \bigcup_{\beta < \alpha} \mathcal{K}_\beta \) and \( \mathcal{K}_\beta \subseteq \mathcal{K}_{\beta'} \) if \( \beta \leq \beta' \) then

\[
C(\mathcal{K}) = \bigcup_{\beta < \alpha} C(\mathcal{K}_\beta).
\]

**Lemma 1.4.** If \( \mathcal{K} \) is a countable family of compact subsets of \( G \) then \( C(\mathcal{K}) \) is also a family of compact subsets of \( G \).

**Lemma 1.5.** Let \( \mathcal{K} \) be a countable family of compact subsets of \( G \). Let us introduce a new topology on \( G \). We put \( U \subseteq G \) open if and only if \( U \cap K \) is relatively open for every \( K \in C(\mathcal{K}) \). Denote \( G \) with this topology as \( G(\mathcal{K}) \). Then:

1. if \( \vec{a} \in Q^n \) then the mapping \( p : G(\mathcal{K})^n \to G(\mathcal{K}), p(\vec{b}) = \langle \vec{a}, \vec{b} \rangle \) is continuous;
2. \( G(\mathcal{K}) \) is a \( k_\omega \)-space.

**Lemma 1.6.** For any \( \mathcal{K} \) being a countable family of compact subsets of \( G \) and any \( U \) being a countable family of open subsets of \( G(\mathcal{K}) \) one can fix a topology \( \tau(\mathcal{U}, \mathcal{K}) \) such that:

1. the mapping \( p : G^n \to G \) where \( p(\vec{a}) = \langle \vec{b}, \vec{a} \rangle \), \( \vec{b} \in Q^n \) is continuous in \( \tau(\mathcal{U}, \mathcal{K}) \),
2. \( \tau(\mathcal{U}, \mathcal{K}) \) is a Hausdorff group topology with a countable base,
3. \( U \in \tau(\mathcal{U}, \mathcal{K}) \) for any \( U \in \mathcal{U} \),
4. \( \tau(\mathcal{U}, \mathcal{K}) \) is finer than the usual topology of \( G \) and coarser than the topology of \( G(\mathcal{K}) \), and
(17) if $U \supseteq \tau_0(U', K')$ then $\tau(U, K)$ is finer than $\tau(U', K')$, where $K$ and $K'$ are countable families of compact subsets of $G$ and $\tau_0(U, K)$ is a fixed countable base at $0 \in G$ in $\tau(U, K)$.

The first part of the following lemma is a corollary of (16). The second part follows from the definition of $C(K)$.

**Lemma 1.7.** Let $K$ be a countable family of compact subsets of $G$. Then

(18) if $K \in C(K)$ then the topology of $K$ as a subspace of $G(K)$ coincides with the topology induced by $G$ or $\tau(U, K)$ for any countable family $U$ of open subsets of $G(K)$.

(19) if $K' \subseteq C(K \cup \{K\})$ then there are $K'' \in K$ and $\bar{a} \in \mathbb{Q}^\infty$ such that $K' \subseteq \bar{a}(K) + K''$.

The next lemma may be easily proved by induction using countability of $C(K)$ and the definition of $G(K)$.

**Lemma 1.8.** If $K$ is a countable family of compact subsets of $G$ and $Z \subseteq G$ then either there exists $K \in C(K)$ such that $Z \subseteq K$ or there exists an infinite $D \subseteq Z$ such that $D$ is a closed discrete subset of $G(K)$.

Let $K$ be a countable family of compact subsets of $G$, $t: \omega^2 \to G$ be an injection. Let us say that $t$ is a correct table in $G(K)$ if properties (20)$_t$–(23)$_t$ hold

(20)$_t$ $t(m, n) \to t_m$ as $n \to \infty$ in $G(K)$ where $t_m \notin t(\omega^2)$ for $m \in \omega$ and $0 \notin t(\omega^2)$ and either $t_m \neq t_k \neq 0$ if $m \neq k$ or $t_m = 0$ for all $m \in \omega$.

(21)$_t$ $t_m \to 0$ as $m \to \infty$ in $G(K)$;

(22)$_t$ $t(\omega^2) \cup \{t_m \mid m \in \omega\} \cup \{0\}$ is a compact subset of $G$ in the usual topology of $G$. So $t(\omega^2) = t(\omega^2) \cup \{t_m \mid m \in \omega\} \cup \{0\}$ where the closure is taken either in the usual topology of $G$ or in $G(K)$;

(23)$_t$ for any thin $\theta \subseteq \omega^2$ the set $t(\theta) \cup \{0\}$ is a convergent sequence in $G$.

The following property follows from the definition of $G(K)$ and (20)$_t$–(21)$_t$:

(24)$_t$ for every $i \in \omega$, $K'_i = \{t(i, n) \mid n \in \omega\} \cup \{t_i\} \subseteq K_i \in C(K)$ for some $K_i$ and $B'_i = \{t_m \mid m \in \omega\} \cup \{0\} \subseteq K \subseteq C(K)$ for some $K$.

Using first-countability of $G$ it is easy to show that if $t': \omega^2 \to G$ is an injection such that $t'(m, n) \to t'_m$ as $n \to \infty$ in $G(K)$ and $t'_m \to 0$ as $m \to \infty$ in $G(K)$ then there exists a thick set $\sigma \subseteq \omega^2$ such that $t = t' \circ \psi_\sigma$ is a correct table.

Like Lemma 1.8 the lemma below may be proved by induction.

**Lemma 1.9.** If $K$ is a countable family of compact subsets of $G$ and $s: \omega^2 \to G$ is a correct table in $G(K)$ then there exists a thick $\sigma \subseteq \omega^2$ such that either $s(\sigma) \subseteq K$ for some $K \in C(K)$ or for any thin $\theta \subseteq \omega^2$ the set $s(\psi_\sigma(\theta))$ is a closed discrete subset of $G(K)$. 
2. Technical lemmas

In what follows $G$ will denote $\mathbb{Q} \times \mathbb{Q}$. We will write $G_\mathbb{Q}(K)$ and $C_\mathbb{Q}(K)$ when we need to specify the group if both $G$ and $\mathbb{Q}$ are considered.

Lemma 2.1. Let $K$ be a countable family of compact subsets of $G$, $s: \omega^2 \to G$ be a correct table in $G(K)$ and $\mathcal{U}$ be a countable family of open subsets of $G(K)$. Let $t_\nu: \omega^2 \to G$ where $\nu \in \{0, 1\}$ have the property:

For any $K \in C(K)$ and any $a \in \mathbb{Q}$ the set

$$s^{-1}\left( \bigcup_{\nu \in \{0, 1\}} \left( t_\nu(\omega^2) - K \right) \cdot a^{-1} \right)$$

is small.

Then there is $\theta \subseteq \omega^2$ such that $\theta = \{(n_i, m_i) \mid i \in \omega\}$ where $n_{i+1} > n_i$, $s(n_i, m_i) \to 0$ as $i \to \infty$ in $\tau(\mathcal{U}, K)$ and for any $K \in C(K)$, any $\vec{a} \in \mathbb{Q}^\infty$

$$\left( \vec{a}(s(\theta) \cup \{0\}) + K \right) \cap t_\nu(\omega^2) = P \cap t_\nu(\omega^2), \quad \nu \in \{0, 1\},$$

for some $P \in C(K)$.

Proof. Let $C(K) = \{K_i\}_{i \in \omega}$. Using (25) choose $e_{-1} = 0$ and $e_k = s(n_k, m_k)$ for $k \in \omega$ so that $n_{k+1} > n_k$ for any $k \in \omega$ and for $k \in \omega \cup \{-1\}$

$$e_k \notin \bigcup_{\nu \in \{0, 1\}} \left( t_\nu(\omega^2) - \left( \bigcup_{i \leq k} K_i + \vec{a}(\{e_i \mid i < k\}) \right) \right) \cdot a^{-1},$$

and

$$e_k \in U_k \cap s(\omega^2),$$

where $\tau_0(\mathcal{U}, K) = \{U_k\}_{k \in \omega \cup \{-1\}}$. Here (20)–(21) and (16) are used. Now put $\theta = \{(n_i, m_i) \mid i \in \omega\}$ and consider the set $R = \vec{a}(s(\theta) \cup \{0\}) + K_n$ for some $n \in \omega$.

We have $\vec{a} = (a_1, \ldots, a_k)$ for some $k \in \omega$, and $\vec{a} \in \mathbb{Q}(i_\vec{a})$ for some $i(\vec{a}) \in \omega$. The set $A = \{(\vec{a}, \vec{b}) \mid \vec{b} \in \{0, 1\}^k \setminus \{0\}\}$ is finite so there exists $r = \max\{n_\mathbb{Q}(a) \mid a \in A\} < \infty$.

Let $M = \max\{i(\vec{a}), r, n\}$. Then

$$R = \bigcup_{(i_1, \ldots, i_k) \in (\omega \cup \{-1\})^k} a_1 \cdot e_{i_1} + \cdots + a_k \cdot e_{i_k} + K_n.$$
Denote

\[ L = \bigcup_{(i_1, \ldots, i_k) \in \Omega \setminus \{ i | i \leq M \}^k} a_1 \cdot e_{i_1} + \cdots + a_k \cdot e_{i_k} + K_n. \]

Now

\[ \bar{a}(s(\theta) \cup \{0\}) + K_n = R = (\bar{a}(\{e_i | i \leq M\}) + K_n) \cup L. \]

Suppose that there is a point \( t \in t_\nu(\omega^2) \cap L \) then

\[ t = a_1 \cdot e_{i_1} + \cdots + a_k \cdot e_{i_k} + f, \]

where \((i_1, \ldots, i_k) \in \Omega \setminus \{ i | i \leq M \}^k\) and \( f \in K_n \). We may assume that \( i_1 = \max\{i_j | j \leq k\} \). Then

\[ \left( \sum_{\nu=1}^{\tau} a_\nu \right) \cdot e_{i_1} = t - (a_1 \cdot e_{j_1} + \cdots + a_k \cdot e_{j_k} + f). \]

where \( j_i < i_1 \) for \( i \leq k \). Thus \( a - \sum_{\nu=1}^{\tau} a_\nu \neq 0 \) (because \((i_1, \ldots, i_k) \in \Omega\)) and

\[ e_{i_1} = \left( t - (f + a_1 \cdot e_{j_1} + \cdots + a_k \cdot e_{j_k}) \right) \cdot a^{-1}, \]

where \( \bar{a} \in \mathbb{Q}_M, M < i_1, n_\mathbb{Q}(a) \leq r \leq M < i_1, f \in K_n \) and \( n \leq M < i_1 \) which contradicts (26). So \( t_\nu(\omega^2) \cap L = \emptyset \). Put \( P = (\bar{a}(\{e_i | i \leq M\}) + K_n) \). Then \( P \in C(K) \) and \( R \cap t_\nu(\omega^2) = (P \cup L) \cap t_\nu(\omega^2) = P \cap t_\nu(\omega^2). \)

**Lemma 2.2.** Let \( K \) be a countable family of compact subsets of \( G \). Let \( t : \omega^2 \rightarrow G \) and \( s : \omega^2 \rightarrow G \) be correct tables in \( G(K) \) such that for any \( K \in C(K) \) the set \( t^{-1}(K) \) is small. Then there is a thick \( \sigma \subseteq \omega^2 \) such that for any \( \theta = \{(n_i, m_i) | i \in \omega\} \subseteq \sigma \) where \( n_{i+1} > n_i \) for \( i \in \omega \), any \( \bar{a} \in \mathbb{Q}_\infty \) and any \( K \in C(K) \) the set

\[ t^{-1}(\bar{a}(s(\theta) \cup \{0\}) + K) \]

is small.

**Proof.** Let \( C(K) = \{ K_i \}_{i \in \omega} \). Suppose that for some \( 0 \neq a \in \mathbb{Q} \), \( P \in C(K) \) and \( \bar{a} \in \mathbb{Q}_\infty \) the set

\[ s^{-1}\left( \left( \bigcup_{i \leq k} K_i^t \cup B^t \right) - \left( \bigcup_{i \leq k} K_i + \bar{a}(P) \right) \right) \cdot a^{-1} \]  

is thick. Denote \((\ldots) \cdot a^{-1} = K' \). Then \( K' \subseteq L \in C(K) \) for some \( L \) by (24) and for any \( \theta \subseteq \omega^2 \) such that \( \theta \subseteq s^{-1}(L) \) and any \( K \in C(K) \) the set

\[ t^{-1}(\bar{a}(s(\theta) \cup \{0\}) + K) \subseteq t^{-1}(\bar{a}(L) + K) \]

is small. So assume that every set of the form (27) is small. Then choose by induction \((n_k, m_k) \in \omega^2 \) so that \( n_{k+1} > n_k \) and
\{ s(n_k, m) \mid m \geq M^k \} \subseteq G \setminus \left( \bigcup_{n \in \mathbb{Q}(a) \leq k} \left( \bigcup_{i \leq k} K_i^t \cup B^t \right) - \left( \bigcup_{i \leq k} K_i + \vec{a} \langle \bigcup_{i < k} K_{n_i}^* \cup \{0\} \rangle \right) \right) \cdot a^{-1} \tag{28}

and put \( \sigma = \{(n_k, m) \in \omega^2 \mid k \in \omega, \ m \leq M^k \} \). If \( \theta \subseteq \sigma \) is such that \( \theta = \{(n_i', m_i') \mid i \in \omega \} \) where \( n_{i+1}' > n_i' \) then making \( \theta \) larger if necessary we may assume that \( \theta = \{(n_i, m_i) \mid i \in \omega \} \) where \( m_i \geq M^i \). Put \( e_{-1} = 0 \) and \( e_k = s(n_k, m_k) \) for \( k \in \omega \) then (28) gives

\[ e_k \not\in \bigcup_{n \in \mathbb{Q}(a) \leq k} \left( \bigcup_{i \leq k} K_i^t \cup B^t \right) - \left( \bigcup_{i \leq k} K_i + \vec{a} \langle \{ e_i \mid i < k \} \cup \{0\} \rangle \right) \cdot a^{-1}. \tag{29} \]

Consider the set \( R = \vec{a} \langle s(\theta) \cup \{0\} \rangle + K_n = \vec{a} \langle \{ e_k \mid k \in \omega \} \cup \{0\} \rangle + K_n \) where \( \vec{a} \in \mathbb{Q}_n, \ n \in \omega. \) Now \( \vec{a} = (a_1, \ldots, a_k) \) for some \( k \in \omega \) so \( \vec{a} \in \mathbb{Q}_{(\vec{a})} \) for some \( i(\vec{a}) \in \omega. \) The set \( A = \{ (\vec{a}, \vec{b}) \mid \vec{b} \in \{0, 1\}^k \} \setminus \{0\} \) is finite so there exists \( r = \max \{ n_{\mathbb{Q}(a)} \mid a \in A \} < \infty. \) Put \( M = \max \{ i(\vec{a}), r, n \}. \) Now

\[ R = \bigcup_{(i_1, \ldots, i_k) \in \Omega} a_1 \cdot e_{i_1} + \cdots + a_k \cdot e_{i_k} + K_n, \]

where \( \sum_{i=1}^{n_k} a_\nu \neq 0 \) or \( e_i = 0 \) for \( (i_1, \ldots, i_k) \in \Omega \) and \( \Omega \) is as in Lemma 2.1. So

\[ \vec{a} \langle \{ e_k \mid k \in \omega \} \cup \{0\} \rangle + K_n = R = \vec{a} \langle \{ e_i \mid i \leq M \} \cup \{0\} \rangle + K_n \cup L, \]

where

\[ L = \bigcup_{(i_1, \ldots, i_k) \in \Omega \setminus \{ (i \leq M) \}} a_1 \cdot c_{i_1} + \cdots + a_k \cdot c_{i_k} + K_n. \]

If for some \( m > M \) the set \( t(\{m\} \times \omega) \cap L \) is infinite then

\[ a_1 \cdot e_{i(1,l)} + \cdots + a_k \cdot e_{i(k,l)} + f^l = t(m, k_l), \quad l \in \omega, \]

where \( k_{l+1} > k_l, f^l \in K_n, (i(1,l), \ldots, i(k,l)) \in \Omega \setminus \{i \mid i \leq M\}^k. \) If there are \( s, l \in \omega \) such that \( i(s, l) > m > M \) and \( i(s, l) = \max \{ i(s', l) \mid s' < k \} \) then

\[ \left( \sum_{i(\nu, l) = i(s, l)} a_\nu \right) \cdot e_{i(s, l)} = t(m, k_l) - (f^l + a_1 \cdot e_{j(1,l)} + \cdots + a_k \cdot e_{j(k,l)}), \]

where \( j(\nu, l) < i(s, l) \) if \( \nu < k \) and \( \sum_{i(\nu, l) = i(s, l)} a_\nu = a \neq 0, a \in A \) and thus \( n_{\mathbb{Q}(a)} \leq r < M < m < i(s, l) \) so

\[ e_{i(s, l)} \in \left( \bigcup_{i \leq i(s, l)} K_i^t - \left( \bigcup_{i \leq i(s, l)} K_i + \vec{a} \langle \{ e_i \mid i < i(s, l) \} \cup \{0\} \rangle \right) \right) \cdot a^{-1}. \]
contradicting (29). So
\[ a_1 \cdot e_{i(1,l)} + \cdots + a_k \cdot e_{i(k,l)} + f^l = t(m, k_l), \quad l \in \omega, \]
where \( k_{l+1} > k_l \) and \( i(s, l) \leq m \), \( (i(1,l), \ldots, i(k,l)) \in \Omega \setminus \{i \mid i \leq M\}^k \). The set
\[ \bigcup_{l \in \omega} a_1 \cdot e_{i(1,l)} + \cdots + a_k \cdot e_{i(k,l)} \subseteq \tilde{a}\{e_i \mid i \leq m\} \cup \{0\} \]
is finite so the set
\[ K' = \bigcup_{l \in \omega} \langle \tilde{a}, (e_{i(1,l)}, \ldots, e_{i(k,l)}) \rangle + K_n \]
is compact in \( G(K) \). But the set
\[ K' \cap \{t(m, n) \mid n \in \omega\} \]
is infinite so by (20), and (24) there is \( b^l \in B^l \) such that
\[ a_1 \cdot e_{i(1,l)} + \cdots + a_k \cdot e_{i(k,l)} + f = b^l \]
for some \( l \in \omega, f \in K_n \). Let \( j = \max\{i(j', l) \mid j' \leq k\} \) then \( j > M \) and
\[ \left( \sum_{i(v,l)=j} a_{i(v,l)} \right) \cdot e_j = b' - (f + a_1 \cdot e_{i_1} + \cdots + a_k \cdot e_{i_k}) \]
and
\[ a_j \in \left( B^l - \left( \bigcup_{i \leq j} K_i + \tilde{a}\{e_i \mid i \leq j\} \cup \{0\}\right) \right) \cdot a^{-1}, \]
where \( a = \sum_{i(v,l)=j} a_{i(v,l)} \neq 0, n_{\mathbb{Q}}(a) < j, \tilde{a} \in \mathbb{Q}_j \) contradicting (29). So \( \{t(m, n) \mid n \in \omega\} \cap L \) is finite for \( m > M \) and the set \( t^{-1}(L) \) is small. Now put \( N = \tilde{a}\{e_i \mid i \leq M\} \cup \{0\} + K_n \). Then \( N \in C(K) \) and \( t^{-1}(N) \) is small. So \( t^{-1}(R) = t^{-1}(N \cup L) \) is small. \( \Box \)

**Lemma 2.3.** Let \( K \) be a countable family of compact subsets of \( G \), let \( s : \omega^2 \to G \) be a correct table in \( G(K) \), \( t_\nu : \omega^2 \to G, \nu \in \{0, 1\} \) be injections, such that for any \( K \in C(K), \) any \( \tilde{a} \in \mathbb{Q}^\infty \) the set \( t^{-1}_0(\tilde{a}(\langle s(\omega^2) \rangle + K)) \) is small, \( t^{-1}_1(K) \cap t^{-1}_0(K) \) is short. Then there is \( f : \omega \to \omega \) such that
\[ t^{-1}_0(\tilde{a}(\langle s(f) \rangle) + K) \cap t^{-1}_1(\tilde{a}(\langle s(f) \rangle) + K) \]
is short for any \( \tilde{a} \in \mathbb{Q}^\infty, K \in C(K) \).

**Proof.** Let \( \{(\tilde{a}_i, K_i) \mid i \in \omega\} \) be the set of all pairs such that \( \tilde{a}_i \in \mathbb{Q}^\infty, K_i \in C(K) \).
If \( \sigma \subseteq \omega^2 \) is short let \( sh(\sigma) = \min\{n \in \omega \mid \{(i \mid i \geq n) \times \omega) \cap \sigma = \emptyset\} \).
For a small \( \theta \subseteq \omega^2 \) let \( sm(\theta) = \min\{n \in \omega \mid \{k \times \omega \cap \theta \text{ is finite for } k \geq n\} \).
Let also \( S_m = s(\{m\} \times \{i \mid i \geq n\}) \cup \{s_m\} \) where \( s(m, n) \to s_m \) as \( n \to \infty \) in \( C(K) \).
Let us choose by induction \( f(k) \in \omega \) such that if \( n < k \) and
\[ P^n = \bigcup_{i \leq n} S_i^{f(i)} \cup B^n, \]

\[ M_n = \max \left\{ \min \left( t^{-1}_\nu (\overline{a_n (s(\omega^2))) + K_n}) \right), \, sh \left( \bigcap_{\nu \in \{0,1\}} t^{-1}_\nu (\overline{a_n (P^n)} + K_n) \right) \right\}, \quad (30) \]

then

\[ sh \left( \bigcap_{\nu \in \{0,1\}} t^{-1}_\nu (\overline{a_n (P^k)} + K_n) \right) \leq M_n. \quad (31) \]

Suppose that \( f(k) \in \omega \) have been chosen so that (30)–(31) hold for \( n \leq k \leq N \) where \( N \in \omega \) and let \( n \leq N. \) Put \( P_{j+1}^N = S_{j+1}^N \cup P^N. \) Then both \( P^N \) and \( P_{j+1}^N \) are compact subsets of \( G(K), \cap_{j \in \omega} P_{j+1}^N = P^N \) and

\[ t^{-1}_\nu (\overline{a_n (P^N)} + K_n) = t^{-1}_\nu (\overline{\bigcap_{j \in \omega} P_{j+1}^N}) + K_n \]

\[ = \bigcap_{j \in \omega} t^{-1}_\nu (\overline{a_n (P_{j+1}^N)} + K_n), \quad \nu \in \{0,1\}, \quad (32) \]

and by Lemma 1.8 and (20), (21), \( P_{0}^{N+1} \subseteq K \subseteq C(K) \) for some \( K \) so the set

\[ \bigcap_{\nu \in \{0,1\}} t^{-1}_\nu (\overline{a_n (P_{0}^{N+1})} + K_n) \]

is short. Thus the set

\[ \left( \{ i \mid i \geq M_n \} \times \omega \right) \cap \bigcap_{\nu \in \{0,1\}} t^{-1}_\nu (\overline{a_n (P_0^{N+1})} + K_n) \]

\[ \subseteq \left( \{ i \mid i \geq M_n \} \times \omega \right) \cap t^{-1}_0 (\overline{a_n (s(\omega^2))} + K_n) \]

is finite by (30). So using (30), (31) for \( k = N, \) (32) and the fact that \( P_i^{N+1} \subseteq P_j^{N+1} \) if \( i \geq j \) one can find \( j(n) \in \omega \) such that

\[ \left( \{ i \mid i \geq M_n \} \times \omega \right) \cap \bigcap_{\nu \in \{0,1\}} t^{-1}_\nu (\overline{a_n (P_{j(n)}^{N+1})} + K_n) = \emptyset. \]

If \( f(N+1) = \max \{ j(n) \mid n \leq N \} \) then (30) and (31) hold for \( n \leq N, \) \( n \leq k \leq N+1 \) and thus by the definition of \( M_n \) for \( n \leq k \leq N+1. \) Let \( \tilde{a} = (a_0, \ldots, a_m) \) for some \( m \in \omega, \) \( K \subseteq C(K) \) then \( (\tilde{a}, K) = (\overline{a_n (K_n)}, K_n) \) for some \( n \in \omega \) and if \( b_i \in s(\omega^2) \) where \( i \leq m \) then \( \{ b_0, \ldots, b_m \} \subseteq P^k \) for \( k \in \omega \) and by (31)

\[ \left( \{ i \mid i \geq M_n \} \times \omega \right) \cap \bigcap_{\nu \in \{0,1\}} t^{-1}_\nu (\overline{a_n (\{ b_0, \ldots, b_m \})} + K_n) = \emptyset. \quad \Box \]

**Lemma 2.4.** Let \( K \) be a countable family of compact subsets of \( G, \) \( s : \omega^2 \to G \) be a correct table in \( G(K) \) such that for any thin \( \theta \subseteq \omega^2 \) the set \( s(\theta) \) is a closed discrete subset of \( G(K). \) Let \( K \subseteq C(K), \) \( a \in Q \) and \( t : \omega^2 \to G \) be a correct table in \( G(K) \) such that

\[ \overline{s(\omega^2)} \subseteq \overline{t(\omega^2)} + \overline{K} \cdot a^{-1}. \]
Then there exists $s': \omega^2 \to G$ a correct table in $G(K)$ such that $s'(\omega^2) \subseteq \overline{t(\omega^2)}$ and for any infinite thin $\delta \subseteq \omega^2$ there is an infinite thin $\theta \subseteq \omega^2$ such that

$$s(\theta) \subseteq (s'(\delta) + K) \cdot a^{-1}.$$ 

**Proof.** Let us first consider the case when

$$\overline{s(\omega^2)} \subseteq \overline{t(\omega^2)} + K.$$ 

Then choose $s''(m, n) \in \overline{t(\omega^2)}$ and $f(m, n) \in K$ for $m, n \in \omega$ such that

$$s''(m, n) + f(m, n) = s(\omega^2).$$ 

Using compactness of $K$ one can choose a thick $\sigma \subseteq \omega^2$ such that $s(\psi_\sigma(i, j)) \to f_i$ as $j \to \infty$, $f_i \to f$ as $i \to \infty$. Since by (20), $s(m, n) \to s_m$ as $n \to \infty$ one has that $s''(\psi_\sigma(i, j)) \to s'_i$ as $j \to \infty$ for some $s'_i \in \overline{t(\omega^2)}$ and $s'_i + f_i = s_m(i)$ where $\psi_\sigma(i \times \omega) \subseteq \{m(i)\} \times \omega$. Similarly $s'_i \to s'$ as $i \to \infty$ and $s' + f = 0$. If infinitely many sets of the form $\{s''(\psi_\sigma(i, j)) \mid j \in \omega\}$ are finite then there are $n(i), k(i) \in \omega$ where $i \in \omega$ such that $n(i + 1) > n(i)$ and $s''(\psi_\sigma(n(i), k(i))) = s_{n(i)}$. Then

$$s'_{n(i)} + f(n(i), k(i)) = \{s''(\psi_\sigma(n(i), k(i))) \mid i \in \omega\}$$ 

is an infinite closed discrete subset of $G(K)$ but

$$\{s'_{n(i)} + f(n(i), k(i)) \mid i \in \omega\} \subseteq \{(s''_{n(i)} \mid i \in \omega) \cup \{-f\}) + K$$ 

with the last set compact. So changing $\sigma$ if necessary we may assume that if $(i, j) \neq (i', j')$ then $s''(\psi_\sigma(i, j)) \neq s''(\psi_\sigma(i', j'))$ and either $s'_i = s'$ for every $i \in \omega$ or $s'_i \neq s'$ if $i \neq i'$. Put $s'(i, j) = s''(\psi_\sigma(i, j))$. Similar considerations give that every subset of the form

$$\{s'(m_i, n_i) \mid i \in \omega\} = \{s(\psi_\sigma(m_i, n_i)) - f(\psi_\sigma(m_i, n_i)) \mid i \in \omega\},$$

where $m_{i+1} > m_i$ is a closed discrete subset of $G(K)$. It follows that $S = s'(\omega^2) \cup \{s'_i \mid i \in \omega\}$ is a closed subspace of $G(K)$ which is not locally compact at $s'$. Since at every point except 0 the subspace $\overline{t(\omega^2)} \subseteq G(K)$ is locally compact one has that $s' = 0$. Using first-countability of $G$ we may assume, making $\sigma$ smaller if necessary, that $S$ is compact in $G$ and for any thin $\theta \subseteq \omega^2$ the set $s'(\theta) \cup \{0\}$ is a convergent sequence in $G$. Now if $\delta \subseteq \omega^2$ is infinite and thin then $\theta = \psi_\sigma(\delta)$ is infinite and thin and

$$s(\theta) = \{s''(\psi_\sigma(m, n)) + f(\psi_\sigma(m, n)) \mid (m, n) \in \delta\} \subseteq s'(\delta) + K.$$ 

In general case it is enough to consider the map $a \cdot s$ instead of $s$. \qed

The next lemma is used in the construction of Example 3.2.

**Lemma 2.5.** Let $K$ be a countable family of compact subsets of $\mathbb{Q}$. Let $s: \omega^2 \to \mathbb{Q}$ be an injection such that $s(m, n) \to 0$ as $n \to \infty$ in $G(K)$ and for any thin $\theta \subseteq \omega^2$ the set
s(θ) is a closed discrete subset of G(K). Let S ⊆ Q be such that S ∪ {0} is compact in G(K). Let M ∈ ω and K be a compact subset of G(K) such that K ∩ S = ∅. Then there is a thick σ ⊆ ω2 such that (v(s(σ) ∪ {0}) + K) ∩ S = ∅ for any v ∈ Q_M.

**Proof.** Denote

\[ E = \{ (v, ε) \mid v ∈ Q_M, v = (v_1, \ldots, v_L), ε ∈ \{0, 1\} \} \setminus \{0\}. \]

Suppose that for any n < N an infinite set I_n ⊆ ω and i_n ∈ ω have been constructed so that i_n > i_m for m < n < N and for any w ∈ Q_M

\[ \left( \bar{w} \left( \bigcup_{n < N} s(\{i_n\} × I_n) \cup \{0\} \right) + K \right) ∩ S = ∅. \]

Let \( \bar{u}, \bar{w} ∈ Q_M, p ∈ E. \) Denote

\[ P = \bar{w} \left( \bigcup_{n < N} s(\{i_n\} × I_n) \cup \{0\} \right) + K. \]

Then P is a compact subset of G(K). Suppose that there are infinitely many k > i_{N-1} such that there exists l_k ∈ ω such that for any j > l_k

\[ \left( p ∗ s(k, l_k) + \bar{u} \left( \{s(k, j') \mid j' > j\} \cup \{0\} \right) + P \right) ∩ S ≠ ∅. \]

Let k_i > i_{N-1}, l_i ∈ ω where i ∈ ω be such that k_{i+1} > k_i and for any i ∈ ω and any j > l_i

\[ \left( p ∗ s(k_i, l_i) + \bar{u} \left( \{s(k_i, j') \mid j' > j\} \cup \{0\} \right) + P \right) ∩ S ≠ ∅. \quad (33) \]

Consider the set D = \( \{ p ∗ s(k_i, l_i) \mid i ∈ ω \}. \) Then D is a closed discrete subset of G(K). Since P and S ∪ {0} are compact there is i ∈ ω such that

\[ \left( p ∗ s(k_i, l_i) + P \right) ∩ (S ∪ \{0\}) = ∅. \quad (34) \]

Now

\[ \bigcap_{j ∈ ω} u \left( \{s(k_i, j') \mid j' > j\} \cup \{0\} \right) = u \left( \bigcap_{j ∈ ω} \{s(k_i, j') \mid j' > j\} \cup \{0\} \right) = \{0\}, \]

so since S ∪ {0} is compact, then by (34) there exists j ∈ ω such that

\[ \left( p ∗ s(k_i, l_i) + \bar{u} \left( \{s(k_i, j') \mid j' > j\} \cup \{0\} \right) + P \right) ∩ (S ∪ \{0\}) = ∅ \]

contradicting (33). Thus there is R(\bar{w}, \bar{u}, p) ∈ ω such that for any k > R(\bar{w}, \bar{u}, p), any l ∈ ω there is j > l such that

\[ \left( p ∗ s(k, l) + \bar{u} \left( \{s(k, j') \mid j' > j\} \cup \{0\} \right) + \bar{w} \left( \bigcup_{n < N} s(\{i_n\} × I_n) \cup \{0\} \right) + K \right) ∩ S = ∅. \quad (35) \]
Let $R = \max\{R(\bar{w}, \bar{u}, p) \mid \bar{w}, \bar{u} \in \mathbb{Q}_M, p \in E\}$. Then for any $\bar{w}, \bar{u} \in \mathbb{Q}_M, p \in E, k > R$, any $l \in \omega$ there is $j > l$ such that (35) holds. Put $i_N = R + 1$ and choose $n_i \subset \omega$ for $i \in \omega$ such that $n_{i+1} > n_i$ and for any $\bar{w}, \bar{u} \in \mathbb{Q}_M, p \in E, i \in \omega$

\[
\left( p \cdot s(i_N, n_i) + \bar{u}\left\{s(i_N, n_{i'}) \mid i' > i\right\} \cup \{0\} \right) + \bar{u}\left( \bigcup_{n < N} s\{i_n\} \times I_n \cup \{0\} \right) + K \right) \cap S = \emptyset, \quad (36)
\]

and put $I_N = \{n_i \mid i \in \omega\}$. Let $\tilde{w} \in \mathbb{Q}_M, \tilde{w} = (w_1, \ldots, w_L)$ and $\{q_1, \ldots, q_L\} \subseteq \bigcup_{n < N+1} s\{i_n\} \times I_n \cup \{0\}$. Assume without loss of generality that $\{q_1, \ldots, q_L\} \subseteq \bigcap_{i = N+1} s\{i_N\} \times I_N = \{q_1, \ldots, q_L\}, q_1 = s(i_N, n_1)$ and $\{q_1, \ldots, q_L\} \cap \{s(i_N, n_{j'}) \mid j' < j\} = \emptyset$. Put $p = \sum_{q_i = q_i} w_i$. We may assume without loss of generality that $p \neq 0$ (otherwise we would consider the set $\{q_1, \ldots, q_L\}$ where $q'_1 = 0$ if $q_i = q_1$ and $q'_l = q_l$ otherwise for $l \leq L'$ so $p \in E$. Put $\bar{u} = (w_{L'+1}, \ldots, w_L)$ and $\bar{u} = (u_1, \ldots, u_L)$ where $u_l = 0$ if $q_i = q_1$ and $u_l = u_l$ otherwise for and $l \leq L'$. Then $\tilde{w}, \tilde{u} \in \mathbb{Q}_M$. Now

\[
(w_1 \cdot q_1 + \cdots + w_L \cdot q_L) \in p \cdot s(i_N, n_j) + \bar{u}\left\{s(i_N, n_i) \mid i > j\right\} \cup \{0\} \right) + \bar{u}\left( \bigcup_{n < N} s\{i_n\} \times I_n \cup \{0\} \right) + K.
\]

so by (36)

\[(w_1 \cdot q_1 + \cdots + w_L \cdot q_L) \notin S.
\]

Thus for any $\bar{w} \in \mathbb{Q}_M$

\[
\left( \bar{u}\left( \bigcup_{n < N+1} s\{i_n\} \times I_n \cup \{0\} \right) + K \right) \cap S = \emptyset.
\]

so one can construct by induction for any $n \in \omega$ an infinite set $I_n \subseteq \omega$ and $i_n \in \omega$ such that for any $\bar{w} \in \mathbb{Q}_M$

\[
\left( \bar{u}\left( \bigcup_{n \in \omega} s\{i_n\} \times I_n \cup \{0\} \right) + K \right) \cap S = \emptyset.
\]

Put $\sigma = \bigcup_{n \in \omega} \{i_n\} \times I_n$. □

Let $S'_1 = \{1/n \mid n \in \mathbb{N}\} \cup \{0\} \subseteq Q, S'_1 - \{S'_1\}$. By Lemma 1.1 and (12) there exists an injection $t : \omega^2 \to Q$ such that $t(\omega^2)$ is homeomorphic to $S_2$ in $G_2(S'_1)$ so for any $K \in C_2(S'_1)$ the set $t^{-1}(K)$ is short. We may also assume without loss of generality that $t$ is a correct table in $G_2(S'_1)$. Let $t_0 : \omega^2 \to Q \times Q$ and $t_1 : \omega^2 \to Q \times Q$ be defined as follows: $t_0(m, n) = (t(m, n), 0), t_1(m, n) = (0, t(m, n))$. Then $t_0$ and $t_1$ are correct tables in $G(S^2)$ where $S^2 = (S'_1 \times \{0\} \cup \{0\} \times S'_1)$ and $S^2 = \{S_2\}$. Denote $T_\nu = t_\nu(\omega^2)$ where $\nu \in \{0, 1\}$. Then by (22), $T_\nu$ is a compact subset of $G = \mathbb{Q} \times Q$. Let $S'_0 = \{(S'_1 \times \{0\})\}$ and $S'_1 = \{(\{0\} \times S'_1)\}$. Then the identity on $Q \times Q$ gives the homeomorphisms $G(S^2) = G(S'_0 \cup S'_1) = G(S'_1)^2$. Using this and the property of $t$ mentioned above it can be shown that for any $K \in C(S^2)$ the set $t_0^{-1}(K) \cap t_1^{-1}(K)$ is
short and for any $K \in C(S^2 \cup \{T_\nu\})$ the set $t_{1-\nu}^{-1}(K)$ is short and thus is small where $\nu \in \{0, 1\}$.

**Lemma 2.6.** (CH) Let $\{O_\alpha\}_{\alpha \in \omega_1}$ list all subsets of $G$ and $\{Z_\alpha\}_{\alpha \in \omega_1}$ list all subsets of $G^2$ so that $O_0 = Z_0 = \emptyset$, $\{s_\alpha\}_{\alpha \in \omega_1}$ list all injections $s: \omega^2 \to G$ such that each $s \in \{s_\alpha\}_{\alpha \in \omega_1}$ repeats $\omega_1$ times. Then there are countable families $K_\alpha$ of compact subsets of $G$ and countable families $U_\alpha$ of subsets of $G$ such that for every $\alpha \in \omega_1$:

1. If $\beta \leq \alpha$ then $K_\beta \subseteq K_\alpha$, $S^2 \in K_\alpha$.
2. If $s_\alpha$ is a correct table in $G(K_\beta)$ for some $\beta < \alpha$ then $s_\alpha(\omega^2) \cup \{0\}$ contains a nontrivial sequence converging to 0 in $G(K_\alpha)$.
3. If $U \in U_\beta$ for some $\beta \leq \alpha$ then $U$ is open in $G(K_\alpha)$.
4. If $O_\alpha$ is open in $G(K_\alpha)$ then $O_\alpha \subseteq U_\alpha$.
5. For every $\beta \leq \alpha$ the topology of $G(K_\alpha)$ is finer than $\tau(U_\beta, K_\beta)$.
6. $Z_\alpha$ is either not closed in $G(K_\alpha)^2$ or is closed in $\tau(U_\alpha, K_\alpha)^2$.
7. If $K \in C(K_\beta \cup \{T_\nu\})$ where $\nu \in \{0, 1\}$ and $\beta \leq \alpha$ then $t_{1-\nu}^{-1}(K)$ is small.
8. For every $K \in C(K_\beta)$ where $\beta \leq \alpha$ the set $t_{1-\nu}^{-1}(K) \cap t_{1}^{-1}(K)$ is short.

**Proof.** Put $K_0 = \{S^2\}$, $U_0 = \{\emptyset\}$. Then (37)–(44) are easy to check. Suppose that the families $K_\alpha$ and $U_\alpha$ have been constructed so that they satisfy conditions (37)–(44) for all $\alpha < \kappa$ where $\kappa \in \omega_1$. Put

$$U(1) = \bigcup_{\alpha < \kappa} \tau_0(U_\alpha, K_\alpha) \cup \bigcup_{\alpha < \kappa} U_\alpha, \quad K = \bigcup_{\alpha < \kappa} K_\alpha. \tag{45}$$

If $Z_\kappa$ is closed in $G(K)^2$ let $\{L_\iota\}_{\iota \in \omega}$ be a family of open subsets of $G(K)$ that witnesses that $Z_\kappa$ is closed. Then $U(1)$ is open in $G(K)$.

Otherwise put $U(2) = U(1) \cup \{L_\iota\}_{\iota \in \omega}$. \tag{46}

If $U_\kappa$ is open in $G(K)$ put

$$U_\kappa = U(2) \cup \{O_\kappa\}. \tag{47}$$

If $U_\kappa$ is close in $C(K)$ and $U \subseteq U_\kappa$ is open in $G(K)$. It is enough to show that for any $U \in U(1)$ and any $K \in C(K)$ the intersection $U \cap K$ is relatively open in $G$. Now by (45) $U$ is open in $\tau(U_\alpha, K_\alpha)$ for some $\alpha < \kappa$ and by (45), (37), (41) and Lemma 1.3 we may assume without loss of generality that $K \subset C(K_\alpha)$. So by (16), (18) and the definition of $G(K)$ the set $U \cap K$ is relatively open in $C$. Thus the topology $\tau(U_\alpha, K)$ is well-defined.

Assume that $s_\kappa$ is correct table in $G(K_\kappa)$ for some $\alpha < \kappa$. Then $s_\kappa$ is correct table in $G(K)$. We will look for a thin set $\theta \subseteq \omega^2$ such that $K_\kappa = K \cup \{(s(\theta) \cup \{0\})\}$ satisfies (37) and (39)–(44) where $s$ is some correct table in $G(K)$. Then if there exists $K \in C(K_\kappa)$ such that $s_\kappa^{-1}(K)$ contains an infinite thin set then by (23) and (38) holds. So if $s$ is chosen so that for any infinite thin $\theta \subseteq \omega^2$ there is $K \in C(K_\kappa)$ such that $s_\kappa^{-1}(K)$ contains an infinite thin subset then (37)–(44) will hold. One of the examples of such $s$ is $s = s_\kappa \circ \psi_\sigma$ for some thick $\sigma \subseteq \omega^2$. Another example is given in Lemma 2.4. For the sake of convenience we will assume in such cases that $s_\kappa$ has the
additional properties of $s$ instead of introducing a new correct table. Let us consider two cases.

**Case 1.** There are $a \in Q$ and $K \in C(K)$ such that the set

$$s^{-1}_\mu \left( \bigcup_{\nu \in \{0,1\}} (t_{\nu}(\omega^2) - K) \cdot a^{-1} \right)$$

is thick.

Then without loss of generality we may assume that $s_\kappa(\omega^2) \subseteq (t_1(\omega^2) - K) \cdot a^{-1}$ and thus

$$s_\kappa(\omega^2) \subseteq (T_1 - K) \cdot a^{-1}.$$  

By Lemma 1.9 we may also assume that for any thin $\theta \subseteq \omega^2$ the set $s_\kappa(\theta)$ is closed and discrete in $G(K)$ so by Lemma 2.4 we may assume that

$$s_\kappa(\omega^2) \subseteq T_1$$

and thus by (43), (37) and Lemma 1.3 for any $K \in C(K,)$ and any $\bar{a} \in Q^\infty$ the set

$$t_0^{-1}(\bar{a} \langle s_\kappa(\omega^2) \rangle + K)$$

is small. By (44) and Lemma 1.3 for any $K \in C(K,)$ the set

$$t_0^{-1}(K) \cap t_1^{-1}(K)$$

is short. By Lemma 2.3 we may assume now that for any $\bar{a} \in Q^\infty$ and any $K \in C(K,)$ the set

$$t_0^{-1}(\bar{a} \langle s_\kappa(\omega^2) \rangle + K) \cap t_1^{-1}(\bar{a} \langle s_\kappa(\omega^2) \rangle + K)$$

is short. Using (43) and applying Lemma 2.2 twice one may assume that for any $K \in C(K \cup \{T_0\},)$ any $\theta \subseteq \omega^2$ such that $\theta = \{(n_i, m_i) \mid i \in \omega \} \subseteq \omega^2$ where $n_{i+1} > n_i$, any $\bar{a} \in Q^\infty$ the set

$$t_0^{-1}(\bar{a} \langle s_\kappa(\theta) \cup \{0\} \rangle + K)$$

is small where $\nu \in \{0,1\}$. Choose $\theta = \{(n_i, m_i) \mid i \in \omega \} \subseteq \omega^2$ so that for $i \in \omega$ $n_{i+1} > n_i$ for any $i \in \omega$ and $s_\kappa(n_i, m_i) \rightarrow 0$ as $i \rightarrow \infty$ in $\tau(U_\kappa, K)$ and put

$$K_\kappa = K \cup \{(s_\kappa(\theta) \cup \{0\}) \}.  \tag{50}$$

Then (37) holds by (45) and (50). To prove (39) it is enough to show that for any $U \in U_\kappa$, any $K \in C(K_\kappa)$ the intersection $U \cap K$ is relatively open in $G$. By (18) and (19) we may assume without loss of generality that $K = \bar{a} \langle s_\kappa(\theta) \cup \{0\} \rangle + K'$ for some $\bar{a} \in Q^\infty$, $K' \in C(K)$. Then by the choice of $\theta$ $K$ is compact in $\tau(U_\kappa, K)$ and since $U \in U_\kappa$, then $U$ is open in $\tau(U_\kappa, K)$ and $U \cap K$ is relatively open in $\tau(U_\kappa, K)$. Now since $K$ is compact in $\tau(U_\kappa, K)$ the intersection $U \cap K$ is relatively open in $G$ by (16). So (39) holds. (40) holds by (45)–(47) and the fact that the topology of $G(K)$ is finer than the topology of $G(K_\kappa)$. If $\beta \subseteq \kappa$ then by (45), $\tau_0(U_\beta, K_\beta) \subseteq U_\kappa$ so by (17), $\tau(U_\kappa, K_\kappa)$ is finer than $\tau(U_\beta, K_\beta)$. Since by (16) the topology of $G(K_\kappa)$ is finer than $\tau(U_\kappa, K_\kappa)$ (41) holds. (42) holds by the choice of $\{L_i\}_{i \in \omega}$, (46)–(47) and the fact that the topology of $G(K)$ is finer.
than the topology of $G(K)$. If $K \in C(K_{\kappa} \cup \{T_{\nu}\}) = C(K \cup \{T_{\nu}\} \cup \{(s_{\kappa}(\theta) \cup \{0\})\})$ then by (19)

$$K \subseteq \alpha(s_{\kappa}(\theta) \cup \{0\}) + K'$$

for some $\alpha \in Q^\infty$, $K' \in C(K \cup \{T_{\nu}\})$. Then by (49) and the choice of $\theta$ the set $t_{\nu}^{-1}(K)$ is small. So (43) holds. Similarly (44) holds by (19) and (48). Now (38) holds by the choice of $\theta$ and the remark before Case 1.

**Case 2.** For any $a \in Q$, any $K \in C(K)$ the set $s_{\kappa}^{-1}\left(\bigcup_{\nu \in \{0,1\}} (t_{\nu}(\omega^2) - K) \cdot a^{-1}\right)$ is small.

Using (43) and applying Lemma 2.2 twice one may assume without loss of generality that for any $K \in C(K \cup \{T_{\nu}\})$, any $\theta \subseteq \omega^2$ such that $\theta = \{(n_i, m_i) \mid i \in \omega\}$ where $n_{i+1} > n_i$ for any $i \in \omega$, any $\alpha \in Q^\infty$ the set $t_{\nu}^{-1}(\alpha(s_{\kappa}(\theta) \cup \{0\}) + K)$ is small. By Lemma 2.1 choose $\theta = \{(n_i, m_i) \mid i \in \omega\}$ so that $n_{i+1} > n_i$ for any $i \in \omega$ and $s_{\kappa}(n_i, m_i) \to 0$ as $i \to \infty$ in $\tau(U_{\kappa}, K)$ and for any $K \in C(K_{\kappa})$, any $\alpha \in Q^\infty$

$$(\alpha(s_{\kappa}(\theta) \cup \{0\}) + K) \cap t_{\nu}(\omega^2) = P \cap t_{\nu}(\omega^2), \quad \nu \in \{0,1\},$$

for some $P \in C(K)$. Put $K_{\kappa} = K \cup \{(s_{\kappa}(\theta) \cup \{0\})\}$. Then (37)–(44) may be checked similarly to Case 1. □

Let $S^1$, $S^1$ be as in the remark before Lemma 2.6. Then by Lemma 1.1 and [16, Lemma 2.1] there is an injection $t: \omega^2 \to Q$ such that $t(\omega^2) \cup \{0\}$ is a closed subset of $G(S)$ homeomorphic to $S_{\omega}$ so $t(m, n) \to 0$ as $n \to \infty$ for every $m \in \omega$ in $G(S^1)$ and for any $K \in G(S^1)$ the set $t^{-1}(K)$ is short.

**Lemma 2.7.** (CH) Let $\{O_{\alpha}\}_{\alpha \in \omega_1}$ and $\{Z_{\alpha}\}_{\alpha \in \omega_1}$ list all subsets of $Q$ so that $O_0 = Z_0 = \emptyset$, $\{s_{\alpha}\}_{\alpha \in \omega_1}$ list all injections $s: \omega^2 \to Q$ such that each $s \in \{s_{\alpha}\}_{\alpha \in \omega_1}$ repeats $\omega_1$ times.

Then there are countable families $K_{\alpha}$ of compact subsets of $Q$, countable families $U_{\alpha}$ of subsets of $Q$, compact subsets $K_{\alpha}$ of $Q$, subsets $D_{\alpha}$ of $Q$ such that for every $\alpha \in \omega_1$:

1. $K_{\alpha} = \bigcup_{\beta < \alpha} K_{\beta} \cup \{K_{\alpha}\}$, $S^1 \in K_{\alpha}$,
2. if $s_{\alpha}$ is such that $s(m, n) \to 0$ as $n \to \infty$ in $G(K_{\beta})$ for some $\beta < \alpha$ then $K_{\alpha} \subseteq s_{\alpha}(\omega^2) \cup \{0\}$ and $s_{\alpha}^{-1}(K_{\alpha})$ is thick; otherwise $K_{\alpha} = S^1$,
3. $K_{\alpha}$ is a nontrivial convergent sequence with the limit point $0$ in $G(K_{\alpha})$,
4. if $O_{\alpha}$ is open in $G(K_{\alpha})$ then $O_{\alpha} \subseteq U_{\alpha}$,
5. if $U \subseteq U_{\beta}$ for some $\beta \leqslant \alpha$ then $U$ is open in $G(K_{\alpha})$,
6. for every $\beta \leqslant \alpha$ the topology of $G(K_{\alpha})$ is finer than $\tau(U_{\beta}, K_{\beta})$,
7. if there is no $K \in C(K_{\alpha})$ such that $Z_{\alpha} \subseteq K$ then $D_{\alpha} \subseteq Z_{\alpha}$ is a closed infinite discrete subset in $\tau(U_{\alpha}, K_{\alpha})$; otherwise $D_{\alpha} = \emptyset$. 
(58) for any $K \in C(K_\alpha)$ there is $n \in \omega$ such that

$$K \cap t(\{n\} \times \omega) = \emptyset.$$  

**Proof.** Put $K_0 = \{S^1\}$, $K_0 = S^1$, $U_0 = \{\emptyset\}$, $D_0 = \{\emptyset\}$. Then (51)-(58) are easy to check. Suppose that families $K_\alpha$, $U_\alpha$ and the sets $K_\alpha$, $D_\alpha$ have been constructed so that they satisfy conditions (51)-(58) for all $\alpha < \kappa$ where $\kappa \in \omega_1$. Put

$$U_{(1)} = \bigcup_{\alpha < \kappa} \tau_0(U_\alpha, K_\alpha) \cup \bigcup_{\alpha < \kappa} U_\alpha, \quad K = \bigcup_{\alpha < \kappa} K_\alpha. \quad (59)$$

If there is no $K \in C(K)$ such that $Z_K \subseteq K$ then by Lemma 1.8 there is $D_\kappa \subseteq Z_\kappa$ such that $D_\kappa$ is an infinite closed discrete subset of $G(K)$. Let $\{L_i\}_{i \in \omega}$ be a family of open subsets of $G(K)$ that witnesses that and

$$U_{(2)} = U_{(1)} \cup \{L_i\}_{i \in \omega}. \quad (60)$$

If $O_\kappa$ is open in $G(K)$ put

$$U_\kappa = U_{(2)} \cup \{O_\kappa\}. \quad (61)$$

Otherwise $U_\kappa = U_{(2)}$. Assume that $s_\kappa$ is such that $s(m, n) \to 0$ as $n \to \infty$ in $G(K_\alpha)$ for some $\alpha < \kappa$. Using first-countability of $\mathbb{Q}$ we may assume that for any thick $\sigma \subseteq \omega^2$ the set $s_\kappa(\sigma) \cup \{0\}$ is a convergent sequence in $\mathbb{Q}$. Let $\{(\tilde{a}, K) \mid \tilde{a} \in \mathbb{Q}^\infty, \ K \in C(K)\} = \{(\tilde{a}_n, K_n) \mid n \in \omega\}$. Let us construct by induction $i_n, j_n \in \omega$, infinite sets $J_n \subseteq \omega$ and thick sets $s_n \subseteq \omega^2$ such that for any $n \in \omega$, $\sigma_{n+1} \subseteq \sigma_n$, $i_{n+1} > i_n$, $j_{n+1} > j_n$ and

$$\bigcup_{k < n} \{j_k\} \times J_k \subseteq \sigma_n \quad (62)$$

and for any $l \leq n$

$$(\tilde{a}_l(s_\kappa(\sigma_l) \cup \{0\}) + K_l) \cap t(\{i_n\} \times \omega) = \emptyset. \quad (63)$$

Suppose that $i_n, j_n \in \omega$, $J_n \subseteq \omega$, $\sigma_n \subseteq \omega^2$ have been constructed for every $n < N$. $N \in \omega$ so that (62)-(63) hold. Let

$$s = s_\kappa \circ \psi_{\sigma_{N-1}}.$$

Denote

$$P = \bigcup_{l < N+1} \left( \tilde{a}_l \left( \bigcup_{k < N} s_\kappa(\{j_k\} \times J_k) \cup \{0\} \right) + K_l \right). \quad (64)$$

Then we may assume without loss of generality that $P \subseteq P'$ for some $P' \in C(K_\alpha)$. By (58) one can choose $i_N \in \omega$ such that $i_N > i_n$ for $n < N$ and $P \cap t(\{i_N\} \times \omega) = \emptyset$. Now by Lemma 2.5 there is thick $\sigma \subseteq \omega^2$ such that

$$(\tilde{a}_n(s(\sigma) \cup \{0\}) + P) \cap t(\{i_N\} \times \omega) = \emptyset \quad (65)$$

for any $n < N + 1$. Let $j_N \in \omega$ be such that $j_N > j_n$ for $n < N$, $\psi_{\sigma_{N-1}}(\{j'\} \times \omega) \subseteq \{j_N\} \times \omega$ for some $j' \in \omega$ and $J_N = \pi_2(\psi_{\sigma_{N-1}}(\{j'\} \times \omega \cap \sigma))$ is infinite. Put

$$\sigma_N = \psi_{\sigma_{N-1}}(\sigma) \cup \bigcup_{k < N} \{j_k\} \times J_k.$$
Then (62) holds. Let \( l < N + 1 \), \( \tilde{a}_l = (a_1, \ldots, a_m) \) for some \( m \in \omega \), \( \{q_1, \ldots, q_m\} \subseteq s_{k_l}(\sigma_N) \cup \{0\} \), \( p \in K_l \). We may assume without loss of generality that for some \( m' \leq m \) \( \{q_1, \ldots, q_{m'}\} \subseteq s(\sigma) \) and \( \{q_{m'+1}, \ldots, q_m\} \subseteq \bigcup_{k<N} s_{k_l}(\{j_k \times j_k\} \cup \{0\}) \). Then

\[
\tau = a_1 \cdot 0 + \cdots + a_{m'} \cdot 0 + a_{m'+1} \cdot q_{m'+1} + \cdots + a_m \cdot q_m + p \in P
\]

and

\[
q = a_1 \cdot q_1 + \cdots + a_{m'} \cdot q_{m'} + a_{m'+1} \cdot 0 + \cdots + a_m \cdot 0 \in \tilde{a}_l(s(\sigma) \cup \{0\}).
\]

So

\[
a_1 \cdot q_1 + \cdots + a_m \cdot q_m + p = q + r \in \tilde{a}_l(s(\sigma) \cup \{0\}) + P
\]

and by (65)

\[
(a_1 \cdot q_1 + \cdots + a_m \cdot q_m) \notin \tau(\{i_N\} \times \omega).\]

So (63) holds. Put \( \sigma = \bigcap_{n \in \omega} \sigma_n. \) Then by (62) \( \sigma \) is thick. Put \( K_\sigma = s_\sigma(\sigma) \cup \{0\}. \) \( K_\sigma = K \cup \{K_\sigma\}. \) Then (51)-(57) may be checked as in Lemma 2.6. If \( K \in C(K_\sigma) \) then by (19) \( K \subseteq \tilde{a}(K_{\sigma}) \) for some \( K' \in C(K), \) \( \tilde{a} \in \mathbb{Q}^\infty. \) If \( (\tilde{a}, K') = (\tilde{a}_l, K_l) \) for some \( l \in \omega \) then by (63)

\[
K \cap \tau(\{i_n\} \times \omega) = \emptyset
\]

for \( n > l. \) So (58) holds. □

3. Examples

In the example below a countable Fréchet topological group with sequential non-Fréchet square is constructed.

Example 3.1. (CH) Let \( K = \bigcup_{\alpha<\omega_1} C(K_\alpha) \) where \( K_\alpha \) are countable families of compact subsets of \( \mathbb{Q} \times \mathbb{Q} \) constructed in Lemma 2.6. Let \( \mathcal{G} \) be the set \( \mathbb{Q} \times \mathbb{Q} \) equipped with the topology defined as follows: \( U \subseteq \mathcal{G} \) is open if \( U \cap K \) is relatively open for all \( K \in K. \) The following fact easily follows from (41) and the definition of \( G(K_\alpha). \)

Fact. For any \( \alpha \in \omega_1 \) the topology of \( \mathcal{G} \) is finer than \( \tau(U_\alpha, K_\alpha). \)

Consider now an arbitrary \( O \) open in \( \mathcal{G}. \) Then \( O = O_\alpha \) for some \( \alpha \in \omega_1 \) and \( O_\alpha \) is open in the topology of \( G(K_\alpha) \) which is finer than the topology of \( \mathcal{G}. \) Thus by (40) and (15) \( O_\alpha \) is open in \( \tau(U_\alpha, K_\alpha). \) It follows from the fact and what we have proved above that the topology of \( \mathcal{G} \) is the common refinement for the family \( \{\tau(U_\alpha, K_\alpha) \mid \alpha \in \omega_1\}. \)

So \( \mathcal{G} \) is a topological group.

Let \( Z \subseteq \mathcal{G}^2 \) be an arbitrary subset. Then \( Z = Z_\alpha \) for some \( \alpha \in \omega_1. \) If \( Z \) is a nonclosed subset of \( G(K_\alpha)^2 \) then since \( G(K_\alpha)^2 \) is sequential being a product of two \( k_\omega \)-spaces there is a sequence in \( Z \) converging to a point outside \( Z \) in the topology of \( G(K_\alpha)^2 \) and thus in the coarser topology of \( \mathcal{G}^2. \) If \( Z \) is closed in \( G(K_\alpha)^2 \) then \( Z \) is closed in \( \tau(U_\alpha, K_\alpha)^2 \) by (42) and thus \( Z \) is closed in the finer topology of \( \mathcal{G}^2. \) Thus \( \mathcal{G}^2 \) is sequential.
Suppose that $\mathcal{G}$ is not Fréchet. Then there exists an injection $s : \omega^2 \to \mathbb{Q} \times \mathbb{Q}$ such that $s(m, n) \to t_m$ as $n \to \infty$ in $\mathcal{G}$ and $t_m \to 0$ as $m \to \infty$ in $\mathcal{G}$ and there is no sequence in $s(\omega^2)$ converging to $0$ in $\mathcal{G}$. Then by the definition of the topology of $\mathcal{G}$ for every $n \in \omega$ there is $K_n \in C(K_{\alpha_n})$ for some $\alpha_n \in \omega_1$ such that $K_n \cap s(\{n\} \times \omega)$ is infinite and $K_0 \cap \{t_m \mid m \in \omega\}$ is infinite. Taking $\beta = \sup\{\alpha_n \mid n \in \omega\}$ and passing to the map $s \circ \psi_\sigma$ for some thick $\sigma \subseteq \omega^2$ if necessary we may assume without loss of generality that $s$ is a correct table in $G(K_{\beta})$. Then $s = s_\alpha$ for some $\alpha > \beta$ and $s(\omega^2) \cup \{0\}$ contains a nontrivial convergent sequence with the limit point $0$ in $G(K_{\alpha})$ by (38), a contradiction. So $\mathcal{G}$ is Fréchet.

Now $(t_0(n, k), t_1(n, k)) \to (s_0(n, k), s_1(n, k))$ as $n \to \infty$ in $\mathcal{G}^2$ and $(s_0(n, k), s_1(n, k)) \to (0, 0)$ as $n \to \infty$ in $\mathcal{G}^2$. Suppose that $(t_0(n_i, k_i), t_1(n_i, k_i)) \to (0, 0)$ as $i \to \infty$ in $\mathcal{G}^2$. Then passing to a subsequence if necessary we may assume without loss of generality that
\[
\{t_0(n_i, k_i) \mid i \in \omega\} \cup \{t_1(n_i, k_i) \mid i \in \omega\} \subseteq K \subseteq K
\]
for some $K$. Then $K \in C(K_{\alpha})$ for some $\alpha \in \omega_1$. The set $\{(n_i, k_i) \mid i \in \omega\}$ is infinite and thin and $t_0^{-1}(K) \cap t_1^{-1}(K) \supseteq \{(n_i, k_i) \mid i \in \omega\}$ contradicting (44). So $\mathcal{G}^2$ is not Fréchet.

In the next example a construction of an $\alpha_3$ non-$\alpha_2$ topological group is given.

Example 3.2. (CH) Let $\mathcal{K} = \bigcup_{\alpha \in \omega_1} C(K_{\alpha})$ where $K_{\alpha}$ are countable families of compact subsets of $\mathbb{Q}$ constructed in Lemma 2.7. Let $\mathcal{G}$ be the set $\mathbb{Q}$ equipped with the topology defined as follows: $U \subseteq \mathcal{G}$ is open if $U \cap K$ is relatively open for any $K \in \mathcal{K}$. Then as in Example 3.1 $\mathcal{G}$ is a topological group and the following holds.

Fact 1. For any $\alpha \in \omega_1$ the topology of $\mathcal{G}$ is finer than $\tau(U_\alpha, K_{\alpha})$.

Now the fact above and (57) give the following fact.

Fact 2. If $K \subseteq \mathcal{G}$ is compact then $K \subseteq K'$ for some $K' \in C(K_{\alpha})$, $\alpha \in \omega_1$.

The definition of $\mathcal{G}$ gives that $\mathcal{G}$ is a quotient space of a topological sum of a family of metrizable compact spaces so $\mathcal{G}$ is sequential.

If $s : \omega^2 \to \mathcal{G}$ is any injection such that $s(m, n) \to 0$ as $n \to \infty$ for every $m \in \omega$ then passing to the map $s \circ \psi_\sigma$ for some thick $\sigma \subseteq \omega^2$ if necessary we may assume without loss of generality that $s(m, n) \to 0$ as $n \to \infty$ in some $G(K_{\beta})$, $\beta \in \omega_1$. Then $s = s_\alpha$ for some $\alpha > \beta$ and as in Example 3.1 (52), (53) give that there is a convergent sequence $K_\alpha$ with the limit point $0$ in $G(K_{\alpha})$ such that $s^{-1}(K_{\alpha})$ is thick. So $\mathcal{G}$ is $\alpha_3$ by (5).

Now $t(m, n) \to 0$ as $n \to \infty$ for every $m \in \omega$. If there is a compact set $K \subseteq \mathcal{G}$ such that $t^{-1}(K) \cap \{i\} \times \omega$ is infinite for every $i \in \omega$ then by Fact 2 we may assume that $K \subseteq C(K_{\alpha})$ for some $\alpha \in \omega_1$. Then by (58) there is $i \in \omega$ such that $K \cap t(\{i\} \times \omega) = \emptyset$. A contradiction. So $\mathcal{G}$ is not $\alpha_2$.

To prove the equivalence of $\alpha_{1.5}$ and $\alpha_1$ for Fréchet topological group we need some preliminary definitions and results.
Let $H$ be a topological group. Let us say that a sequence $S \subseteq H$ is normal if $e \notin S$, $S$ is infinite and $S$ converges to $e$. If $S_1, S_2, S_3$ are normal sequences then let us say that $S_1$ is $S$-reducible to $S_2$ if for any $x \in S_1$ there are $x' \in S_2$ and $x'' \in S$ such that $(x'')^{-1} \cdot x' = x$.

**Lemma 3.3.** Let $H$ be a topological group, $S, S', S'' \subseteq H$ be normal sequences. Then there is a normal sequence $S''' \subseteq H$ such that $S'$ is $S$-reducible to $S'''$ and $S''' \cap S'' = \emptyset$.

**Proof.** Let $S = \{x_i\}_{i \in \omega}, S' = \{x'_i\}_{i \in \omega}, S'' = \{x''_i\}_{i \in \omega}$. Suppose that for $n < N, N \in \omega$ $i_n \in \omega$ have been chosen so that $i_{n+1} > i_n, x_{i_n} \cdot x''_{i_n} \notin S'$. Consider the sequence $S \cdot x'_N$. It has $x'_N \neq e$ as its limit point and is infinite. So there is $i_N \in \omega$ such that $i_N > i_n$ for $n < N$ and $x_{i_N} \cdot x'_N \notin S''$. Now put $S''' = \{x_{i_n} \cdot x''_{i_n}\}_{n \in \omega}$. □

Let us call a sheaf $\{S_n\}_{n \in \omega}$ closed if $e$ is the only cluster point of $\bigcup_{n \in \omega} S_n$ in $H$. If $e \in H$ is a $G_\delta$-point and $\{S_n\}_{n \in \omega}$ is a normal sheaf then there is a normal closed sheaf $\{S'_n\}_{n \in \omega}$ such that every $S'_n$ is a cofinite subset of $S_n$. Indeed if $\{U_n\}_{n \in \omega}$ is a family of open subsets of $H$ such that $\overline{U_{n+1}} \subseteq U_n$ and $\bigcap_{n \in \omega} U_n = e$ then put $S'_n = S_n \cap U_n$. To prove that a group $H$ is $\alpha_1$ it is enough to check that for every normal sheaf $\{S_n\}_{n \in \omega}$ with vertex $e \in H$ there is a sequence $S$ converging to $e$ such that for every $n \in \omega$ $S \cap S_n$ is cofinite in $S_n$. Now if $\{S_n\}_{n \in \omega}$ is closed it is enough for $S$ to be compact.

**Proposition 3.4.** If $H$ is a Fréchet topological group and $H$ is an $\alpha_{1.5}$-space then $H$ is an $\alpha_1$-space.

**Proof.** Let $S \subseteq H$ be a normal sequence. It is enough to consider a countable $H$. Then by the remark above it is enough to prove that if $\{S_n\}_{n \in \omega}$ is a normal closed sheaf then there is a compact subset $K \subseteq H$ such that $K$ intersects each $S_n$ in a cofinite set. Using Lemma 3.3 one can construct a normal sheaf $\{S'_n\}_{n \in \omega}$ such that $\bigcup_{i \leq n} S_i$ is $S$-reducible to $S'_n$. Now if $P$ is compact and intersects infinitely many $S'_n$ in a cofinite set then $K = (S \cup \{e\})^{-1} \cdot P$ intersects each $S_n$ in a cofinite set. Since $H$ is an $\alpha_{1.5}$-space such $P$ exists. □

**Acknowledgements**

The author wishes to thank Professor G. Gruenhage for his advice.

**References**


