HORSESHOES, ENTROPY AND PERIODS FOR GRAPH MAPS

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1. INTRODUCTION

In recent years, many papers (and even some books) have appeared, where topological entropy and cycles (periodic orbits) are studied for maps of compact one-dimensional spaces into themselves. However, usually those one-dimensional spaces are an interval or a circle. Maps of other graphs are studied much less (see e.g. [11, 12, 1, 3, 4, 16, 15, 6, 7]).

Here we would like to contribute to filling up this gap by applying some of the machinery developed to study interval maps, to the case of graphs (see Section 2 for a precise definition). Our main goal is to understand the properties of topological entropy of the graph maps and its relation to horseshoes and periods of cycles.

For continuous maps of an interval and a circle into itself it is well known that the magnitude of the topological entropy $h(f)$ of such a map $f$ depends on the horseshoes (see [18, 17, 2]). The same turns out to be true in our case. We get also the usual corollary about lower semi-continuity of topological entropy. We define an $s$-horseshoe for $f$ in much the same way as for interval maps: there is an interval $I$ and $s$ its subintervals with pairwise disjoint interiors, each of them mapped by $f$ onto the whole $I$. Here we require that an "interval" is contained in an edge of the graph. For precise definitions, see Section 2.

**Theorem A.** If a continuous map $f$ of a graph into itself has an $s$-horseshoe then $h(f) \geq \log s$.

**Theorem B.** If a continuous map $f$ of a graph into itself has positive topological entropy then there exist sequences $(k_n)_{n=1}^{\infty}$ and $(s_n)_{n=1}^{\infty}$ of positive integers such that for each $n$ the map $f^{k_n}$ has an $s_n$-horseshoe and

$$\limsup_{n \to \infty} \frac{1}{k_n} \log s_n = h(f).$$

**Theorem C.** Topological entropy, as a function of a continuous map of a graph into itself, is lower semi-continuous.

By the same reasons as for interval maps, Theorem C cannot be improved, that is the entropy is not upper semi-continuous (see e.g. [2]).

As for other spaces, also for graphs the existence of horseshoes implies the existence of cycles of many periods. Thus, using Theorem B we are able to give various characterizations of maps with positive entropy or maps with zero entropy via the set $\text{Per}(f)$ of periods of cycles of $f$. Those results generalize the existing theorems for interval and circle maps (see [8, 18, 17, 2]).
By \( \mathbb{N} \) we shall denote the set of all natural numbers (here 0 is not a natural number). Fix \( s \in \mathbb{N} \). For any odd prime number \( p \) denote by \( \gamma_s(p) \) the smallest integer larger than \( \log(2s)/\log(p/2) \). If \( p > 4s \) then \( \gamma_s(p) = 1 \), so only for finitely many \( p \) we have \( \gamma_s(p) \neq 1 \). Therefore the number

\[
\Gamma_s = \prod_{p \text{ odd prime}} \gamma_s(p)
\]

is finite.

For \( i \in \mathbb{N} \) we shall denote by \( \text{god}(i) \) the greatest odd divisor of \( i \). Then, for a subset \( X \subseteq \mathbb{N} \), the set of gods of \( X \) (that is, \( \{ \text{god}(i): i \in X \} \) will be called the pantheon of \( X \). The following theorem shows that existence of too many gods for the set of periods of a graph map implies chaos (that is positive entropy; we also get horseshoes, so there is also chaos in the sense of Li and Yorke, see [14]). This result can be considered as the authors’ contribution to mathematical theology (see e.g. [10, 19]).

**Theorem D.** Let \( f \) be a continuous map of a graph with \( s \) edges into itself. If the pantheon of \( \text{Per}(f) \) has more than \( s \Gamma_s \) elements then \( h(f) > 0 \).

Of course the estimate \( s \Gamma_s \) is not the best possible; we have not tried to optimize it, concentrating rather on minimizing the length of the proof. Thus it remains an open question: for a given graph, how many gods are necessary to make its map chaotic? For the interval and the circle, the answer is known: two (see e.g. [2]).

Theorem D can be restated as follows: If \( f \) is a continuous map of a graph with \( s \) edges into itself and \( h(f) = 0 \) then there exist \( k \leq s \Gamma_s \) different odd numbers \( n_1, \ldots, n_k \) such that \( \text{Per}(f) \) is contained in \( \bigcup_{i=1}^k \bigcup_{j=0}^\infty 2^j n_i \). In this case, the results of [6] give us an additional information. Namely, for each \( i \) one of the sets \( \bigcup_{j=0}^\infty 2^j n_i \cap \text{Per}(f) \) and \( \bigcup_{j=0}^\infty 2^j n_i \setminus \text{Per}(f) \) is finite.

For any set \( X \subseteq \mathbb{N} \) we shall denote by \( \rho(X) \) its upper density, that is

\[
\rho(X) = \limsup_{n \to \infty} \frac{1}{n} \operatorname{Card} \{ k \in X : k \leq n \}.
\]

As an easy corollary to previous results we get the following theorem.

**Theorem E.** Let \( f \) be a continuous map of a graph into itself. Then the following statements are equivalent.

(i) \( h(f) > 0 \).

(ii) There is \( m \in \mathbb{N} \) such that \( \{ mn : n \in \mathbb{N} \} \subseteq \text{Per}(f) \).

(iii) \( \rho(\text{Per}(f)) > 0 \).

(iv) The pantheon of \( \text{Per}(f) \) is infinite.

Theorem E has been proved by Blokh [6] (the equivalence of (i), (ii) and (iii) is actually proved there; the equivalence of (i) and (iv) follows from the results of [6]). However, our proof is different from Blokh’s one (although some of the techniques used are similar). In particular, to prove the implications (i) \( \Rightarrow \) (ii) and (iv) \( \Rightarrow \) (i) Blokh uses the spectral decomposition theorem. This seems to be a more complicated way of proof than our horseshoe methods for (i) \( \Rightarrow \) (ii) and definitely is much more complicated than our classical [5] methods for (iv) \( \Rightarrow \) (i).

The implications (i) \( \Rightarrow \) (ii) and (iv) \( \Rightarrow \) (i) in Theorem E are quite strong. In general we cannot say much more; we explain it in Section 6.
At the end of the paper we also give an asymptotic estimate on the number of cycles in the terms of entropy (Theorem 6.4).

It is well known that our results cannot be generalized to the continuous maps of spaces of higher dimension. There are examples of a minimal homeomorphism of the 2-dimensional torus [20] and of an analytic diffeomorphism in dimension 4 [9] with positive topological entropy. A minimal map has no cycles and therefore cannot have any kind of horseshoes. On the other hand, it is known [13] that for $C^{1+\varepsilon}$ diffeomorphisms of surfaces the analogues of Theorems A, B and C hold.

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2. DEFINITIONS AND NOTATION

Since the word "graph" is used by different authors in different meanings, we have to specify what we mean by it. For us, a graph is not directed (oriented), is finite and connected. By edges we mean closed edges. For each edge $e$ there is a continuous map $\varphi$ from $[0, 1]$ onto $e$ such that $\varphi((0, 1))$ is a homeomorphism of $(0, 1)$ onto the set $\varphi((0, 1))$, which will be called the interior of $e$ and denoted $\text{Int}(e)$. The point(s) $\varphi(0)$ and $\varphi(1)$ will be called the endpoint(s) of $e$. Thus, $e$ is homeomorphic either to $[0, 1]$ or to a circle. Each endpoint of each edge is a vertex.

Thus, a graph $G$ is a connected compact Hausdorff space which is a union of finitely many disjoint sets: interiors of edges and the set of vertices.

The number of edges having a vertex as an endpoint (with the edges homeomorphic to a circle counted twice) will be called the valence of this vertex (more common names for it are degree and order; we choose valence since the words "degree" and "order" have many other meanings).

Usually, when we speak about a graph, we mean not just a topological space, but also its graph structure, that is the set of vertices and edges (however, notice that once we know the vertices, we know also edges).

Let $G$ be a graph. A set $I \subset G$ will be called an interval if there is a homeomorphism $h: J \to I$, where $J$ is $[0, 1]$, $(0, 1)$, $[0, 1)$, or $(0, 1)$, and no points of $I$, except perhaps its endpoints (that is the points $h(0)$ and $h(1)$) are vertices. The set $h((0, 1))$ will be called the interior of $I$ and will be denoted $\text{Int}(I)$. If $J = [0, 1]$, the interval $I$ will be called closed; if $J = (0, 1)$, it will be called open. It may happen that the above terminology does not coincide with the one used when we think about $I$ as a subset of $G$ (the same applies to the edges of $G$). For example, if $G = I = [0, 1]$ and $h = \text{id}$, then for $I$ regarded as a subset of the topological space $G$, $I$ is both closed and open and the interior of $I$ is $I$.

We shall consider additionally any set consisting of one point to be an interval (a closed one; then $\text{Int}(I) = \emptyset$).

Let $s \geq 2$. An $s$-horseshoe for $f$ is a closed interval $I \subset G$ and closed subintervals $J_1, \ldots, J_s$ of $I$ with pairwise disjoint interiors, such that $f(J_j) = I$ for $j = 1, \ldots, s$. We shall denote this horseshoe by $(I; J_1, \ldots, J_s)$. An $s$-horseshoe is strong if in addition the intervals $J_1, \ldots, J_s$ are contained in $\text{Int}(I)$ and are pairwise disjoint.

If $X$ is a compact topological space and $f: X \to X$ a continuous map then we will use the following notation. If $\mathcal{A}$ is a family of subsets of $X$ then

$$\mathcal{A}^n = \mathcal{A}^n_f = \left\{ \bigcap_{i=0}^{n-1} f^{-i}(A_i) : A_i \in \mathcal{A} \text{ for } i = 1, \ldots, n - 1 \text{ and } \bigcap_{i=0}^{n-1} f^{-i}(A_i) \neq \emptyset \right\}$$

and

$$\mathcal{A}^n|_B = \{ A \cap B : A \in \mathcal{A}^n \text{ and } A \cap B \neq \emptyset \}.$$
If $\mathcal{A}$ is a cover of $X$ then $\mathcal{N}(\mathcal{A})$ is the minimal cardinality of a cover chosen from $\mathcal{A}$. Then

$$h(f, \mathcal{A}) = \lim_{n \to \infty} \frac{1}{n} \log \mathcal{N}(\mathcal{A}^n)$$

is the entropy of $f$ on the cover $\mathcal{A}$. Recall that the topological entropy of $f$ is

$$h(f) = \sup \{ h(f, \mathcal{A}) : \mathcal{A} \text{ is an open cover of } X \}.$$ 

We shall often use the well known formula $h(f^n) = nh(f)$ for $n \geq 0$.

A finite subset $P$ of $X$ is a cycle if $f(P) = P$ and $f|_P$ is a cyclic permutation. The number of elements of a cycle $P$ is the period of $P$. We denote the set of periods of all cycles of $f$ by $\text{Per}(f)$.

Let $G$ be a graph and $f : G \to G$ a continuous map. Let $I, J \subset G$ be closed intervals. We shall say that $I$ $f$-covers $J$ if there is a closed subinterval $K \subset I$ with $f(K) = J$. We shall need several simple lemmas about $f$-covering. They are well known for interval or circle maps (cf. [S, 21]). Their proofs are very simple and we leave them to the reader.

**Lemma 2.1.** Let $I, J \subset G$ be closed intervals. Assume that $f(I)$ simply connected and $J$ is contained in $f(I)$. Then $I$ $f$-covers $J$.

**Lemma 2.2.** If $I, J, K \subset G$ are closed intervals, $I$ $f$-covers $J$ and $J$ $g$-covers $K$ then $I$ $(g \circ f)$-covers $K$.

**Lemma 2.3.** Let $I, J \subset G$ be closed intervals such that $I \subset J$ and $I$ $f$-covers $J$. Then there exists $x \in I$ with $f(x) = x$.

The intervals which we consider here are homeomorphic to the real intervals, and these homeomorphisms give the orderings on them. Thus we will use for the points $x, y$ of such an interval the notations $x < y$, $[x, y]$, etc.

The following lemma is also very simple, so again we leave its proof to the reader.

**Lemma 2.4.** Let $I, J \subset G$ be closed intervals. Assume that $J = [a, b]$ and $f(J)$ contains a point $x \in (a, b)$ and a point outside $(a, b)$. Then $I$ $f$-covers either $[a, x]$ or $[x, b]$.

### 3. HORSESHOES

In this section we investigate some consequences of the existence of horseshoes. In particular, we prove Theorem A and an auxiliary result on periods.

Let $G$ be a graph and $f : G \to G$ a continuous map. Since in the definition of a horseshoe we demand that $f(J_i) = I$ rather than just $J_i$ $f$-covers $I$, we shall not need the language of $f$-coverings here. Nevertheless we shall often use Lemma 2.1.

**Lemma 3.1.** Let $(I; J_1, \ldots, J_s)$ be an $s$-horseshoe for $f$. Then for any finite sequence $\mathcal{T} = (j_0, j_1, \ldots, j_{n-1})$ of elements of $\{1, \ldots, s\}$ there is a closed interval $J_{\mathcal{T}}$ such that $f^i(J_{\mathcal{T}}) = J_{j_i}$ for $i = 0, 1, \ldots, n - 2$ and $f^{n-1}(J_{\mathcal{T}}) = J_{j_n}$.

**Proof.** We use induction. For $n = 1$ we just take $J_{\mathcal{T}} = J_{j_0}$ if $\mathcal{T} = (j_0)$. Assume that the lemma holds for all sequences of length $n$ and take $\mathcal{T} = (j_0, j_1, \ldots, j_n)$. Then for $\mathcal{X} = (j_1, j_2, \ldots, j_n)$ there is a closed interval $J_{\mathcal{X}}$ such that $f^i(J_{\mathcal{X}}) = J_{j_{i+1}}$ for $i = 0, 1, \ldots, n - 2$ and $f^{n-1}(J_{\mathcal{X}}) = J_{j_n}$. By Lemma 2.1, there is a closed interval $K \subset J_{j_0}$...
such that \( f(K) = J_x \). We set \( J_x = K \) and then \( f^i(J_x) \subset J_h \) for \( i = 0, 1, \ldots, n - 1 \) and \( f^n(J_x) = J_x \).

**Lemma 3.2.** If \( f \) has an \( s \)-horseshoe then \( f^n \) has an \( s^n \)-horseshoe for all \( n \in \mathbb{N} \).

**Proof.** Assume that \( f \) has an \( s \)-horseshoe \((I; J_1, \ldots, J_s)\). Fix \( n \). For every \( T \in \{1, \ldots, s\}^n \) we choose a closed interval \( J_T \) satisfying the conditions of Lemma 3.1. Clearly, \( J_T \) is a closed subinterval of \( I \) and \( f^n(J_T) = I \). Let \( J_x \) be a minimal (with respect to inclusion) subinterval of \( J_T \) with those properties. To show that \((I; (J_x)_{x \in \{1, \ldots, s^n\}})\) is an \( s^n \)-horseshoe for \( f^n \), we have to show only that if \( T \neq T' \in \{1, \ldots, s\}^n \) then the interiors of \( J_T \) and \( J_{T'} \) are disjoint. Suppose they are not disjoint. Since \( T \neq T' \), for some \( i \in \{0, 1, \ldots, n - 1\} \) the sets \( f^i(J_T) \) and \( f^i(J_{T'}) \) are contained in some intervals \( J_k \) and \( J_{k'} \) respectively, which have disjoint interiors. Therefore the common part of the interiors of \( J_T \) and \( J_{T'} \) is mapped by \( f \) to a point (the common point of \( J_k \) and \( J_{k'} \)). This contradicts the minimality of \( J_k \) and \( J_{k'} \).

**Lemma 3.3.** If \( f \) has an \( s \)-horseshoe and \( s \geq 4 \) then it has a strong \((s - 2)\)-horseshoe.

**Proof.** Assume that \( I \subset G \) is a closed interval and \( J_1, \ldots, J_s \) are closed subintervals of \( I \) with pairwise disjoint interiors, such that \( f(J_j) = I \) for \( j = 1, \ldots, s \). We can assume that each \( J_j \) is minimal with this property, that is no proper closed subinterval \( J \) of \( J_j \) satisfies \( f(J) = I \). Then clearly, each endpoint of \( J_j \) is mapped to an endpoint of \( I \). We can assume also that an orientation of \( I \) is given, so that speaking of left and right makes sense, and that \( J_1 \) is the leftmost and \( J_s \) the rightmost of the intervals \( J_1, \ldots, J_s \). Choose points \( a \in \text{Int}(J_1) \) and \( b \in \text{Int}(J_s) \) and let \( I' \) be the closed subinterval of \( I \) with endpoints \( a \) and \( b \). Clearly, \( J_2, \ldots, J_{s - 1} \subset \text{Int}(I') \). By Lemma 3.1, there are closed subintervals \( J_2, \ldots, J_{s - 1} \) respectively such that \( f(J_j) = I' \) for \( j = 2, \ldots, s - 1 \). If \( i, j \in \{2, \ldots, s - 1\}, i \neq j \) and \( J_i \cap J_j \neq \emptyset \), then \( J_i \cap J_j \) consists of the common endpoint of \( J_i \) and \( J_j \). However, this endpoint is mapped by \( f \) to an endpoint of \( I \), which does not belong to \( f(J_i) = f(J_j) = I' \), a contradiction. Therefore the intervals \( J_2, \ldots, J_{s - 1} \) are pairwise disjoint.

**Lemma 3.4.** If \( f \) has a strong \( s \)-horseshoe then \( h(f) \geq \log s \).

**Proof.** Assume that \((I; J_1, \ldots, J_s)\) is a strong \( s \)-horseshoe for \( f \). Then \( \mathcal{A} = \{A_1, \ldots, A_s\} \), where \( A_i = (G \setminus \bigcup_{j=1}^{s-1} J_j) \cup J_i \) is an open cover of \( G \). For any finite sequence \( T = (j_0, j_1, \ldots, j_{n-1}) \) of elements of \( \{1, \ldots, s\} \) we choose an interval \( J_T \) from Lemma 3.1. Let \( x \in J_T \). Since \( f_i(x) \in J_{j_i} \) for \( i = 0, 1, \ldots, n - 1 \), the only element of \( \mathcal{A}^n \) containing \( x \) is \( \bigcap_{i=0}^{n-1} f^{-1}(A_{j_i}) \). Therefore \( \mathcal{A}^n(\mathcal{A}^n) = \mathcal{A} \). Thus,

\[
h(f) \geq h(f, \mathcal{A}) = \lim_{n \to \infty} \frac{1}{n} \log s^n = \log s.
\]

Now we can prove Theorem A.

**Theorem A.** If a continuous map \( f \) of a graph into itself has an \( s \)-horseshoe then \( h(f) \geq \log s \).

**Proof.** Let \( G \) be a graph and \( f: G \to G \) a continuous map which has an \( s \)-horseshoe. By the definition of an \( s \)-horseshoe, \( s \geq 2 \). By Lemma 3.2, \( f^n \) has an \( s^n \)-horseshoe for all \( n \in \mathbb{N} \),
Jaume Llibre and Michal Misiurewicz

so by Lemma 3.3, \( f^n \) has a strong \((s^n - 2)\)-horseshoe. Therefore, by Lemma 3.4, 
\[
h(f) = (1/n)h(f^n) \geq (1/n)\log(s^n - 2).
\]
Thus,
\[
h(f) \geq \lim_{n \to \infty} \frac{1}{n} \log(s^n - 2) = \log s.
\]

If \( f \) has an \( s \)-horseshoe for some \( s \geq 2 \) and we do not want to specify \( s \), then we shall just say that \( f \) has a horseshoe.

**Proposition 3.5.** If \( f^k \) has a horseshoe for some \( k \geq 1 \) then \( 4k^2n \in \text{Per}(f) \) for all \( n \in \mathbb{N} \).

**Proof.** Assume that \( f^k \) has an \( s \)-horseshoe. By Lemma 3.2, \( f^{2k} \) has an \( s^2 \)-horseshoe, so by Lemma 3.3 it has a strong \((s^2 - 2)\)-horseshoe (since \( s^2 - 2 \geq 2 \), we have \( s^2 - 2 = 2 \)). Let \( (I_1, J_1, \ldots, J_{2^{2k-2}}) \) be this strong horseshoe. Fix \( n \geq 1 \) and set \( m = 2kn \), \( g = f^{2k} \). Let \( \mathcal{F} = (1, 2, 2, \ldots) \) be the sequence of length \( m \) consisting of one 1 and \( m - 1 \) 2's. By Lemma 3.1 there exists a closed interval \( J_i \subset J_{2^i} \) such that \( g^i(J_i) \subset J_{2^{i-1}} \) for \( i = 1, 2, \ldots, m - 2 \), and \( g^{m-1}(J_i) = J_{2^1} \). Then \( g^m(J_i) = g(J_{2^1}) = I \). By Lemma 2.3 there is \( x \in \mathcal{F} \) such that \( g^n(x) = x \). We have \( x \in J_1 \), \( g^i(x) \in J_{2i} \) for \( i = 1, 2, \ldots, m-1 \) and \( J_1 \cap J_{2^1} = \emptyset \). Therefore \( x \) is a periodic point of \( g = f^{2k} \) of period \( m = 2kn \). Then \( x \) is a periodic point of \( f \) and \( f^{4k^2n}(x) = x \). Let \( r \) be the period of \( x \) for \( f \). Since for \( 2k \)-th iterate of \( f \) the period of \( x \) is \( 2kn \), then \( 2kn \) divides \( r \). Therefore \( r = 2knl \) for some \( l \). Thus, the period of \( x \) for \( f^{2k} \) is \( nl \). Since \( nl = 2kn \), we get \( l = 2k \), so \( r = 4k^2n \).

**Existence of Horseshoes for Positive Entropy Maps**

The aim of this section is to prove Theorem B. We follow closely the proof of [18] and [17] (see also [2]), making necessary modifications. We need a couple of new definitions first.

Let \( G \) be a graph. A set \( D \subset G \) will be called a dwarf if there is a connected set \( \tilde{D} \), containing at most one vertex, such that \( \tilde{D} \subset D \subset \text{Cl}(\tilde{D}) \) (Cl denotes the closure). In particular, every interval is a dwarf.

In this section we shall assume that if \( X \subset G \) is homeomorphic to a circle, then there are at least 4 vertices in \( X \). Notice that with this assumption, no subset of a dwarf is homeomorphic to a circle. Moreover, the intersection of a dwarf and an interval is an interval (unless it is empty).

When we want to prove Theorem B, we may make the above restriction. Indeed, if we start with an arbitrary graph \( G \) and there is a subset of \( G \), homeomorphic to a circle with less than 4 vertices in it, then we can divide some edges of \( G \), by introducing new vertices, so that such subset exists no more. The new graph \( G' \) will be different from \( G \) as a graph, but equal to \( G \) as a topological space. Any interval in \( G' \) will be an interval in \( G \). Therefore any horseshoe for \( f^k \) as a map of \( G' \) will be a horseshoe for \( f^k \) as a map of \( G \).

Let \( f: G \to G \) be a continuous map. A partition \( \mathcal{A} \) of \( G \) will be called \( f \)-proper if \( \mathcal{A} \) is finite, every element of \( \mathcal{A} \) is an interval, and \( f(A) \) is a dwarf for every \( A \in \mathcal{A} \).

**Lemma 4.1.** For every open cover \( \mathcal{B} \) of \( G \) there exists an \( f \)-proper partition \( \mathcal{A} \) of \( G \), finer than \( \mathcal{B} \).

**Proof.** Consider the following conditions for a set \( A \subset G \):

(i) \( A \) is open.
(ii) \( A \) is connected.
(iii) $A$ is contained in some element of $A$.
(iv) $A$ is a dwarf.
(v) $f(A)$ is a dwarf.

Clearly, for every $x \in G$ there exists a neighborhood $U_x$ of $x$ satisfying (i)–(v). From the cover $\{U_x\}_{x \in G}$ we can choose a finite cover $\{U_i\}_{i=1}^n$ of $G$. Notice that an intersection of two sets satisfying (i)–(v) also satisfies (i)–(v) and a difference of two sets satisfying (ii)–(v) is a union of finitely many disjoint sets which satisfy (ii)–(v). Therefore, if we denote $U_i^0 = U_i$, $U_i^1 = G \setminus U_i$ then for every $\eta_1, \ldots, \eta_n \in \{0, 1\}$ the set $\bigcap_{i=1}^n U_i^{\eta_i}$ is a union of finitely many disjoint sets which satisfy (ii)–(v) (it may happen that this set is empty; in particular, this is the case if all $\eta_i$ are 1).

Let $C$ be the family of all sets of the form $\bigcap_{i=1}^n U_i^{\eta_i}$. If $(\eta_1, \ldots, \eta_n) \neq (\eta_1, \ldots, \eta_n)$ then $\bigcap_{i=1}^n U_i^{\eta_i} \cap \bigcap_{i=1}^n U_i^{\eta_i} = \emptyset$. Therefore the components of elements of $C$ form a finite partition of $G$; each of those components satisfies (ii)–(v). Now we modify this partition by removing the empty set, and partitioning the elements of the partition containing a vertex into finitely many intervals (if necessary). In such a way we obtain a partition $A$ which is $f$-proper and is finer than $A$.

**LEMMA 4.2.** Let $A$ be an $f$-proper partition of $G$. Let $n$ be a positive integer and let $A \in A^n$. Then:

(a) there exists an interval $K \subset A$ such that $f^n(K) = f^n(A)$,
(b) the set $f^n(A)$ is a dwarf.

**Proof:** We use induction. If $n = 1$ then we can take simply $K = A$. Both conditions (a) and (b) hold by the definition of an $f$-proper partition. Assume now that the conclusion of the lemma holds for some $n$ and prove it for $n + 1$ instead of $n$. Let $A \in A^{n+1}$. Then there exist $B \in A$ and $C \in A^n$ such that $A = C \cap f^{-n}(B)$. We get $f^n(A) = f^n(\{x : x \in C \text{ and } f^n(x) \in B\}) = f^n(C) \cap B$. By the induction hypothesis, there exists an interval $L \subset C$ such that $f^n(L) = f^n(C)$. We have now $f^n(A) = f^n(L) \cap B$.

Since $A \in A^{n+1}$, by the definition we have $A \neq \emptyset$, so $f^n(A) \neq \emptyset$. The set $f^n(C)$ is a dwarf (by the induction hypothesis) and $B$ is an interval, so $f^n(A)$ is an interval. Since $L$ is also an interval and $f^n(A) \subset f^n(L)$, there exists a subinterval $K$ of $L$ such that $f^n(K) = f^n(A)$. Since $L \subset C$, we get $K \subset C$. On the other hand, $f^n(K) = f^n(A) \subset B$, so $K \subset f^{-n}(B)$. Therefore $K$ is an interval contained in $C \cap f^{-n}(B) = A$ and $f^{n+1}(K) = f(f^n(K)) = f(f^n(A)) = f^{n+1}(A)$. This proves (a). Since $f^n(A)$ is an interval, it is connected, so $f^{n+1}(A)$ is also connected. Since $f^n(A) \subset B$, the set $f^{n+1}(A)$ is contained in the set $f(B)$, which is a dwarf as an image of $D \in A$. Therefore $f^{n+1}(A)$ is a connected subset of a dwarf, so it is a dwarf itself. This proves (b).

The following three analytic lemmas are very simple. The reader can find their proofs in [18] and [2].

**LEMMA 4.3.** Let $(\alpha_n)_{n=0}^\infty$ and $(\beta_n)_{n=0}^\infty$ be two sequences of real numbers. Then

$$\lim_{n \to \infty} \frac{1}{n} \log \left( \sum_{k=0}^n \exp(\alpha_k + \beta_{n-k}) \right) \leq \max \left( \lim_{n \to \infty} \frac{\alpha_n}{n}, \lim_{n \to \infty} \frac{\beta_n}{n} \right).$$

**LEMMA 4.4.** If $a_{n,i}$, $i = 1, \ldots, k$, $n = 0, 1, \ldots$, are non-negative numbers then

$$\lim_{n \to \infty} \frac{1}{n} \log \sum_{i=1}^k a_{n,i} = \max_{1 \leq i \leq k} \lim_{n \to \infty} \frac{1}{n} \log a_{n,i}.$$
Lemma 4.5. Let \((a_n)_{n=1}^\infty\) be a sequence of real numbers and let \(b, u \in \mathbb{R}, p \in \mathbb{N}\) be such that

(i) \(u > 0\),
(ii) \(a_{n+1} \leq a_n + b\) for all \(n\),
(iii) if \(n \geq p\) and \(a_n/n \geq u\) then \(a_{n+1} \leq a_n + u\).

Then \(\lim_{n \to \infty} \frac{a_n}{n} \leq u\).

Let \(\mathcal{A}\) be an \(f\)-proper partition of \(G\). For any dwarf \(J \subset G\) we have

\[
\text{Card}\{A \in \mathcal{A} : A \cap J \neq \emptyset \text{ and } A \setminus J \neq \emptyset\} \leq q,
\]

where \(q\) is the maximal valence of vertices of \(G\) (or \(q = 2\) if \(G\) is an interval).

Set

\[
\mathcal{E} = \left\{ A \in \mathcal{A} : \limsup_{n \to \infty} \frac{1}{n} \log \text{Card}(\mathcal{A}^n|_A) = h(f, \mathcal{A}) \right\}.
\]

Since \(\mathcal{A}\) is a partition, we have \(\mathcal{N}(\mathcal{A}^n) = \text{Card}\mathcal{A}^n\) for all \(n\). By Lemma 4.4, we have

\[
h(f, \mathcal{A}) = \limsup_{n \to \infty} \frac{1}{n} \log \sum_{A \in \mathcal{A}} \text{Card}(\mathcal{A}^n|_A) = \max_{A \in \mathcal{A}} \limsup_{n \to \infty} \frac{1}{n} \log \text{Card}(\mathcal{A}^n|_A),
\]

and therefore the family \(\mathcal{E}\) is non-empty.

Lemma 4.6. For any \(A \in \mathcal{E}\),

\[
\limsup_{n \to \infty} \frac{1}{n} \log \text{Card}(\mathcal{E}^n|_A) = h(f, \mathcal{A}).
\]

Proof. Clearly,

\[
\limsup_{n \to \infty} \frac{1}{n} \log \text{Card}(\mathcal{E}^n|_A) \leq h(f, \mathcal{A}).
\]

Denote \(\alpha_n = \log \text{Card}(\mathcal{E}^n|_A)\) and \(\beta_n = \log(\sum_{B \in \mathcal{E}} \text{Card}(\mathcal{A}^n|_B))\) for \(n = 1, 2, \ldots\), and \(\alpha_0 = \beta_0 = 0\).

We can divide the set \(\mathcal{A}^n|_A\) into sets \(T_k, k = 1, \ldots, n\), where

\[
T_k = \left\{ B \in \mathcal{A}^n|_A : B = \bigcup_{i=0}^{n-1} f^{-i}(B_i), B_i \in \mathcal{A} \text{ for } i = 0, \ldots, n-1, \right\}
\]

for \(k < n\) and \(T_n = \mathcal{E}^n\). From the definition of \(T_k\) it follows easily that \(\text{Card} \ T_k \leq \exp(\alpha_n) \exp(\beta_{n-k})\) for all \(k\). Therefore

\[
\text{Card} \ (\mathcal{A}^n|_A) \leq \sum_{k=0}^{n} \exp(\alpha_k + \beta_{n-k}).
\]

For any \(B \in \mathcal{A} \setminus \mathcal{E}\) we have \(\limsup_{n \to \infty} (1/n) \log \text{Card}(\mathcal{A}^n|_B) < h(f, \mathcal{A})\), and hence by Lemma 4.4, \(\limsup_{n \to \infty} (\beta_n/n) < h(f, \mathcal{A})\). From the definition of \(\mathcal{E}\) it follows that

\[
\limsup_{n \to \infty} \frac{1}{n} \log \text{Card}(\mathcal{A}^n|_A) = h(f, \mathcal{A}),
\]

and thus, by Lemma 4.3, \(\limsup_{n \to \infty} (\alpha_n/n) \geq h(f, \mathcal{A})\).
Denote for arbitrary $A, B \in \mathcal{E}$,
\[
\gamma(A, B, n) = \text{Card}\{E \in \mathcal{E}^n|_A : f^*(E) \supset B\}.
\]

**Lemma 4.7.** Assume that $h(f, \mathcal{A}) > \log (q + 1)$. Then there exists $A_0 \in \mathcal{E}$ such that
\[
\limsup_{n \to \infty} \frac{1}{n} \log \gamma(A_0, A_0, n) \geq h(f, \mathcal{A}).
\]

*Proof.* Fix an interval $A \in \mathcal{E}$ and a real number $u$ such that $\log (q + 1) < u < h(f, \mathcal{A})$. Suppose that there exists $p$ such that for all $n \geq p$,
\[
\frac{1}{n} \log \text{Card}(\mathcal{E}^n|_A) > u \quad \text{implies} \quad \text{Card}(\mathcal{E}^{n+1}|_A) < (q + 1)\text{Card}(\mathcal{E}^n|_A).
\]

Clearly, $\log \text{Card}(\mathcal{E}^{n+1}|_A) \leq \log \text{Card}(\mathcal{E}^n|_A) + \log \text{Card} \mathcal{E}$, so by Lemma 4.5,
\[
\limsup_{n \to \infty} \frac{1}{n} \log \text{Card}(\mathcal{E}^n|_A) \leq u.
\]

This contradicts Lemma 4.6. Therefore we have
\[
\text{for every number } p \text{ there exists an integer } n \geq p \text{ such that}
\frac{1}{n} \log \text{Card}(\mathcal{E}^n|_A) > u \text{ and } \text{Card}(\mathcal{E}^{n+1}|_A) \geq (q + 1)\text{Card}(\mathcal{E}^n|_A).
\]

Fix an element $E \in \mathcal{E}^n|_A$. By Lemma 4.2(b), the set $f^n(E)$ is a dwarf and therefore, in view of (1), if it meets $r$ elements of $\mathcal{E}$ then it contains at least $r - q$ of them. But in view of the definition of $\mathcal{E}^{n+1}$ we have $r = \text{Card}(\mathcal{E}^{n+1}|_E)$. Hence,
\[
\text{Card}\{B \in \mathcal{E} : f^n(E) \supset B\} \geq \text{Card}(\mathcal{E}^{n+1}|_E) - q.
\]

Summing over $E \in \mathcal{E}^n|_A$ we obtain
\[
\sum_{B \in \mathcal{E}} \gamma(A, B, n) \geq \text{Card}(\mathcal{E}^{n+1}|_A) - q \text{Card}(\mathcal{E}^n|_A).
\]

In view of (2) we conclude that for every $p$ there exists $n \geq p$ such that
\[
\frac{1}{n} \log \sum_{B \in \mathcal{E}} \gamma(A, B, n) \geq \frac{1}{n} \log \text{Card}(\mathcal{E}^n|_A) > u.
\]

Therefore
\[
\limsup_{n \to \infty} \frac{1}{n} \log \sum_{B \in \mathcal{E}} \gamma(A, B, n) \geq u.
\]

The number $u$ can be chosen arbitrarily close to $h(f, \mathcal{A})$ and hence
\[
\limsup_{n \to \infty} \frac{1}{n} \log \sum_{B \in \mathcal{E}} \gamma(A, B, n) \geq h(f, \mathcal{A}).
\]

By Lemma 4.4, since $\mathcal{E}$ is finite, for each $A \in \mathcal{E}$ there is $\varphi(A) \in \mathcal{E}$ such that
\[
\limsup_{n \to \infty} \frac{1}{n} \log \gamma(A, \varphi(A), n) \geq h(f, \mathcal{A}).
\]
The transformation \( \varphi : \mathcal{E} \to \mathcal{E} \) has to have a periodic point. Denote it by \( A_0 \), and its period by \( m \). Look at the definition of \( \gamma(A, B, n) \). If \( A, B, C \in \mathcal{E}, D \in \mathcal{E}^n(A), f^n(D) \supseteq B, D' \in \mathcal{E}^m(A) \) and \( f^n(D') \supseteq C \) then \( D \cap f^{-n}(D') \in \mathcal{E}^{m+n} \) and \( f^{-1}(D \cap f^{-n}(D')) \supseteq C \). Therefore \( \gamma(A, C, n + k) \geq \gamma(A, B, n) \cdot \gamma(B, C, k) \). Using this formula \( m - 1 \) times we obtain
\[
\gamma(A_0, A_0, \sum_{i=0}^{m-1} n_i) \geq \prod_{i=0}^{m-1} \gamma(\varphi^i(A_0), \varphi^{i+1}(A_0), n_i)
\]
for any \( n_i, i = 0, 1, \ldots, m - 1 \). From this and (3) it follows that
\[
\limsup_{n \to \infty} \frac{1}{n} \log \gamma(A_0, A_0, n) \geq h(f, \mathcal{E}).
\]

**Theorem B.** If a continuous map \( f \) of a graph into itself has positive topological entropy then there exist sequences \((k_n)_{n=1}^\infty\) and \((s_n)_{n=1}^\infty\) of positive integers such that for each \( n \) the map \( f^{k_n} \) has an \( s_n \)-horseshoe and
\[
\limsup_{n \to \infty} \frac{1}{k_n} \log s_n = h(f).
\]

**Proof.** Let \( G \) be a graph and \( f : G \to G \) a continuous map with positive entropy. Take an integer \( r > (\log(q + 1) + 1)/h(f) \) (if \( h(f) = \infty \) then take \( r = 1 \)). For a fixed \( n \) take a finite open cover \( \mathcal{B} \) for which \( h(f', \mathcal{B}) \geq h(f') - 1/n > \log(q + 1) \) (if \( h(f) = \infty \), then we replace this by \( h(f, \mathcal{A}) > n + \log(q + 1) \)). By Lemma 4.1 we can find a finite partition \( \mathcal{A} \) of \( I \) into intervals, finer than \( \mathcal{B} \). By Lemma 4.7, applied to \( f' \) and \( \mathcal{A} \), there exist an interval \( A_0 \in \mathcal{E} \) and a positive integer \( m_n \) such that
\[
\frac{1}{m_n} \log \gamma(A_0, A_0, m_n) \geq h(f', \mathcal{A}) - \frac{1}{n}.
\]
Set \( s_n = \gamma(A_0, A_0, m_n) \). Then by the definition of \( \gamma \), there are elements \( E_1, E_2, \ldots, E_n \) of \( \mathcal{E}^n(A) \) such that \( f'^{m_n}(E_i) \supseteq A_0 \) for each \( i \) and by Lemma 4.2(a), for each \( i \) there is an interval \( K_i \subset E_i \) with \( f'^{m_n}(K_i) = f'^{m_n}(E_i) \). Therefore we get disjoint subintervals \( K_1, K_2, \ldots, K_n \) of \( A_0 \) such that \( f'^{m_n}(K_i) \supseteq A_0 \) for each \( i \). Let \( I \) be the closure of \( A_0 \). The closures of the sets \( K_i \) are closed intervals contained in the closed interval \( I \) and their images under \( f^{k_n} \) (where \( k_n = rm_n \)) contain \( I \). Since the partition \( \mathcal{A} \) is proper, for every \( i \) the set \( f^{k_n}(K_i) \) is contained in a dwarf. Therefore we can use Lemma 3.1 and we find subintervals \( J_i \) of the closures of \( K_i \) such that \( f^{k_n}(J_i) = I \). Since the intervals \( K_i \) are pairwise disjoint, the interiors of the intervals \( J_i \) are also pairwise disjoint. In such a way we obtain a horseshoe \( (I; J_1, J_2, \ldots, J_n) \) for \( f^{k_n} \).

We have
\[
\frac{1}{k_n} \log s_n = \frac{1}{m_n} \log \gamma(A_0, A_0, m_n) \geq \frac{1}{r} \left( h(f', \mathcal{A}) - \frac{1}{n} \right) \geq \frac{1}{r} \left( h(f') - \frac{2}{n} \right).
\]
If \( h(f) \) is finite, then
\[
\liminf_{n \to \infty} \frac{1}{k_n} \log s_n \geq \lim_{n \to \infty} \frac{1}{r} \left( h(f') - \frac{2}{n} \right) = h(f).
\]
If \( h(f) = \infty \) then
\[
\liminf_{n \to \infty} \frac{1}{k_n} \log s_n \geq \lim_{n \to \infty} \left( n + \log(q + 1) - \frac{1}{n} \right) = \infty.
\]
In both cases
\[ \lim_{n \to \infty} \frac{1}{k_n} \log s_n \geq h(f). \]
The inequality
\[ h(f) \geq \lim_{n \to \infty} \sup \frac{1}{k_n} \log s_n \]
follows from Theorem A. This completes the proof. 

5. LOWER SEMI-CONTINUITY OF ENTROPY

As in the case of interval or circle maps (see [18, 17, 21]), a simple consequence of Theorems A and B is a theorem on lower semi-continuity of entropy (Theorem C). We prove it in this section.

Let \( G \) be a graph. We consider the space \( C(G,G) \) of all continuous maps from \( G \) into itself with the \( C^0 \) topology (that is, topology of uniform convergence). Then topological entropy can be regarded as a function from this space to the set \([0, \infty)\) (we compactify \([0, \infty)\) by adding a point at infinity).

A function \( \varphi : X \to [0, \infty) \) is lower semi-continuous if for every \( x \in X \) and \( a < \varphi(x) \) there is a neighborhood \( U \) of \( x \) in \( X \) such that \( \varphi(y) > a \) for every \( y \in U \). Let us recall that Theorem C says that topological entropy \( h : C(G,G) \to [0, \infty) \) is lower semi-continuous.

**Lemma 5.1.** Assume that \( f \in C(G,G) \) has a strong \( s \)-horseshoe. If \( g \in C(G,G) \) is sufficiently close to \( f \) then it has also a strong \( s \)-horseshoe.

**Proof.** Let \( (I; J_1, \ldots, J_s) \) be a strong \( s \)-horseshoe for \( f \). We can choose closed intervals \( I' \subset \text{Int}(I) \) and \( I'' \subset \text{Int}(I') \) such that each \( J_j \) is contained in \( \text{Int}(I'') \). By Lemma 2.1 there exist closed intervals \( J'_{j'} \subset J_j \) (j = 1, \ldots, s) such that \( f(J'_{j}) = I' \). If \( g \) is sufficiently close to \( f \) then for each \( j \) we have \( I'' \subset g(J'_{j}) \subset I \). By Lemma 2.1 there exist closed intervals \( J''_{j'} \subset J'_{j'} \) (j = 1, \ldots, s) such that \( g(J''_{j'}) = I'' \). Then \( (I''; J''_1, \ldots, J''_s) \) is a strong \( s \)-horseshoe for \( g \).

**Theorem C.** Topological entropy, as a function of a continuous map of a graph into itself, is lower semi-continuous.

**Proof.** Let \( f \in C(G,G) \). We prove that \( h \) is lower semi-continuous at \( f \). If \( h(f) = 0 \) then there is nothing to prove. Assume that \( h(f) > 0 \). Then by Theorem B, for every \( a < h(f) \) there exists \( k \geq 1 \) and \( s \geq 2 \) such that \( f^k \) has an \( s \)-horseshoe and
\[ \frac{1}{k} \log s > a. \]
By Lemma 3.2, for each \( n, f^{kn} \) has an \( s^n \)-horseshoe, so by Lemma 3.3, it has an \((s^n - 2)\)-horseshoe. If \( n \) is sufficiently large then also
\[ \frac{1}{kn} \log(s^n - 2) > a. \]
If \( g \in C(G,G) \) is sufficiently close to \( f \) then \( g^{kn} \) is close to \( f^{kn} \). Hence, by Lemma 5.1 \( g^{kn} \) has also a strong \((s^n - 2)\)-horseshoe. Therefore, by Theorem A,
\[ h(g) = \frac{1}{kn} h(g^{kn}) \geq \frac{1}{kn} \log(s^n - 2) > a. \]
In this section we study the dependence between the positive entropy of \( f \) and the set \( \text{Per}(f) \). In particular, we prove Theorems D and E. Let \( G \) be a graph, \( f: G \to G \) a continuous map. We start with a technical lemma.

**Lemma 6.1.** Let \( I, J_1, J_2 \subset G \) be closed intervals such that \( J_1 \) and \( J_2 \) have disjoint interiors. Assume that there are positive integers \( r, s, t \) such that \( I f^r \)-covers \( J_1 \) and \( J_2 \), \( J_1 f^s \)-covers \( I \), and \( J_2 f^t \)-covers \( I \). Then \( h(f) > 0 \).

**Proof.** By Lemma 2.2, \( J_1 f^r \)-covers \( J_1 \) and \( J_2 \), and \( J_2 f^t \)-covers \( J_1 \) and \( J_2 \). Again by Lemma 2.2, each of \( J_1, J_2, f^{(s+t)(t+r)} \) covers \( J_1 \) and \( J_2 \). Therefore \( f^{(s+t)(t+r)} \) has a 2-horseshoe, so by Theorem A,

\[
h(f) = \frac{1}{(s + r)(t + r)} h(f^{(s+t)(t+r)}) > 0.
\]

Let \( P \) be a cycle of \( f \) of odd period \( n \). We shall say that \( \{x_1, x_2, x_3\} \subset P \) is a prime triple if \( x_i \neq x_j \) for \( i \neq j \) and whenever \( k \) is such that \( f^k(x_i) = x_j \) and \( i \neq j \) then all three points \( x_1, x_2, x_3 \) lie on the \( f^k \)-orbit of \( x_i \). This amounts to the fact that if \( f^k(x_1) = x_2, f^{k_2}(x_2) = x_3 \) and \( f^{k_3}(x_3) = x_1 \) then \( k_1, k_2, k_3 \in \{0, 1, \ldots, n - 1\} \) then \( k_1, k_2 \) and \( k_3 \) generate the same subgroup of \( \mathbb{Z}/n \) (different than \( \{0\} \)).

**Lemma 6.2.** Assume that at least one endpoint of an edge \( e \) of \( G \) is a fixed point of \( f \) and that there is a prime triple in \( P \cap e \). Then \( h(f) > 0 \).

**Proof.** Let \( v \) be an endpoint of \( e \) such that \( f(v) = v \). As before, we can consider an ordering on \( e \) and use the notation \( x < y, [x, y], \) etc. With this ordering, we can assume that \( v \) is the left endpoint of \( e \). Let us take the prime triple \( \{x, y, z\} \) in \( P \cap e \) closest to \( v \), that is such that the smallest interval containing \( v, x, y, z \) does not contain any other prime triple in \( P \). We can assume that \( v < x < y < z \), and then the above condition can be stated as

(i) There is no prime triple in \( (v, z) \cap P \).

There are positive integers \( i, j, k \) such that \( f^i(x) = y, f^j(y) = x \) and \( f^k(z) = x \) (see Fig. 1).

Since \( \{x, y, z\} \) is a prime triple, we get

(ii) \( i, j, k \) (mod \( n \)) generate the same subgroup in \( \mathbb{Z}/n \).

Set \( I = [v, x], J = [x, y] \) and \( K = [y, z] \) (see Fig. 1).

Since \( I = [v, x], f^i(x) = y \) and \( f^j(y) = x \), by Lemma 2.4 we obtain

(iii) \( I f^i \)-covers either \( I \) and \( J \) or \( K \).

Since \( n \) is odd and \( f^j(y) = x \), the triple \( \{y, x, f^j(x)\} \) is prime. By (i), we get \( f^j(x) \notin (v, z) \).

Therefore, by Lemma 2.4,

\[
\begin{align*}
&v \quad i \quad x \quad j \quad y \quad K \quad z \\
&f^j & f^i & f^k
\end{align*}
\]

**Fig. 1.**
(iv) \( J f^j \)-covers either \( I \) or \( J \) and \( K \).

By (ii), \( \{x, f^j(x), f^{j+k}(x)\} = \{x, y, f^k(y)\} \) is a prime triple. Hence,

(v) \( K f^k \)-covers either \( I \) or \( J \) and \( K \).

Assume now that \( h(f) = 0 \). Suppose that \( K f^k \)-covers \( J \) and \( K \).

By Lemma 6.1, \( J \) does not \( f^j \)-cover \( K \) and does not \( f^{j+i} \)-cover \( K \). Hence, by (iv), \( J f^j \)-covers \( I \). Thus, by Lemma 6.1 and (iii), \( I f^j \)-covers \( K \), so by Lemma 2.2, \( J f^{j+i} \)-covers \( K \), a contradiction. Hence, \( K \) does not \( f^k \)-cover \( J \) and \( K \), and by (v) we get

(vi) \( K f^k \)-covers \( I \).

Suppose that \( I f^i \)-covers \( I \) and \( J \). By Lemma 6.1, \( J \) does not \( f^j \)-cover \( I \). Hence, by (iv), \( J f^j \)-covers \( J \) and \( K \). However, by (vi), our assumption and Lemma 2.2, \( K f^k+i \)-covers \( J \), contrary to Lemma 6.1. Hence, \( I \) does not \( f^i \)-cover \( I \) and \( J \), and by (iii) we get

(vii) \( I f^i \)-covers \( K \).

From (vi), (vii) and Lemma 2.2 we obtain

(viii) \( K f^{k+i} \)-covers \( K \).

By (ii), there is \( m \) divisible by \( k + i \) such that \( m \equiv k \pmod{n} \). Since \( f^m(x) = x \), the same proof that gave us (vi), shows that \( K f^m \)-covers \( I \). On the other hand, since \( m \) is divisible by \( k + i \), from (viii) and Lemma 2.2 we get that \( K f^m \)-covers \( K \). Thus, by Lemma 6.1, \( I \) does not \( f^i \)-cover \( K \), contrary to (vii). This shows that \( h(f) > 0 \).

The next lemma is in fact purely number-theoretical.

**Lemma 6.3.** Let \( P \) be a cycle of period \( p^k \), where \( p \) is an odd prime number. Then any subset of \( P \) of cardinality larger than \( 2^k \) contains a prime triple.

**Proof.** We can reformulate our statement in a number-theoretical language: every subset of \( \mathbb{Z}/p^k \) of cardinality larger than \( 2^k \) contains distinct elements \( a, b, c \) such that \( a - b \), \( b - c \) and \( c - a \) generate the same subgroup. We shall prove this statement by induction on \( k \). For \( k = 1 \) there is nothing to prove, since \( p^1 = p \) is prime and every \( d \neq 0 \) generates the whole \( \mathbb{Z}/p \). Suppose that our statement is true for \( k - 1 \) replacing \( k \). Let \( X \) be a subset of \( \mathbb{Z}/p^k \) of cardinality larger than \( 2^k \). If there are three elements \( a, b, c \) of \( X \) which are different \( \pmod{p} \) then each of \( a - b \), \( b - c \) and \( c - a \) generates the whole \( \mathbb{Z}/p \) and we are done. Otherwise, there are subsets \( X_1, X_2 \subset X \) such that \( X_1 \cup X_2 = X \) and for each \( i = 1, 2 \) if \( a, b \in X_i \) then \( a \equiv b \pmod{p} \). One of the sets \( X_1, X_2 \) (say, \( X_1 \)) has more than \( 2^{k-1} \) elements. Take \( a \in X_1 \) and set \( Y = \{(b - a)/p : b \in X_1 \} \subset \mathbb{Z}/p^{k-1} \). By the induction hypothesis there are distinct \( c, d, e \in Y \) such that \( c - d \), \( d - e \) and \( e - c \) generate the same subgroup of \( \mathbb{Z}/p^{k-1} \). Then \( c' = p(c + a) \), \( d' = p(d + a) \) and \( e' = p(e + a) \) are three distinct elements of \( X \) and \( c' - d' = p(c - d) \), \( d' - e' = p(d - e) \) and \( e' - c' = p(e - c) \) generate the same subgroup of \( \mathbb{Z}/p^k \).

Now we are ready to prove Theorems D and E.

**Theorem D.** Let \( f \) be a continuous map of a graph with \( s \) edges into itself. If the pantheon of \( \text{Per}(f) \) has more than \( s \Gamma_s \) elements then \( h(f) > 0 \).

**Proof.** Let \( X \) be the pantheon of \( \text{Per}(f) \). Assume that \( \text{Card}(X) > s \Gamma_s \). Take \( X' \subset X \) such that \( \text{Card}(X') = s \Gamma_s + 1 \). For each \( j \in X' \) choose a periodic point \( x_j \) of period \( i_j \) such
that \( \text{gcd}(i) = j \). Take \( m \) so large that \( 2^m > i/j \) for all \( j \in X' \). Then the period of \( x_j \) for \( \tilde{f} = f^{2^m} \) is \( j \). For each \( j \in X' \) there is an edge \( e_j \) such that if \( P_j \) is the orbit of \( x_j \) under \( \tilde{f} \) then \( \text{Card}(P_j \cap e_j) \geq j/s \). Since \( \text{Card}(X') > s\Gamma_s \), there is an edge \( e \) and a set \( Y \subset X' \) with \( \text{Card}(Y) > \Gamma_s \) such that \( e_j = e \) for all \( j \in Y \).

For any \( j \in Y \) and an odd prime number \( p \) denote by \( \gamma_j(p) \) the exponent with which \( p \) appears in the decomposition of \( j \) into prime factors. Every element of \( Y \) is odd, so if each \( \gamma_j(p) \) attains at most \( \gamma_j(p) \) values as \( j \) runs over \( Y \), then \( Y \) has at most \( \prod_p \gamma_j(p) = \Gamma_s \) elements, a contradiction. Therefore there is an odd prime number \( p \) and elements \( j_1, j_2, \ldots, j_k \in Y \), with \( k > \gamma_j(p) \), such that in the decomposition of \( j \) into prime factors \( p \) appears with the exponent \( x_1 < x_2 < \ldots < x_k \). Then \( x_k - x_1 \geq k - 1 \geq \gamma_j(p) \). Set \( \theta = f^k \), where \( \tau = j_1/j_1 \). For \( \theta \) all elements of \( P_j \) are fixed points and \( P_j \cdot \theta \) decomposes into a union of cycles of period \( n = p^{x_k-x_1} \).

Since \( \text{Card}(P_j \cap e) \geq j/s > 0 \), there is a fixed point \( \nu \) of \( \theta \) in \( e \). If \( \nu \) is not a vertex of \( G \), then we make it a vertex artificially. In other words, we consider the graph \( G' \), which is equal to \( G \) as a topological space, but has additionally a new vertex \( v \). Of course, then \( e \) becomes the union of two new edges \( e_1 \) and \( e_2 \) with a common endpoint \( v \).

Since \( \text{Card}(P_m \cap e)/\text{Card}(P_m) = \text{Card}(P_m \cap e)/j_k \geq 1/s \), there is a cycle \( Q \subset P_m \) of \( g \) such that \( \text{Card}(Q \cap e)/n = \text{Card}(Q \cap e)/\text{Card}(Q) \geq 1/s \). Therefore there is an edge \( d \) of \( G \) (if \( \nu \) is an endpoint of \( d \), then \( d = e \) or \( G' \) otherwise; then \( d = e_1 \) or \( d = e_2 \)) such that \( \text{Card}(Q \cap d) \geq n/(2s) \). By the definition of \( \gamma_j(p) \), we have \( \gamma_j(p) \cdot \log(p/2) > \log(2s) \), so \( (p/2)^{\gamma_j(p)} > 2s \). Since \( n = p^{x_k-x_1} \) and \( x_k - x_1 \geq \gamma_j(p) \), we obtain \( (p/2)^{x_k-x_1} > 2s \), so \( n > 2s \cdot 2^{x_k-x_1} \). Thus, \( \text{Card}(Q \cap d) > 2^{x_k-x_1} \), and by Lemma 6.3, \( Q \cap d \) contains a prime triple. Therefore by Lemma 6.2, \( h(\theta) > 0 \), so

\[
\text{h}(f) = \frac{1}{2^m} \cdot h(f^{2^m}) = \frac{1}{2^m} \cdot h(\theta) > 0.
\]

**Theorem E.** Let \( f \) be a continuous map of a graph into itself. Then the following statements are equivalent.

1. \( h(f) > 0 \).
2. There is \( m \in \mathbb{N} \) such that \( \{mn: n \in \mathbb{N}\} \subset \text{Per}(f) \).
3. \( \rho(\text{Per}(f)) > 0 \).
4. The pantheon of \( \text{Per}(f) \) is infinite.

**Proof.** Let \( G \) be a graph and \( f: G \to G \) a continuous map. If \( f \) has positive entropy then from Theorem B and Proposition 3.5 it follows that for some \( m \in \mathbb{N} \) the set \( \{mn: n \in \mathbb{N}\} \) is contained in \( \text{Per}(f) \). This proves that (i) \( \Rightarrow \) (ii).

Clearly, (ii) \( \Rightarrow \) (iii), since \( \rho(\{mn: n \notin \mathbb{N}\}) = 1/m > 0 \). Also, (iii) \( \Rightarrow \) (iv), since any set with a finite pantheon is contained in a finite union of the sets of the form \( \{m2^n: n = 0, 1, 2, \ldots\} \), which have upper density zero. Finally, (iv) \( \Rightarrow \) (i) by Theorem D.

Now we give an explanation (promised in the introduction) why the implications (i) \( \Rightarrow \) (ii) and (iv) \( \Rightarrow \) (i) in Theorem E cannot be improved in general. Let \( G \) be a circle and let \( m \in \mathbb{N} \). Then we can easily construct a continuous map \( f: G \to G \) with \( \text{Per}(f) = \{m2^n: n \in \mathbb{N}\} \) (cf. [17, 2]). We start with the rotation by \( 1/m \) (that is, the angle \( 2\pi/m \)), choose an arc \( I \) of length less than \( 1/m \) (if the length of the whole circle is 1) and replace the map there by a map which gives a 3-horseshoe for \( f^m \) (see Fig. 2). Then \( h(f) = (1/m)\log 3 \), and \( \text{Per}(f) = \{mn: n \in \mathbb{N}\} \). In fact, we can make this type of construction whenever \( G \) is not a tree. On the other hand, we can replace the map on \( I \) by a map which gives periods \( 2^t \), \( t = 0, 1, 2, \ldots \), for \( f^m \) and has entropy zero. If \( G \) has many disjoint subsets homeomorphic
to a circle, we can make this construction independently on each of them with different \( m \)'s and then define the map on the rest of the graph in such a way that it stays continuous and has entropy zero.

We end our paper by noting that as for interval maps, Theorem B allows us to show that if \( h(f) > 0 \) then not only \( \text{Per}(f) \) is large but also the number of cycles of a given period is frequently very large.

**Theorem 6.4.** Let \( G \) be a graph and \( f: G \to G \) a continuous map with positive topological entropy. Then

\[
\limsup_{n \to \infty} \frac{1}{n} \log \alpha_n \geq h(f),
\]

where \( \alpha_n \) is the number of cycles of period \( n \).

**Proof.** Let \( n \in \mathbb{N} \). By Theorem B there exists \( k \) such that \( f^k \) has an \( s \)-horseshoe for some \( s \) such that \( (1/k) \log s > h(f) - 1/(2n) \) (if \( h(f) = \infty \) then \( (1/k) \log s > n + 1 \)). Take \( m \) such that \( km > n \) and \( s^m - (s/e^{1/(2n)})^m > 2 \). By Lemma 4.2, \( f^{km} \) has an \( s^m \)-horseshoe, so by Lemma 4.3, \( f^{km} \) has a strong \( (s^m - 2) \)-horseshoe. We get

\[
\frac{1}{km} \log(s^m - 2) > \frac{1}{km} \log s^m - \frac{1}{2n} > h(f) - \frac{1}{n}
\]

(if \( h(f) = \infty \) then \( (1/(km)) \log(s^m - 2) > n \)). By Lemmas 3.3 and 3.4, \( f^{km} \) has at least \( s^m - 2 \) fixed points.

In such a way we see that

\[
(5) \quad \limsup_{n \to \infty} \frac{1}{n} \log \text{Card} \{x:f^n(x) = x\} \geq h(f).
\]

Suppose that \( \limsup_{n \to \infty} (1/n) \log \alpha_n < h(f) \). Then there is \( n_0 \in \mathbb{N} \) and \( \beta < h(f) \) such that for all \( n \geq n_0 \) we have \( \alpha_n < e^{\beta n} \). Set \( b_n = \text{Card} \{x:f^k(x) = x\} \) for some \( k \leq n \). Then for every \( n \geq 2 \) we have \( b_n = b_{n-1} + n\alpha_n \), so \( b_n \leq b_{n-1} + ne^{\beta} \) for all \( n \geq n_0 \). Thus,

\[
b_n \leq b_{n_0} + \sum_{k=n_0+1}^{n} ke^{\beta} \text{ for } n \geq n_0.
\]

Therefore,

\[
\limsup_{n \to \infty} \frac{1}{n} \log b_n \leq \limsup_{n \to \infty} \frac{1}{n} \log \left(b_{n_0} + \sum_{k=n_0+1}^{n} ke^{\beta}\right) = \beta.
\]

Consequently, by the definition of \( b_n \) we get

\[
\limsup_{n \to \infty} \frac{1}{n} \log \text{Card} \{x:f^n(x) = x\} \leq \beta < h(f),
\]

contrary to (5). Thus \( \limsup_{n \to \infty} \frac{1}{n} \log \alpha_n \geq h(f) \). \( \blacksquare \)
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