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Quasi-convex groups of isometries of negatively curved spaces ☆

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Abstract

Let H be a properly discontinuous group of isometries of a negatively curved (Gromov hyperbolic) metric space X. We give equivalent conditions on H to be quasi-convex. The main application of this is to give alternate definitions of quasi-convex, or rational subgroups of negatively curved (word hyperbolic) groups. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

The main purpose of this paper is to extend the results of Bowditch [4,5] about equivalent definitions of geometric finiteness to the setting of a general negatively curved (Gromov hyperbolic) metric space. Because of problems with finite generation [3], we will restrict ourselves to the case where there are no parabolic elements. In the cases already covered by Bowditch, quasi-convex will be the same as geometrically finite without parabolics. All of the actions we are interested in are *properly discontinuous* (a set S of homeomorphisms acts properly discontinuously on X if for each compact $K \subset X$, $\{g \in S\colon\ g(K) \cap K \neq \emptyset\}$ is finite).

Main Theorem. *For X a negatively curved space and H a properly discontinuous group of isometries of X, the following conditions are equivalent.*

(1) *For any $a \in X$, the set Ha, of translates of a by H is quasi-convex.*

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(2) *H acts cocompactly on the weak convex hull of its limit set $\Lambda H \subset \partial X$ (the union of all lines in X joining limit points of H).*
(3) *If Ω is the domain of discontinuity of H acting on ∂X, then H acts cocompactly on $X \cup \Omega$.*
(4) *All limit points of H are conical.*
(5) *All limit points of H are horospherical.*

We will show $(2) \Rightarrow (1) \Rightarrow (4) \Rightarrow (5) \Rightarrow (3) \Rightarrow (2)$. The implication $(2) \Rightarrow (4)$ is contained in [9].

Definition. If H satisfies any of the above we say H is *quasi-convex*.

The most important application of this result is in the case where H is a subgroup of a negatively curved group G, and so H acts properly discontinuously on the Cayley graph of G, where (1) is the standard definition of what has been called a rational or quasi-convex subgroup. In a negatively curved group, there are never parabolic subgroups, so the result extends the results of Bowditch fully in this case. The proof that $(1) \Rightarrow (2)$ is similar to the proofs in [20,13,15] which deal only with the case where H is a subgroup of a negatively curved group G.

Condition (5) is rather interesting. It is known in the case of a Kleinian group, that if every limit point is bounded parabolic or horospherical, then the group is geometrically finite. The author has been unable to find an explicit statement of this result, but it is implicit in [16, 2.6.1, 2.6.2]. This will also be true in the setting of [5] when the group is acting on a simply connected complete manifold of pinched negative curvature. This will follow from [5] by replacing conical with horospherical and using Lemma 7 which we prove latter. (In fact, the only place which Bowditch uses conical limit points is to get the corresponding result for conical limit points.)

The second possible application of this result is when M is a manifold (orbifold) with $\pi_1(M)$ negatively curved as a group. This need not imply that M has a metric of nonpositive curvature. This result will give us information about the universal cover of M acted on by subgroups of $\pi_1(M)$ and also by other groups of homeomorphism which preserve the structure obtained from M.

Also we show that if H is a quasi-convex group of isometries of X, then H is negatively curved as a group and ∂H (as a group) is homeomorphic to ΛH. We also extend the results of [20,18,13,15] to this slightly more general setting. In particular:

Corollary. *If G is a properly discontinuous group of isometries of a negatively curved metric space X and $H < G$ is a quasi-convex group of isometries of X with $\Lambda H = \Lambda G$, then G is quasi-convex and H is of finite index in G.*

Proof. The fact that G is quasi-convex follows from (2). The fact that H is of finite index in G follows from the fact that they both act cocompactly and properly discontinuously on $WCH(\Lambda G)$. That is if K is a compact set whose translates under H cover $WCH(\Lambda G)$, then every element of G can be written as an element of H times an element of the set $\{g \in G\colon\ g(K) \cap K \neq \emptyset\}$ which is finite. □

2. Negatively curved spaces

Let X be a proper geodesic metric space with metric d. When the words interval, segment, ray, line, triangle, polygon, etc. are used it is to be understood that they are geodesic. We will assume that all intervals (rays, lines, or segments) are *parameterized by arc length*. Unless otherwise stated, closed rays will have domain $[0, \infty)$.

Definition. A triangle in X is said to be δ-*thin* if any point on the triangle is within δ of one of the other two sides of the triangle.

Definition. We say X is *negatively curved* if there is a $\delta > 0$ such that all triangles in X are δ-thin.

Notation. *For* $a \in X$ *we define* $B(a, n) \equiv \{x \in X\colon\ d(x, a) \leqslant n\}$. *For* $A \subset X$ *we define* $\mathrm{Nbh}(A, n) \equiv \{x \in X\colon\ d(x, A) \leqslant n\}$.

Remark. For the remainder of the paper, X will be a proper geodesic negatively curved metric space with thin triangle constant δ, and H will be a properly discontinuous group of isometries of X.

Definition. Two rays $R, S \subset X$ are equivalent if there is an $N > 0$ such that $R \subset \mathrm{Nbh}(S, N)$.

Remark. If R and S are equivalent rays then, for $r \gg 0$, $d(R(r), S) \leqslant 2\delta$.

Definition. We define ∂X to be the set of equivalence classes of rays. The elements of ∂X are called *points at* ∞.

Remark. If all triangles are δ-thin, then all n-gons are $(n-2)\delta$-thin and ideal n-gons, n-gons with one or more vertices on ∂X, are $2(n-2)\delta$-thin.

Definition. Let T be a closed set of X, and $x \in X$. Define

$$\pi_T(x) \equiv \bigl\{t \in T\colon\ d(t, x) = d(T, x)\bigr\}.$$

Notice that in general $\pi_T(x)$ is not a single point. For $t \in T$ we define

$$\pi_T^{-1}(t) \equiv \bigl\{x \in X\colon\ t \in \pi_T(x)\bigr\}$$

and we extend this to ∂X by defining $x \in \partial X$ to be in $\pi_T^{-1}(t)$ if and only if there is some ray R representing x with $R \subset \pi_T^{-1}(t)$.

Definition. Let T be some geodesic interval (segment, ray, or line) and $t \in$ domain T. Define the *half-space*

$$H(T,t) \equiv \{x \in X\colon\ T(a) \in \pi_T(x) \text{ for some } a \geqslant t\}.$$

Define the corresponding *disk at* ∞,

$$D(T,t) \equiv \Big\{[S] \in \partial X\colon\ \lim_{s\to\infty} d(S(s), X - H(T,t)) = \infty\Big\}.$$

The disks defined above form the basis of a natural topology (equivalent to Gromov's) on ∂X so that ∂X is compact metrizable [2], and in the case where the isometry group of X acts cocompactly on X, ∂X is finite-dimensional [19]. Also the union of a half-space with its corresponding disk forms a neighborhood of every point of the disk in the natural compactification $\overline{X} \equiv X \cup \partial X$ of X.

Definition. For $A \subset X$ we define the *limit set* of A, $\Lambda(A) \equiv \overline{A} \cap \partial X$, where $\overline{A}$ is the closure of A in $\overline{X}$. Notice that the limit set is always closed.

Remark. A point at ∞, x, is in ΛA if and only if for any $b \in X$ there is a sequence $[b, a_n]$ of closed intervals with $a_n \in A$ such that $[b, a_n]$ converges, on compact subsets, to a ray R emanating from b with $[R] = x$ [7, 3.10, 3.15, and 3.17].

Remark. If (x_i) and (y_i) are sequences of elements of X with $d(x_i, y_i) \leqslant N$ for some fixed N, and if $x_i \to x \in \partial X$, then $y_i \to x$ by [7, 3.16].

Thus the following is well defined:

Definition. If G is a group of isometries of X then $\Lambda G \equiv \Lambda Ga$ where $a \in X$ and $Ga \equiv \{g(a)\colon\ g \in G\}$.

Definition. Let X_0 and X_1 be metric spaces. A relation $\mathcal{R} \subset X_0 \times X_1$ is a quasi-Lipschitz equivalence if the following three conditions are satisfied for some $K > 0$ for $i = 0, 1$:

(1) $\forall x_i \in X_i$, $d(x_i, \pi_i(\mathcal{R})) \leqslant K$.
(2) $\forall x_i \subset X_i$, $\pi_i \circ \pi_{|1-i|}^{-1} \circ \pi_{|1-i|} \circ \pi_i^{-1}(x_i) \subset B(x_i, K)$.
(3) $\forall A_i \subset X_i$, K diam$(A_i) + K \geqslant$ diam$(\pi_{|1-i|} \circ \pi_i^{-1}(A_i))$.

The first condition just says that any point of X_i is close to $\pi_i(\mathcal{R})$. The second says that if we move a point from one space to the other and back, we are close to where we started. The third says that the metrics are Lipschitz compatible through the relation. In this case we say X_0 and X_1 are quasi-isometric. It can be shown that this defines an equivalence relation on proper geodesic metric spaces.

Theorem 1 [10]. *Let W and Y be proper geodesic metric spaces with Y negatively curved. If W is quasi-isometric to a subspace Z of Y* (*where Z need be neither proper nor geodesic*), *then W is negatively curved, and the quasi-Lipschitz equivalence gives a topological embedding of ∂W onto ΛZ.*

Corollary. *If the negatively curved spaces W and Y are quasi-isometric, then the corresponding quasi-Lipschitz equivalence gives rise to natural homeomorphisms between ∂W and ∂Y, so that we can define the boundary of a quasi-isometric equivalence class of negatively curved spaces.*

Definition. The *weak convex hull* of a set $A \subset \overline{X}$, denoted $WCH(A)$, is the union of all intervals (segments, rays, or lines) of X which have both endpoints in A.

Definition. A set $A \subset \overline{X}$ is *quasi-convex*(ε) if the weak convex hull $WCH(A) \subset \mathrm{Nbh}(A, \varepsilon)$. We say A is *quasi-convex* if it is *quasi-convex*(ε) for some $\varepsilon \geqslant 0$.

Remark. It is easily shown that for any quasi-convex set $A \subset \overline{X}$, $A \cup \Lambda(A)$ is *quasi-convex*(ε), for some $\varepsilon > 0$.

Definition. An set $A \subset X$ is ∂-*quasi-convex*(ε) if $\Lambda A \neq \emptyset$ and $WCH(\Lambda A) \subset \mathrm{Nbh}(A, \varepsilon)$.

Remark. It should be clear that if H is a group of isometries of X such that Ha is quasi-convex for some $a \in X$, then by thin quadrilaterals the same will be true for any other $x \in X$.

Remark. It should also be clear that Ha is ∂-quasi-convex for some $a \in X$ if and only if H acts cocompactly on $WCH(\Lambda H)$.

The following lemma is from [19].

Lemma 2. *The weak convex hull of a set $A \subset \overline{X}$ is quasi-convex*(4δ).

We are now ready for the first implication in the proof of the main theorem.

(2) $\Rightarrow$ (1)

Proof. We wish to show that Ha is quasi-convex for some (equivalently any) $a \in X$. We are given that H acts cocompactly on $WCH(\Lambda H)$ or equivalently that Ha is ∂-quasi-convex(ε) for some $\varepsilon \geqslant 0$. I.e., $WCH(\Lambda H) \subset \mathrm{Nbh}(Ha, \varepsilon)$. Let $D = d(a, WCH(\Lambda H))$, so there is a line L with endpoints in ΛH and some $x \in L$ with $d(x, a) = D$. Since ΛH is invariant under the action of H, it follows that for any $h \in H$, $h(x) \in h(L)$ where the endpoints of $h(L)$ are in ΛH, and of course $d(h(x), h(a)) = D$. Since $[a, h(a)] \subset \mathrm{Nbh}([x, h(x)], D + 2\delta)$ by [7], and since by Lemma 2 $[x, h(x)] \subset \mathrm{Nbh}(WCH(\Lambda H), 4\delta)$, it follows that $[a, h(a)] \subset \mathrm{Nbh}(Ha, \varepsilon + D + 6\delta)$. Thus Ha is quasi-convex. □

Definition. A limit point p of $A \subset X$ is called a *conical limit point of A* if there is some N such that for all rays R representing p, $\mathrm{Nbh}(R, N) \cap A \neq \emptyset$. It follows that $\mathrm{Nbh}(R, N) \cap A$ is infinite for all R representing p. For a group G of isometries of X, and $p \in \Lambda H$, then p is a *conical limit point of H* if p is a conical limit point of Ha for some (equivalently any) $a \in X$.

Remark. Clearly (1) implies (4). That is, for any $p \in \Lambda H$, there is a sequence $(h_i) \subset H$ such that $[a, h_i(a)] \to R$, a ray representing p. By (1), this ray will be contained in a uniform neighborhood of Ha, and therefore p is conical.

Definition. Let R be a ray in X emanating from the point $y \in X$. The *funnel* about R is

$$F(R) \equiv \{x \in X\colon\ d(x, R) \leqslant d(\pi_R(x), y)\}.$$

For $A \subset X$ we say $p \in \Lambda A$ is a *funneled limit point of* A if for any ray R representing p, $F(R) \cap A \neq \emptyset$. If G is a group of isometries of X, we define $p \in \Lambda G$ to be a *funneled limit point of* G if p is a funneled limit point of Ga for some a. By [7, 3.16], this is independent of choice of a.

It is obvious that all conical limit points are funneled, however the converse is false.

Lemma 3. *There are funneled limit points which are not conical.*

Proof. Let $F = F(x, y)$, the free group on $\{x, y\}$, and T the simplicial tree which is the Cayley graph of F with generating set $\{x, y\}$. Choose two sequences of positive integers (n_i), (m_i) such that $m_i < n_i < m_{i+1}$ for all $i > 0$ and $m_i/n_i \to 0$. Let $B = \{x^{n_i} y x^{m_i}\}$, and $G < F$ the subgroup generated by B. The fact that, in any freely reduced word in B, none of the y terms will cancel, shows that G is free with basis B and $x^n \notin G$ for $n \neq 0$. Since $\langle x \rangle$ is a quasi-convex subgroup of F, it follows by Theorem 13 that x^∞, the point at infinity on the positive x-axis, is not a conical limit point of G. On the other hand, any ray which represents x^∞ has a subray starting at the vertex x^n of T for some n. Since $m_i/n_i \to 0$ it follows that the corresponding funnel contains a vertex $x^{n_i} y x^{m_i}$ for some i. Thus x^∞ is a funneled limit point of G. □

Definition. Let R, S be rays with domains $[a, \infty)$ and $[b, \infty)$, respectively. We say R and S *asymptoticly fellow travel*(N), denoted $R \sim_N S$, if for all $t \gg 0$, $d(R(t), S(t)) \leqslant N$.

Notice that $\sim_N$ is not an equivalence relation as it is not transitive, however we have the following lemma of [7].

Lemma 4. *If R, S are equivalent rays, then there exists $a \in \mathbb{R}$ and an isometry $\rho\colon [a, \infty) \to [0, \infty)$ so that the geodesic rays R and $S \circ \rho$ asymptoticly fellow travel*(6δ).

Definition. Let $R : [0, \infty) \to X$ be a ray. Define the horoball corresponding to R to be

$$H(R) \equiv \bigcup_{\substack{S \sim_{6\delta} R \\ b \geqslant 0}} S([b, \infty)).$$

Notice that in the case where X is hyperbolic n-space, this is in fact the 6δ neighborhood of the horoball about the endpoint of R through the point $R(0)$ (which is itself a horoball). We define a point $x \in \partial X$ to be a horospherical limit point of H if the set Ha meets every horoball about x.

Remark. It is an easy exercise to show that for any ray R, $F(R) \subset H(R)$. Namely given any point $p \in F(R)$ we construct a ray emanating from p equivalent to R [7] and from the definition of $F(R)$, it soon follows that $p \in H(R)$.

Lemma 5. *Let $R:[0,\infty) \to X$ be a ray, and $R' = R|_{[12\delta,\infty)}$ reparameterized to have domain $[0,\infty)$. $H(R') \subset F(R)$.*

Proof. Let $a \in H(R')$, so there exists a ray S emanating from a so that there are points $s \in S$ and $r \in R'$ with $d(r,s) \leqslant 6\delta$ and $0 \ll d(s,a) \leqslant d(r, R(12\delta))$ (remember that $R(12\delta)$ is the first point of R'). Let $p \in \pi_R(a)$. As $[p,a]$ moves geodesically away from R, if z is the point δ units from p on $[p,a]$, then z is no more than δ from a point of $[r,a]$ which in turn is no more than δ from a point of $[a,s] \cup [s,r]$. Thus $d(p,[a,s] \cup [s,r]) \leqslant 3\delta$. Choosing s very far from a we have some $q \in [a,s]$ with $d(q,p) \leqslant 3\delta$. It follows by the triangle inequality that $|d(r,p) - d(s,q)| \leqslant 9\delta$ and that $|d(a,p) - d(a,q)| \leqslant 3\delta$. It follows that $|d(s,a) - [d(r,p) + d(p,a)]| \leqslant 12\delta$. However, since $a \in H(R')$, $d(s,a) \leqslant d(r, R(12\delta))$, so $d(r,p) + d(p,a) \leqslant d(r, R(12\delta)) + 12\delta = d(r, R(0))$. Since $p \in R$ it follows that $d(p,a) \leqslant d(p, R(0))$ and so $a \in F(R)$. $\square$

Corollary. *The funneled limit points are exactly the horospherical limit points, and so* $(4) \Rightarrow (5)$.

Definition. Let H be a properly discontinuous set of isometries of X containing the identity. Choose $\mathbf{0} \in X$, and let $H\mathbf{0} \equiv \{h(\mathbf{0}):\ h \in H\}$. Since H is properly discontinuous, $H\mathbf{0}$ is closed and we may define $D = \pi^{-1}_{H\mathbf{0}}(\mathbf{0})$. Clearly for any $h \in H$, $h(D) = \pi^{-1}_{H\mathbf{0}}(h\mathbf{0})$. Define $\mathcal{D} = \{h(D):\ h \in H\}$.

Remark. By proper discontinuity and the definition of D, $\mathcal{D}$ is locally finite on X.

Remark. Notice that $D \cap X$ is star-like about $\mathbf{0}$, that is if $x \in D \cap \Gamma$ then $[x, \mathbf{0}] \subset D$. Using thin triangles we see that $D \cap X$ is *quasi-convex*(δ).

Lemma 6. *D is the closure of $D \cap X$ in $\overline{X}$ and so quasi-convex in $\overline{X}$.*

Proof. By proper discontinuity of H, $D \cap X$ is closed in X, and by definition of D, if $x \in \partial X \cap D$ then x is a limit point of $D \cap X$. Thus we need only show that if $y \in \Lambda(D \cap X)$ then $y \in D$. Take a sequence of points $a_i \in D$ with $a_i \to y$. Some subsequence of the sequence of segments $[\mathbf{0}, a_i]$ will converge [7, 3.10] to a ray R which represents y and $R \subset D \cap X$ by the fact that $D \cap X$ is star-like about $\mathbf{0}$. Thus by definition $y \in D$. $\square$

Lemma 7. *If y is a horospherical limit point of H then $y \notin \bigcup \mathcal{D}$.*

Proof. By the corollary to Lemma 5, it suffices to show that no funneled limit point is contained in D. Suppose to the contrary that $R \subset D$ is a ray emanating from $\mathbf{0}$ representing a funneled limit point of H. By definition of funneled, there is a $r \in R$ and $x \in H\mathbf{0}$ such that $d(\mathbf{0}, r) > d(x, r)$ and so $r \notin D$ which contradicts $R \subset D$. □

Definition. The domain of discontinuity Ω of H is defined to be $\partial X - \Lambda H$. Since ΛH is closed, Ω will be an open set of ∂X.

By proper discontinuity of H, $\mathcal{D}$ is locally finite on X. To show that it is also locally finite on Ω we need the following technical results.

Lemma 8 [7]. *For any interval R, $\Lambda H(R, r + 8\delta) \subset D(R, r)$.*

Lemma 9 [7]. *For any interval R, any geodesic interval joining $D(R, r + 8\delta)$ to the complement of $D(R, r)$ passes within 2δ of the point $R(r + 4\delta)$ of R.*

Lemma 10. $\Omega \subset \bigcup \mathcal{D}$.

Proof. Let the ray R represent an element of Ω. For some $r > 0$, $H(R, r) \cap H\mathbf{0} = \emptyset$. Let $A = \pi_{H\mathbf{0}}(B(R(r + 4\delta), 2\delta))$, since $\mathcal{D}$ is locally finite on X, A is a finite subset of $H\mathbf{0}$ and by Lemma 9, $\pi_H(H(R, r + 8\delta)) \subset A$. By Lemma 8 any sequence of points of X converging to the point represented by R will have a subsequence all of whose elements project to a single $h(\mathbf{0}) \in A$ and thus the point of Ω represented by R will be in $h(D)$.

Corollary. *$\mathcal{D}$ is locally finite on Ω.*

Proof. Notice that in the proof of Lemma 10 we actually showed that $\pi_H(D(R, r + 16\delta)) \subset A$. Thus only elements of $\mathcal{D}$ of the form hD where $h(\mathbf{0}) \in A$ hit $D(R, r + 16\delta)$, and so $\mathcal{D}$ is locally finite on Ω since A was finite, and H properly discontinuous. □

This corollary provides an alternate proof of a result of Coornaert that the action of a properly discontinuous group of isometries of a negatively curved space is properly discontinuous on Ω, its domain of discontinuity [8].

Definition. Define the quotients $M = X/H$ and $M_\Omega = (X \cup \Omega)/H$.

Theorem 11. *If $D \cap \Lambda H = \emptyset$ then M_Ω is compact.*

Proof. Let $f : X \cup \Omega \to M_\Omega$ be the quotient map of H. We know by Lemma 10 that the translates of D under H cover $X \cup \Omega$, and since $D \cap \Lambda H = \emptyset$ we have that $D \subset X \cup \Omega$. Since D is a compact set (Lemma 6), f is a continuous map, and $f(D) = M_\Omega$, the result follows. □

Corollary. (5) $\Rightarrow$ (3).

Proof. Use Lemma 7. □

We now complete the Main Theorem by showing that (3) implies (2).

(3) $\Rightarrow$ (2)

Proof. Let $\blacktriangle = WCH(\Lambda H)/H$. This is a closed subset of M since it can be shown (using [7, 3.10], for example) that $WCH(\Lambda H)$ is a closed set in X. Since H acts properly discontinuously, M is a metric space with the metric inherited from X. Let y_i be a sequence of points in $\blacktriangle$. To show that $\blacktriangle$ is compact it suffices to show that some subsequence of the y_i converges in $\blacktriangle$. Since $\blacktriangle \subset M_\Omega$ and M_Ω is compact by hypothesis, we may assume that $y_i \to y \in M_\Omega$. It suffices to show that $y \in M$, since $\blacktriangle$ is closed in M.

Suppose $y \notin M$. Let z be a lift of y to Ω. $WCH(\Lambda H) \cup \Lambda H$ is a closed set in $\overline{X}$, and so there is a open set U with $z \in U$ so that $U \cap (WCH(\Lambda H) \cup \Lambda H) = \emptyset$. Projecting U into M_Ω gives an open set containing y which misses $\blacktriangle$, and we have a contradiction. □

We now need the following result from [6].

Quasi-isomorphism Theorem. *If X and Y are geodesic metric spaces and G is a group which acts properly discontinuously and cocompactly on both, then X and Y are quasi-isomorphic.*

Now that we have shown the equivalence of the different definitions of quasi-convex, we will give a proof of the following result from [9].

Theorem 12. *If H is a quasi-convex group of isometries of the negatively curved space X, then H is a negatively curved group and $\Lambda H \cong \partial H$ and this homeomorphism is H equivariant.*

Proof. H is quasi-convex so H acts cocompactly on $WCH(\Lambda H)$. Let

$$Y = \mathrm{Nbh}\big(WCH(\Lambda H), \varepsilon\big),$$

where $\varepsilon \gg 0$ so that $WCH(WCH(\Lambda H)) \subset Y$. Clearly Y is a proper geodesic metric space (since it is a closed path connected set in a proper geodesic metric space). Also notice that Y, as a geodesic metric space, is quasi-isometric to $WCH(\Lambda H)$, as a subspace of X, under the identity function of $WCH(\Lambda H)$. Thus from Theorem 1, Y is negatively curved with $\partial Y \cong \Lambda WCH(\Lambda H) = \Lambda H$, and this homeomorphism is H equivariant, and of course H still acts cocompactly and properly discontinuously on Y. It follows that any locally finite Cayley graph of H will be quasi-isometric to Y (by the Quasi-isometry Theorem) and so, whenever H is finitely generated, H will be negatively curved. It is easy to show that any group which acts cocompactly and properly discontinuously on a proper connected metric space is finitely generated, and so H is finitely generated and therefore negatively curved. □

The following results are abstractions of results in [20,17,13,15]. See [1] for similar results in the classical setting.

Theorem 13. *Let G be a properly discontinuous group of isometries of the negatively curved space X and $H, K < G$. If p is a conical limit point H, and if K is quasi-convex with $p \in \Lambda K$, then p is a conical limit point of $H \cap K$.*

Proof. Choose $a \in X$ and a ray R emanating from a representing p. Since K is quasi-convex, for $N \gg 0$, $R \subset \mathrm{Nbh}(Ka, N)$. Since p is a conical limit point of H, for $N \gg 0$ we can choose a sequence $h_i \in H$ so that $d(h_i(a), R) \leqslant N$ and $d(h_i(a), h_j(a)) > 4N$ for $i \neq j$. Fix N large enough for both. Choose $k_i \in K$ so that $d(k_i(a), h_i(a)) \leqslant 2N$. Define

$$C = \{g \in G:\ g(B(a, 2N)) \cap B(a, 2N) \neq \emptyset\},$$

Since $B(a, 2N)$ is compact and G is properly discontinuous, it follows that C is a finite subset of G. Since $d(h_i(a), k_i(a)) \leqslant 2N$, it follows that $d(a, h_i^{-1}k_i(a)) \leqslant 2N$ and so $h_i^{-1}k_i \in C$. Taking subsequences if necessary, we may assume (since C is finite) that all $h_i^{-1}k_i$ are equal to a single $g \in C$. Thus for all i, $h_1^{-1}k_1 = h_i^{-1}k_i$, or $h_i h_1^{-1} = k_i k_1^{-1} \in H \cap K$. Notice $d(h_i h_1^{-1}(a), h_i(a)) = d(h_1^{-1}(a), a)$ and so there are infinitely many points of $H \cap K$ (specifically $h_i h_1^{-1}(a)$) in $\mathrm{Nbh}(R', N + d(h_1^{-1}(a), a)$ for any subray $R' \subset R$. The result follows. □

Corollary. *Let G be a properly discontinuous group of isometries of the negatively curved space X, and $H, K < G$ be two quasi-convex groups of isometries of X. Then $\Lambda(H \cap K) = \Lambda H \cap \Lambda K$, and $H \cap K$ will also be a geometric group of isometries.*

Proof. Let $J = H \cap K$. Clearly $\Lambda J \subset \Lambda H \cap \Lambda K$, and so by using definition (4), it suffices to show that every $p \in \Lambda H \cap \Lambda K$ is a conical limit point of J. Apply Theorem 13. □

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