

# On the solution of strong nonlinear oscillators by applying a rational elliptic balance method

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## ABSTRACT

A rational elliptic balance method is introduced to obtain exact and approximate solutions of nonlinear oscillators by using Jacobi elliptic functions. To illustrate the applicability of the proposed rational elliptic forms in the solution of nonlinear oscillators, we first investigate the exact solution of the non-homogenous, undamped Duffing equation. Then, we introduce first and second order rational elliptic form solutions to obtain approximate solutions of two nonlinear oscillators. At the end of the paper, we compare the numerical integration values of the angular frequencies with approximate solution results, based on the proposed rational elliptic balance method.

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## 1. Introduction

In general, the exact solution of nonlinear oscillatory systems are unknown and hence, numerical integration, perturbation methods and nonperturbative techniques have been applied to obtain their approximate solutions. These methods are discussed in a great many papers, so we shall not elaborate further. See [1–5], for example. Many additional references dealing with different approaches for approximating solutions to nonlinear oscillatory systems are provided in these articles. Here we introduce an approach based on rational Jacobi elliptic functions to obtain exact and approximate solutions of strongly nonlinear oscillators by following a procedure similar to that of the rational harmonic balance method that provides a general framework for determining higher order corrections [6]. Mickens and Semwogerere showed that the rational harmonic balance functional form has Fourier coefficients that decrease exponentially [7]. They also concluded that the rational harmonic balance representation should provide accurate results for oscillators of the form

$$\frac{d^2x}{dt^2} + f(x) = 0, \quad (1)$$

with initial conditions

$$x(0) = x_{10}; \quad \dot{x}(0) = 0 \quad (2)$$

where  $x$  is the system displacement, and  $f(x)$  is the restoring force. Sarma and Rao introduced a modified rational form to consider mixed-parity restoring forces for the Duffing equation and found good agreement between approximate and exact angular frequency values [8]. In accordance with these results, Mickens concluded that the inappropriate choice of the rational form can lead to large errors in the determination of the angular frequency for periodic solutions of Eq. (1) [7].

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On the other hand, Beléndez et al. in [9] assumed that  $f(x)$  is an odd function and used a rational form

$$x(\tau) = \frac{x_{10}(1 + c_0) \cos \tau}{1 + c_0 \cos 2\tau}, \tag{3}$$

to solve Eq. (1). Here,  $\tau = \omega t$  and  $c_0$  is an undetermined constant that need to be determined by applying the rational harmonic balance method and must satisfied the condition  $|c_0| \ll 1$ . In an attempt to provide better solution methodology, Beléndez et al. used a modified rational harmonic balance method to solve the Duffing oscillator by introducing the new independent variable  $\tau$  to ensure that the solution of Eq. (1) is a periodic function of  $\tau$  with period  $2\pi$  with results that agree well with the exact solution [10].

In this paper we do not introduce a new independent variable for (1) however, we consider that  $f(x)$  can be either an odd or even function of  $x$  and use rational form solutions based on Jacobi elliptic functions instead of trigonometric ones [11]. The main motivation for this assumption comes from the fact that the mixed-parity Helmholtz–Duffing oscillator:

$$\ddot{x} + Ax + B_1x^2 + \varepsilon x^3 + D_1 = 0 \tag{4}$$

has an exact solution of the form [12]

$$x(t) = \frac{a - b + c(a + b)\text{cn}(\omega t + \phi, k^2)}{1 + c\text{cn}(\omega t + \phi, k^2)}. \tag{5}$$

Here,  $A, B_1, \varepsilon$ , and  $D_1$  are system constant parameters,  $\text{cn}(\omega t + \phi, k^2)$  is the  $\text{cn}$  Jacobian elliptic function that has a period in  $\omega t$  equal to  $4K(k^2)$ , and  $K(k^2)$  is the complete elliptic integral of the first kind for the modulus  $k$ ,  $\omega$  is the frequency of oscillation,  $\phi, a, b$ , and  $c$  are unknown constants that are determined by substituting Eq. (5) into Eq. (3) and by using the initial conditions (3). The solution of Eq. (4) is discussed in detail in [12] therefore, we shall not elaborate any further on it.

## 2. Exact solution based on rational Jacobi elliptic forms

Since the aim of this paper is to obtain approximate solutions of nonlinear oscillators based on the usage of rational Jacobi elliptic forms, we first investigate the solution of the non-homogeneous Duffing equation that describes the free vibrational motion of a vehicular body supported by rubber shear mountings with quadratic response [13,14]:

$$\ddot{x} + x + \varepsilon x^3 = -F_0 \tag{6}$$

with initial conditions

$$x(0) = x_{10}; \quad \dot{x}(0) = 0. \tag{7}$$

Here the dots denote the derivative with respect to  $t$ ,  $x$  represents the system displacement,  $\varepsilon$  is a nonlinear material parameter and  $F_0$  is a constant. We next assume that the exact solution of Eq. (6) is prescribed as an elliptic rational function of the form:

$$x(t) = \frac{a + bc\text{cn}(\omega t + \phi, k^2)}{1 + c\text{cn}(\omega t + \phi, k^2)}, \tag{8}$$

where  $a, b, c, k, \omega, \phi$  are unknown constants. Substituting Eq. (8) into Eq. (6) and using the elliptic balance method, we obtain:

$$\begin{aligned} a + F_0 + a^3\varepsilon + 2c(b - ac)(k^2 - 1)\omega^2 + \text{cn}(\omega t + \phi, k^2)(b + 2ac + 3cF_0 + 3a^2b\varepsilon \\ + (b - ac)(2k^2 - 1)\omega^2) + \text{cn}^2(\omega t + \phi, k^2)(2bc + ac^2 + 3c^2F_0 + 3ab^2\varepsilon \\ + c(ac - b)(2k^2 - 1)\omega^2) + \text{cn}^3(\omega t + \phi, k^2)(bc^2 + c^3F_0 + b^3\varepsilon - 2(b - ac)k^2\omega^2) = 0. \end{aligned} \tag{9}$$

This Eq. (9) holds for all time  $t$ , if and only if, each of its coefficient terms vanish i.e.

$$a + F_0 + a^3\varepsilon + 2c(b - ac)(k^2 - 1)\omega^2 = 0, \tag{10}$$

$$b + 2ac + 3cF_0 + 3a^2b\varepsilon + (b - ac)(2k^2 - 1)\omega^2 = 0, \tag{11}$$

$$2bc + ac^2 + 3c^2F_0 + 3ab^2\varepsilon - c(b - ac)(2k^2 - 1)\omega^2 = 0, \tag{12}$$

$$bc^2 + c^3F_0 + b^3\varepsilon - 2(b - ac)k^2\omega^2 = 0. \tag{13}$$

Then, the modulus  $k$  and the frequency  $\omega$  of the elliptic function are given by the following equations:

$$k^2 = \frac{a + 4bc + 2ac^2 + F_0 + 6c^2F_0 + a^3\varepsilon + 6ab^2\varepsilon}{2(2bc + F_0 + 3c^2F_0 + a^3\varepsilon + a(1 + c^2 + 3b^2\varepsilon))}, \tag{14}$$

$$\omega^2 = \frac{2bc + F_0 + 3c^2F_0 + a^3\varepsilon + a(1 + c^2 + 3b^2\varepsilon)}{c(b - ac)}. \tag{15}$$

From the initial conditions given by Eq. (7) and by using Eq. (8), we have that  $\phi = 0$  and

$$c = \frac{a + b - x_{10}}{x_{10}}. \tag{16}$$

To obtain  $b$ , we multiply Eq. (11) by  $c$  and add this to Eq. (12) and after solving for  $b$ , we get

$$b = \frac{-(a - x_{10})(2a + 4F_0 + x_{10} + a^2 \varepsilon x_{10} \pm x_{10} \sqrt{1 - 2a(a + 4F_0)\varepsilon + a^4 \varepsilon^2})}{2(2F_0 + x_{10} + a(1 + \varepsilon x_{10}(a + x_{10})))}. \tag{17}$$

Then, we add Eqs. (11) and (13) and use the expressions for  $k$ ,  $\omega$ ,  $c$ , and  $b$  given by Eqs. (14)–(17), to get after several algebraic operations that

$$\begin{aligned} & (a + F_0 + a^3)^3 [a + 2F_0 + x_{10} + a^2 \varepsilon x_{10} + a \varepsilon x_{10}^2] \{a^6 F_0 \varepsilon^2 + a^5 \varepsilon (4F_0 \varepsilon x_{10} + (1 + \varepsilon x_{10}^2)^2) \\ & - 5a^4 F_0 \varepsilon - 10a^3 F_0^2 \varepsilon + 5a^2 F_0 \varepsilon x_{10} (4F_0 + 2x_{10} + \varepsilon x_{10}^3) - a(2F_0^2 (1 + 8\varepsilon x_{10}^2) + (2 + \varepsilon x_{10}^2)) \\ & \times (x_{10} + \varepsilon x_{10}^3)^2 + 4F_0 (x_{10} + 4\varepsilon x_{10}^3 + 2\varepsilon^2 x_{10}^5)) - F_0 (2F_0^2 + 4F_0 x_{10} + 2x_{10}^2 + \varepsilon x_{10}^4)\} = 0. \end{aligned} \tag{18}$$

Notice that Eq. (18) is a twenty-third order polynomial equation for the constant  $a$ . However, to have real values for the modulus  $k$  and the frequency  $\omega$  of the Jacobi elliptic function, we only need to use the following sixth-order polynomial equation to determine the value of  $a$  i.e.:

$$\begin{aligned} & a^6 F_0 \varepsilon^2 + a^5 \varepsilon (4F_0 \varepsilon x_{10} + (1 + \varepsilon x_{10}^2)^2) - 5a^4 F_0 \varepsilon - 10a^3 F_0^2 \varepsilon + 5a^2 F_0 \varepsilon x_{10} (4F_0 \\ & + 2x_{10} + \varepsilon x_{10}^3) - a(2F_0^2 (1 + 8\varepsilon x_{10}^2) + (2 + \varepsilon x_{10}^2)(x_{10} + \varepsilon x_{10}^3)^2 + 4F_0 (x_{10} + 4\varepsilon x_{10}^3 \\ & + 2\varepsilon^2 x_{10}^5)) - F_0 (2F_0^2 + 4F_0 x_{10} + 2x_{10}^2 + \varepsilon x_{10}^4) = 0. \end{aligned} \tag{19}$$

According to our derived exact solution of Eq. (6), which has not been previously explored in the present context, it is evident that the higher elliptic terms in Eq. (8) have small amplitudes relative to the leading terms. In other words, the condition  $|b| > |a| > |c|$  is satisfied. The same conclusion holds for the exact solution of Eq. (4) [12]. These conditions agree well with those of the harmonic balance method [6]. Once the constants  $a$ ,  $b$ ,  $c$ ,  $k$ ,  $\omega$  are found by using Eqs. (14)–(19), we may compute the corresponding exact period of oscillation  $T$  of the Duffing oscillator (6) which is given by

$$T = \frac{4K(k^2)}{\omega}. \tag{20}$$

Since elliptic rational forms provide the exact solution to some nonlinear oscillators i.e., those given by Eqs. (4) and (6) [12,15], it is clear that by considering approximate solutions based on rational elliptic forms, we could get approximate expressions with a high degree of accuracy. In the next section, we shall investigate the approximate solution of two nonlinear oscillators by applying the rational elliptic balance method.

### 3. Approximate solutions of two nonlinear oscillators

In our study, we first derive the solution of a nonlinear singular oscillator that describes the path  $x$  of the electrons in plasma physics [16,17] and show how our proposed rational elliptic balance solution provides a high degree of accuracy when compared to the exact angular frequency value. Then, we explore the solution of a nonlinear oscillator in which the restoring force has a rational-like form.

#### 3.1. Nonlinear singular oscillator

Here, we obtain the approximate analytical solution of the following nonlinear singular oscillator

$$\ddot{x} + \frac{\varepsilon}{x} = 0, \tag{21}$$

where  $x$  describes the path of the electrons in plasma physics and the parameter  $\varepsilon$  in Eq. (21) does not need not to be small i.e.,  $0 < \varepsilon < \infty$ . Next, we assume that the approximate analytical solution to Eq. (21) is of the form

$$x(t) = \frac{a \operatorname{cn}(\omega_{22}t + \phi, k_{22}^2)}{\{1 + b \operatorname{cn}^2(\omega_{22}t + \phi, k_{22}^2)\}}, \tag{22}$$

where  $a$ ,  $b$ ,  $k_{22}$ ,  $\phi$ , and  $\omega_{22}$  are constants that need to be determined. Substituting Eq. (22) into Eq. (21), yields

$$\begin{aligned} & \varepsilon + \operatorname{cn}^2(\omega_{22}t + \phi, k_{22}^2) (4b\varepsilon - a^2 \omega_{22}^2 - 6a^2 b \omega_{22}^2 + 2a^2 k_{22}^2 \omega_{22}^2 + 6a^2 b k_{22}^2 \omega_{22}^2) \\ & + \operatorname{cn}^4(\omega_{22}t + \phi, k_{22}^2) (6b^2 \varepsilon + 6a^2 b \omega_{22}^2 + 2a^2 b^2 \omega_{22}^2 - 2a^2 k_{22}^2 \omega_{22}^2 - 12a^2 b k_{22}^2 \omega_{22}^2 \\ & - 2a^2 b^2 k_{22}^2 \omega_{22}^2) + b \operatorname{cn}^8(\omega_{22}t + \phi, k_{22}^2) (4b^2 \varepsilon - a^2 b \omega_{22}^2 + 6a^2 k_{22}^2 \omega_{22}^2 + 2a^2 b k_{22}^2 \omega_{22}^2) \\ & + b^4 \varepsilon \operatorname{cn}^8(\omega_{22}t + \phi, k_{22}^2) = 0. \end{aligned} \tag{23}$$

Using the transformation  $\cos \varphi_2 = \text{cn}(\omega_{22}t + \phi, k_{22}^2)$ , we can write Eq. (23) in the following form

$$\begin{aligned} & (128 + b(256 + b(288 + 5b(32 + 7b))))\varepsilon - 8a^2(8 + 12b - 7b^2 + 2k^2(b(b-3) - 2))\omega_{22}^2 \\ & + 4(2b(2 + b)(16 + b(16 + 7b))\varepsilon - a^2(16 - 17b^2 + 2b(3 + b)k_{22}^2)\omega_{22}^2) \cos 2\varphi_2 \\ & + 4(b^2(24 + b(24 + 7b))\varepsilon + 2a^2(b(12 + b) + 2(-2 + (-3 + b)b)k_{22}^2)\omega_{22}^2) \cos 4\varphi_2 \\ & + 4b(2b^2(2 + b)\varepsilon + a^2(-b + 2(3 + b)k_{22}^2)\omega_{22}^2) \cos 6\varphi_2 + b^4\varepsilon \cos 8\varphi_2 = 0. \end{aligned} \quad (24)$$

Setting the coefficients of the constant terms and the coefficients of  $\cos 2\varphi_2$  to zero provides the following expressions for  $k_{22}$  and  $\omega_{22}$ :

$$k_{22}^2 = \frac{2048 - 9856b^2 - 11264b^3 - 5616b^4 - 704b^5 + 189b^6}{2b(77b^5 - 121b^4 - 1664b^3 - 3872b^2 - 3968b - 1408)}, \quad (25)$$

$$\omega_{22}^2 = \frac{b(1408 + 3968b + 3872b^2 + 1664b^3 + 121b^4 - 77b^5)\varepsilon}{a^2(256 - 576b - 48b^2 - 480b^3 + 80b^4)}. \quad (26)$$

By considering the initial conditions given by Eq. (7), we have from Eq. (22) that  $\phi = 0$  and

$$b = \frac{a - x_{10}}{x_{10}}. \quad (27)$$

The remaining equation needed to determine the constant  $a$  of Eq. (22) is obtained by setting the coefficients of the term  $\cos 4\varphi$  to zero in Eq. (24). This yields:

$$\begin{aligned} & 7a^8 - 182a^7x_{10} + 2093a^6x_{10}^2 - 2864a^5x_{10}^3 + 53a^4x_{10}^4 - 3998a^3x_{10}^5 \\ & + 11651a^2x_{10}^6 - 5468ax_{10}^7 + 756x_{10}^8 = 0. \end{aligned} \quad (28)$$

This is an eighth-order polynomial equation that has the following roots:

$$\begin{aligned} & a = x_{10}(-0.9233 \pm 1.1529i); \quad a = x_{10}(0.2746 \pm 0.0767i); \\ & a = x_{10}(1.4373 \pm 0.5194i); \quad a = x_{10}(12.2114 \pm 10.5597i), \end{aligned} \quad (29)$$

where  $i = \sqrt{-1}$ . Since we are expecting real values for the constant parameters of Eq. (44), we now examine the coefficient of the harmonic term  $\cos 6\varphi_2$  and explore if  $a$  can have real values. Then, setting to zero the coefficients of  $\cos 6\varphi_2$  and by recalling Eqs. (A.5) and (A.7), we can get the following polynomial expression for the parameter  $a$  that depends on the initial condition  $x_{10}$ :

$$3a^7 - 60a^6x_{10} + 550a^5x_{10}^2 - 380a^4x_{10}^3 - 229a^3x_{10}^4 + 128a^2x_{10}^5 - 558ax_{10}^6 + 192x_{10}^7 = 0. \quad (30)$$

This polynomial equation (30) has the roots:

$$\begin{aligned} & a = x_{10}(9.5987 \pm 8.7306i); \quad a = x_{10}(0.068 \pm 0.8911i); \\ & a = 1.3759x_{10}; \quad a = -1.0415x_{10}; \quad a = 0.332x_{10}. \end{aligned} \quad (31)$$

We have three real roots for  $a$  but only the root  $a = 1.3759x_{10}$  satisfies the condition  $|a| > |b|$  [11]. By taking  $a = 1.3759x_{10}$ , we can compute from Eqs. (25)–(27) the values of the constant parameters  $b$ ,  $k_{22}$ , and  $\omega_{22}$ :

$$b = 0.3759x_{10}; \quad k_{22}^2 = 0.0226; \quad \omega_{22} = \frac{1.26079\sqrt{\varepsilon}}{x_{10}}. \quad (32)$$

With these parameter values, we next use Eq. (20) to compute the analytical approximate circular frequency  $\Omega_{22}$  from the following equation

$$\Omega_{22} = \frac{\pi \omega_{22}}{2K(k_{22}^2)} = \frac{1.25361\sqrt{\varepsilon}}{x_{10}}. \quad (33)$$

This frequency value is 0.0247% bigger than the exact angular frequency value

$$\omega_{ex}(x_{10}) = \frac{1.2533131\sqrt{\varepsilon}}{x_{10}} \quad (34)$$

determined by Mickens in [18]. The percentage error of 0.0247% is significantly lower than the error of 1.6% obtained by Ramos in [19], 1.275% obtained by Beléndez et al. in [20], or 0.4% obtained by Beléndez et al. in [21]. This result shows that our proposed rational Jacobi elliptic form (22) provides the best analytical estimate value when compared to the exact angular frequency given by Eq. (34). To further investigate on applicability of rational Jacobi elliptic forms to solve nonlinear differential equations, we shall next derive the analytical solution of a Duffing nonlinear oscillator.

### 3.2. Non-homogenous Duffing oscillator: first-order elliptic rational form solution

Here we consider the nonlinear Duffing oscillator of the form

$$\ddot{x} + \frac{\varepsilon x^3}{(Ax^2 + B)} + C = 0 \tag{35}$$

and use the rational Jacobi elliptic form (8) to find its approximate solution. In this case,  $x$  represents motion displacement and  $\varepsilon$ ,  $A$ ,  $B$ , and  $C$  are system constant parameters. Physical applications as well as approximate solutions of Eq. (35) when  $C \equiv 0$  by using several techniques may be found in [22,23] and works cited therein.

For the non-homogeneous nonlinear oscillator considered in this study (35) and in accordance with the rational elliptic balance method, we assume that its solution is given by Eq. (8). Then, substitution of Eq. (8) into Eq. (35) provides the following expression:

$$\begin{aligned} & a^2AC + BC + a^3\varepsilon - 2(a^2A + B)c(ac - b)(k^2 - 1)\omega^2 + \text{cn}(\omega t + \phi, k^2)(2aAbC \\ & + 3a^2AcC + 5BCc + 3a^2b\varepsilon + 2a^3c\varepsilon + (b - ac)(4abcA(k^2 - 1) + a^2A(2k^2 - 1) \\ & + B(2k^2 - 1 + 4c^2(k^2 - 1)))\omega^2) + \text{cn}^2(\omega t + \phi, k^2)(Ab^2C + 6abcAC + 3a^2c^2AC \\ & + 10Bc^2C + 3ab^2\varepsilon + 6a^2bc\varepsilon + a^3c^2\varepsilon + (b - ac)(-B(c + 2c^3) + 2B(c + c^3)k^2 \\ & + a^2cA(1 - 2k^2) + 2Ab^2c(k^2 - 1) + 2aAb(2k^2 - 1))\omega^2) + \text{cn}^3(\omega t + \phi, k^2)(3Ab^2cC \\ & + 6abc^2AC + a^2c^3AC + 10Bc^3C + b^3\varepsilon + 6ab^2c\varepsilon + 3a^2bc^2\varepsilon + (b - ac)(-Ab^2 \\ & + 2aAbc + Bc^2 - 2(a^2A - Ab^2 + B + 2aAbc + Bc^2)k^2)\omega^2) + \text{cn}^4(\omega t + \phi, k^2) \\ & \times (c(5Bc^3C + Abc(3b + 2ac)C + b^2(2b + 3ac)\varepsilon) - (b - ac)(-Bc^3 + 2Bc(2 + c^2)k^2 \\ & + Ab(4ak^2 + bc(2k^2 - 1)))\omega^2) + \text{cn}^5(\omega t + \phi, k^2)(c^2(Ab^2cC \\ & + Bc^3C + b^3\varepsilon) - 2(b - ac)(Ab^2 + Bc^2)k^2\omega^2) = 0. \end{aligned} \tag{36}$$

As usual, we use the Jacobi amplitude function  $\varphi$  of argument  $\omega t + \phi$ ,  $\varphi = \text{am}(\omega_1 t + \phi, k^2)$  so that  $\cos \varphi = \text{cn}(\omega t + \phi, k^2)$ . Thus, substituting the Jacobi amplitude function  $\varphi$  into Eq. (36) leads to

$$\begin{aligned} & 2(8a^2AC + 4Ab^2C + 8BC + 24aAbcC + 12a^2Ac^2C + 9Ab^2c^2C + 40Bc^2C \\ & + 6aAbc^3C + 15Bc^4C + 8a^3\varepsilon + 12ab^2\varepsilon + 24a^2bc\varepsilon + 6b^3c\varepsilon \\ & + 4a^3c^2\varepsilon + 9ab^2c^2\varepsilon + (b - ac)(4aAb(k^2 - 2) + Ab^2c(2k^2 - 5) + 4a^2Ac(2k^2 - 3) \\ & + Bc(2(6 + c^2)k^2 - 5(4 + c^2)))\omega^2) + \cos \varphi(2(16aAbC \\ & + 24a^2AcC + 18Ab^2cC + 40BcC + 36aAbc^2C + 6a^2Ac^3C + 5Ab^2c^3C \\ & + 60Bc^3C + 5Bc^5C + 24a^2b\varepsilon + 6b^3\varepsilon + 16a^3c\varepsilon + 36ab^2c\varepsilon + 18a^2bc^2\varepsilon \\ & + 5b^3c^2\varepsilon + 2(b - ac)(Ab^2(k^2 - 3) + 2a^2A(k^2 - 2) + 2aAbc(2k^2 - 5) \\ & + B(-4 - 13c^2 + (2 + 5c^2)k^2))\omega^2) + \cos 2\varphi(8(A(b^2 + 6abc + 3(a^2 + b^2)c^2 + 2abc^3)C \\ & + (6a^2bc + 2b^3c + a^3c^2 + 3ab^2(1 + c^2))\varepsilon + A(ac - b)(2ab + b^2c + a^2c(2k^2 - 1)) \\ & \times \omega^2 + Bc(5d(2 + c^2)C + (ac - b)(1 + c^2 + 2k^2)\omega^2)) \\ & + \cos 3\varphi(5Bc^3(8 + c^2)C + Ac(24abc + 4a^2c^2 + b^2(12 + 5c^2))C \\ & + 4b^3\varepsilon + bc(24ab + 12a^2c + 5b^2c)\varepsilon - 2A(b - ac)(2b(b - 2ac) \\ & + (b^2 + 4a(a + 2bc))k^2)\omega^2 - 2B(b - ac)(4k^2 + c^2(9k^2 - 2))\omega^2) \\ & + \cos 4\varphi(2(c(5Bc^3C + Abc(3b + 2ac)C + b^2(2b + 3ac)\varepsilon) - (b - ac) \\ & \times (-Bc^3 + 2Bc(2 + c^2)k^2 + Ab(4ak^2 + bc(2k^2 - 1)))\omega^2) + \cos 5\varphi(c^2(Ab^2cC \\ & + Bc^3C + b^3\varepsilon) - 2(b - ac)(Ab^2 + Bc^2)k^2\omega^2) = 0. \end{aligned} \tag{37}$$

By using the initial conditions (7) and by setting the coefficients of the constant term,  $\cos \varphi$ ,  $\cos 2\varphi$  and  $\cos 3\varphi$  equal to zero in Eq. (37), we obtain the relations that are needed to determine the parameters  $a$ ,  $b$ ,  $c$ ,  $k$ ,  $\omega$  and  $\varphi$  of Eq. (8):

$$\begin{aligned} & 2(8a^2AC + 4Ab^2C + 8BC + 24aAbcC + 12a^2Ac^2C + 9Ab^2c^2C \\ & + 40Bc^2C + 6aAbc^3C + 15Bc^4C + 8a^3\varepsilon + 12ab^2\varepsilon + 24a^2bc\varepsilon + 6b^3c\varepsilon \\ & + 4a^3c^2\varepsilon + 9ab^2c^2\varepsilon + (b - ac)(4aAb(k^2 - 2) + Ab^2c(2k^2 - 5) \\ & + 4a^2Ac(2k^2 - 3) + Bc(2(6 + c^2)k^2 - 5(4 + c^2)))\omega^2) = 0; \end{aligned} \tag{38}$$

$$\begin{aligned} & 2(16aAbC + 24a^2AcC + 18Ab^2cC + 40BcC + 36aAbc^2C + 6a^2Ac^3C \\ & + 5Ab^2c^3C + 60Bc^3C + 5Bc^5C + 24a^2b\varepsilon + 6b^3\varepsilon + 16a^3c\varepsilon + 36ab^2c\varepsilon \\ & + 18a^2bc^2\varepsilon + 5b^3c^2\varepsilon + 2(b - ac)(Ab^2(k^2 - 3) + 2a^2A(k^2 - 2) \end{aligned}$$

$$\begin{aligned}
 &+ 2aAbc(2k^2 - 5) + B(-4 - 13c^2 + (2 + 5c^2)k^2)\omega^2 = 0; \tag{39} \\
 &8(A(b^2 + 6abc + 3(a^2 + b^2)c^2 + 2abc^3)C + (6a^2bc + 2b^3c + a^3c^2 \\
 &+ 3ab^2(1 + c^2))\varepsilon + A(ac - b)(2ab + b^2c + a^2c(2k^2 - 1))\omega^2
 \end{aligned}$$

$$\begin{aligned}
 &+ B(5d(2 + c^2)C + (ac - b)(1 + c^2 + 2k^2)\omega^2) = 0; \tag{40} \\
 &5(Bc^3(8 + c^2)C + Ac(24abc + 4a^2c^2 + b^2(12 + 5c^2))C + 4b^3\varepsilon
 \end{aligned}$$

$$\begin{aligned}
 &+ bc(24ab + 12a^2c + 5b^2c)\varepsilon - 2A(b - ac)(2b(b - 2ac) \\
 &+ (b^2 + 4a(a + 2bc))k^2)\omega^2 - 2B(b - ac)(4k^2 + c^2(9k^2 - 2))\omega^2 = 0. \tag{41}
 \end{aligned}$$

Also, from Eqs. (7) and (8), we have that

$$\phi \equiv 0 \quad \text{and} \quad b \equiv x_{10}(1 + c) - a. \tag{42}$$

Then, using Eqs. (38) and (39) and solving for  $k$  and  $\omega$ , yields the following expressions

$$k = \sqrt{\frac{H_1}{H_2}}; \quad \omega = \sqrt{\frac{2H_2}{H_3}}, \tag{43}$$

where the expression of  $H_1$ ,  $H_2$ , and  $H_3$  are given in Appendix.

To find the constants  $a$  and  $c$ , we substitute the expression of  $k$  and  $\omega$  given by Eq. (43) into Eqs. (40) and (41). Hence, for each choice of  $\varepsilon$ ,  $A$ ,  $B$  and  $C$ , the constants  $a$  and  $c$  can be found by numerical solution of Eqs. (40) and (41). During the numerical solution processes, it is important to bear in mind that the rational elliptic balance procedure requires to satisfy the following condition  $|b| > |a| > |c|$  in order to have periodic response.

To further investigate alternative rational elliptic expressions to improve the accuracy of approximate solutions to Eq. (35), we study in the next section a second-order rational elliptic form solution.

### 3.3. Non-homogeneous Duffing oscillator: second-order elliptic rational form solution

We now seek the approximate solution of Eq. (35) by using the following second-order rational elliptic form

$$x(t) = \frac{a + bcn(\omega_2t + \phi, k_2^2)}{\{1 + ccn^2(\omega_2t + \phi, k_2^2)\}}, \tag{44}$$

where  $a$ ,  $b$ ,  $c$ ,  $k_2$ ,  $\phi$ , and  $\omega_2$  are constants that need to be determined. According to the rational elliptic balance method, we substitute Eq. (44) into Eq. (35), this yields

$$\begin{aligned}
 &a^2AC + BC + a^3\varepsilon + 2a(a^2A + B)c(k_2^2 - 1)\omega_2^2 + bcn(\omega_2t + \phi, k_2^2)(2aAC \\
 &+ B(2k_2^2 - 1 + 6c(k_2^2 - 1))\omega_2^2 + a^2(3\varepsilon + A(-1 - 10c + 2(1 + 5c)k_2^2)\omega_2^2)) \\
 &+ cn^2(\omega_2t + \phi, k_2^2)(5BcC + A(b^2 + 3a^2c)C + 3ab^2\varepsilon + 2a^3c\varepsilon + 2aBc(2 + c \\
 &- (4 + c)k_2^2)\omega_2^2 - 2aA(a^2c(-2 - 3c + (4 + 3c)k_2^2) + b^2(1 + 7c - (2 + 7c)k_2^2))\omega_2^2) \\
 &+ bcn^3(\omega_2t + \phi, k_2^2)(b^2\varepsilon + 6ac(AC + a\varepsilon) - (2B(c(5c - 2) + (1 + (4 - 5c)c)k_2^2) \\
 &+ A(b^2(1 + 6c - 2(1 + 3c)k_2^2) + 2a^2(-7c(1 + c) + (1 + 7c(2 + c))k_2^2)))\omega_2^2) \\
 &+ cn^4(\omega_2t + \phi, k_2^2)(c(10BcC + 3A(b^2 + a^2c)C + 6ab^2\varepsilon + a^3c\varepsilon) + 2a(Ac(-2a^2c \\
 &+ b^2(8 + 5c)) + A(a^2c(3 + 4c) - b^2(2 + c(16 + 5c)))k_2^2 + Bc(c(2 + 5c) + (3 \\
 &- c(4 + 5c)k_2^2))\omega_2^2) + bcn^5(\omega_2t + \phi, k_2^2)(c(2b^2\varepsilon + 3ac(2AC + a\varepsilon)) + (Ac(-9a^2c \\
 &+ 2b^2(3 + c)) - 2A(-9a^2c(1 + c) + b^2(1 + c(6 + c)))k_2^2 + 2Bc(k_2^2 + c(5 - c \\
 &+ (c - 10)k_2^2)))\omega_2^2) + ccn^6(\omega_2t + \phi, k_2^2)(c(10cBC + A(3b^2 + a^2c)C + 3ab^2\varepsilon) \\
 &+ 2a(-3Ab^2c + A(-a^2c + b^2(9 + 6c))k_2^2 + Bc(c(-2 + 3c) + (5 + (4 - 3c)c)k_2^2))\omega_2^2) \\
 &+ bccn^7(\omega_2t + \phi, k_2^2)(c(2acAC + b^2\varepsilon) + (-Ab^2c + 2A(-2a^2c + b^2(3 + c))k_2^2 \\
 &+ 2Bc(c(2 + c) - (c - 1)(5 + c)k_2^2))\omega_2^2) + c^2cn^8(\omega_2t + \phi, k_2^2)(Ab^2(cC - 2ak_2^2\omega_2^2) \\
 &+ Bc(5cC + 2a(k_2^2 + c(4k_2^2 - 2))\omega_2^2)) + bc^3B\omega_2^2cn^9(\omega_2t + \phi, k_2^2)(2(3 + c)k_2^2 - c) \\
 &+ Bc^4cn^{10}(\omega_2t + \phi, k_2^2)(cC - 2ak_2^2\omega_2^2) = 0. \tag{45}
 \end{aligned}$$

Then we apply the transformation  $\cos \varphi_2 = cn(\omega_2t + \phi, k_2^2)$  to Eq. (45) and get after using trigonometric identities that

$$\begin{aligned}
 &2(16a^2A(2+c)(8+c(8+5c))C + (2Ab^2(64+c(144+5c(24+7c))) \\
 &+ B(2+c)(128+c(352+7c(32+9c))))C + 48ab^2(8+c(12+5c))\varepsilon \\
 &+ 2a(2Ab^2(8(-8+c(15c-8)) + (32+5(8-7c)c)k_2^2) + Bc(20c(16+c(15+5c)) \\
 &- (-32+c(112+c(90+23)))k_2^2))\omega_2^2 + 32a^3((8+c(8+3c))\varepsilon + Ac(12c+2k_2^2 \\
 &- 5ck_2^2)\omega_2^2) - 4b \cos \varphi_2(-4aA(64+c(144+5c(24+7c)))C - 48a^2(8+c(12 \\
 &+ 5c))\varepsilon - 2b^2(48+5c(16+7c))\varepsilon + (2Ab^2(48+48c-45c^2+2(-8+c(5c-9)))k_2^2) \\
 &+ 8a^2A(16-2c(4+39c) + (-8+c(23c-4)))k_2^2) + B(128+384c+160c^2-120c^3 \\
 &- 77c^4+c(-32+c(-80+c(3+c)(7c-10)))k_2^2))\omega_2^2 + 2 \cos 2\varphi_2(8a(16a^2c(2+c) \\
 &+ 3b^2(4+3c)(4+5c))\varepsilon + Bc(5(128+c(256+c(240+7c(16+3c))))C + 2a(8(2 \\
 &+ c)(16+c(16+17c)) - (128+c(40+c(48+17c)))k_2^2)\omega_2^2) + 8A(a^2c(3(16 \\
 &+ c(16+5c))C + 2a(16(2+c) + (c-16)k_2^2)\omega_2^2) + b^2((16+c(48+c(45+14c)))C \\
 &- 2a(16+c(-16-35c+(9+4c)k_2^2))\omega_2^2)) + 8b \cos 3\varphi_2(6acA(16+c(20+7c))C \\
 &+ B(c(64+c(40+c(44+21c))) - 2(16+c(4+c)(11+c))k_2^2)\omega_2^2 + 4a^2(3c(8 \\
 &+ 5c)\varepsilon + A(c(56+11c) + (c-2)(4+13c)k_2^2)\omega_2^2) + b^2((4+3c)(4+7c)\varepsilon + A(-8(2 \\
 &+ k^2) + c(24+19c+c(c-9)k_2^2))\omega_2^2) + 8 \cos 4\varphi_2(2Ac(6a^2c(2+c) + b^2(12 \\
 &+ c(18+7c)))C + 4aA(2a^2c(-4c+(6+5c)k_2^2) + b^2(2c(16+c) + (-8+c(9c \\
 &- 10))k_2^2))\omega_2^2 + c(5Bc(2+c)(8+c(8+3c))C + 4A(2a^2c+3b^2(4+3c))\varepsilon \\
 &+ 2aB(8c(2+c(2+c)) + (24+c(22+5c)))k_2^2)\omega_2^2) + 8b \cos 5\varphi_2(2aAc^2(12 \\
 &+ 7c)C + Bc(5c(8+c(4+c)) + 2(4+(c-1)c(5+2c))k_2^2)\omega_2^2 + 4a^2c(3c\varepsilon \\
 &+ A(-9c(18+11c)k_2^2)\omega_2^2) + b^2(c(8+7c)\varepsilon + A(c(24+c) + 2(-4+3(c \\
 &- 1)c)k_2^2)\omega_2^2) + c \cos 6\varphi_2(c(16A(a^2c+b^2(3+2c)) + 5Bc(32+c(32+9c)))C \\
 &+ 48ab^2c\varepsilon + 2a(-16c(3Ab^2+Bc(2+c)) + (16A(-a^2c+b^2(9+4c)) \\
 &+ Bc(80+c(96+35c)))k_2^2)\omega_2^2) + 2bc \cos 7\varphi_2(4c(2acAC + b^2\varepsilon) \\
 &+ (-c(4Ab^2+B(c-16)c) + 2(4A(-2a^2c+b^2(3+c)) + Bc(20+c(11 \\
 &+ 5c)))k_2^2)\omega_2^2) + 2c^2 \cos 8\varphi_2(2Ab^2(cC - 2ak_2^2)\omega_2^2) + Bc(5c(2+c)C \\
 &+ 2a(-4c(2+3c)k_2^2)\omega_2^2) + 2bBc^3\omega_2^2 \cos 9\varphi_2(-c+2(3+c)k_2^2) + Bc^4(cC - 2ak_2^2)\omega_2^2 \cos 10\varphi_2 = 0. \tag{46}
 \end{aligned}$$

Next we set in Eq. (46) the coefficients of the two lowest harmonic terms equal to zero to obtain the following expressions for  $k_2$  and  $\omega_2$ :

$$k_2 = \sqrt{\frac{H_4}{2H_5}}; \quad \omega_2 = \sqrt{\frac{H_6}{H_7}}, \tag{47}$$

where the expressions of  $H_4, H_5, H_6$  and  $H_7$  are given in Appendix.

From the initial conditions (7), we have from Eq. (44) that  $\phi = 0$  and

$$b = x_{10}(1+c) - a. \tag{48}$$

The remaining equations needed to determine the constants  $a$  and  $c$  of Eq. (44) are obtained by setting the coefficients of the harmonic terms  $\cos 2\varphi$  and  $\cos 3\varphi$  equal to zero. This provides the following two equations:

$$\begin{aligned}
 &8a(16a^2c(2+c) + 3b^2(4+3c)(4+5c))\varepsilon + Bc(5(128+c(256+c(240+7c(16 \\
 &+ 3c))))C + 2a(8(2+c)(16+c(16+17c)) - (128+c(40+c(48+17c)))k_2^2)\omega_2^2) \\
 &+ 8A(a^2c(3(16+c(16+5c))C + 2a(16(2+c) + (c-16)k_2^2)\omega_2^2) + b^2((16+c(48 \\
 &+ c(45+14c)))C - 2a(16+c(-16-35c+(9+4c)k_2^2))\omega_2^2)) = 0, \tag{49}
 \end{aligned}$$

$$\begin{aligned}
 &6acA(16+c(20+7c))C + B(c(64+c(40+c(44+21c))) - 2(16+c(4+c)(11 \\
 &+ c))k_2^2)\omega_2^2 + 4a^2(3c(8+5c)\varepsilon + A(c(56+11c) + (c-2)(4+13c)k_2^2)\omega_2^2) \\
 &+ b^2((4+3c)(4+7c)\varepsilon + A(-8(2+k_2^2) + c(24+19c+c(c-9)k_2^2))\omega_2^2) = 0. \tag{50}
 \end{aligned}$$

Then, substitution of the expressions of  $k_2, \omega_2$ , and  $b$  given by Eqs. (47) and (48) respectively, into Eqs. (49) and (50) and by numerically solving these equations, we obtain the values of  $a$  and  $c$  that satisfy the condition  $|b| > |a| > |c|$  [6].

## 4. Results

To assess the accuracy of our rational Jacobi elliptic solution forms used to determine analytical solutions of the Duffing nonlinear oscillator given by Eq. (35), we first derive the second-order harmonic balance solution of Eq. (35) and then, we determine the exact angular frequency value from energy considerations.

### 4.1. Second-order harmonic balance solution

Here, we derive the approximate solution of Eq. (35) by applying the rational harmonic balance method by assuming a second-order solution of the form [7,9]:

$$x(t) = \frac{a_B + b_B \cos(\omega_B t + \phi)}{1 + c_B \cos^2(\omega_B t + \phi)} \quad (51)$$

where  $a_B$ ,  $b_B$ ,  $c_B$ , and  $\omega_B$  need to be determined. Substituting Eq. (51) into Eq. (35), expanding the resulting expression in a trigonometric series and by setting the constant terms and the coefficients of  $\cos \omega_B t$  and  $\cos 2\omega_B t$  to zero, respectively, yields the following equations:

$$\begin{aligned} &16a_B^2 A(2 + c_B)(8 + c_B(8 + 5c_B))D_1 + (2Ab_B^2(64 + c_B(144 + 5c_B(24 + 7c_B))) \\ &+ B(2 + c_B)(128 + c_B(256 + c_B(352 + 7c_B(32 + 9c_B))))D_1 + 48a_B b_B^2(8 \\ &+ c_B(12 + 5c_B))\varepsilon + 32a_B(20Bc_B^2(1 + c_B) + Ab_B^2(-8 + c_B(15c_B - 8)))\omega_B^2 \\ &+ 32a^3((8 + c_B(8 + 3c_B))\varepsilon + 12Ac_B^2\omega_B^2) = 0, \end{aligned} \quad (52)$$

$$\begin{aligned} &2a_B A(64 + c_B(144 + 5c_B(24 + 7c_B)))D_1 + 4B(-16 + c_B(-48 \\ &+ 5c_B(3c_B - 4)))\omega_B^2 + b_B^2((48 + 5c_B(16 + 7c_B))\varepsilon + 3A(-16 + c_B(15c_B \\ &- 16))\omega_B^2) + 8a_B^2(3(8 + c_B(12 + 5c_B))\varepsilon + A(-8 + c_B(4 + 39c_B))\omega_B^2) = 0, \end{aligned} \quad (53)$$

$$\begin{aligned} &8a_B(16a_B^2 c_B(2 + c_B) + 3b_B^2(4 + 3c_B)(4 + 5c_B))\varepsilon + Bc_B(5(128 + c_B(256 + c_B(240 \\ &+ 7c_B(16 + 3c_B))))D_1 + 16a_B(2 + c_B)(16 + c_B(16 + 17c_B))\omega_B^2 + 8A(a_B^2 c_B(3(16 \\ &+ c_B(16 + 5c_B))D_1 + 32a_B(2 + c_B)\omega_B^2) + b_B^2((16 + c_B(48 + c_B(45 + 14c_B)))D_1 \\ &+ 2a_B(-16 + c_B(16 + 35c_B))\omega_B^2) = 0. \end{aligned} \quad (54)$$

By taking into account the initial conditions given by Eq. (7), we get:

$$b_B = x_{10}(1 + c_B) - a_B; \quad \phi = 0. \quad (55)$$

After substitution of Eq. (55) into Eqs. (52)–(54), we can numerically obtain the values of  $a_B$ ,  $b_B$ ,  $c_B$ , and  $\omega_B$  that satisfy the condition  $|b_B| > |a_B| > |c_B|$  [6].

### 4.2. Exact angular frequency

To find the exact angular frequency of Eq. (35), we follow Radhakrishnan et al. procedure [24] to get from Eq. (35) that

$$\int_{x=x_i}^{x=x_0} \frac{dx}{\sqrt{I(x_0) - I(x)}} = \int_{t=T/2}^{t=T} dt = \frac{T}{2} = \frac{\pi}{\omega_{ex}}, \quad (56)$$

where  $\omega_{ex}$  is the exact angular frequency of the nonlinear oscillator and

$$I(x) = 2\varepsilon \left( \frac{x^2}{2A} - \frac{B \ln(b + ax^2)}{2A^2} \right) + 2Cx, \quad (57)$$

where the value of  $x_i$  is determined by solving the following equation

$$\dot{x}^2 = I(x_0) - I(x) \equiv 0. \quad (58)$$

To compute the approximate analytical angular frequencies' values for the first and second-order solutions given by Eqs. (8) and (44) respectively, we recall that the Jacobi elliptic function  $\text{cn}(\omega t, k^2)$  has a period in  $\omega t$  equal to  $4K(k^2)$  and thus, the approximate period of oscillation of Eq. (35) can be determined by using Eq. (20). Thus, the corresponding circular frequency  $\Omega$  of the first-order solution of Eq. (35) can be computed from

$$\Omega = \frac{\pi \omega}{2K(k^2)}. \quad (59)$$

Similarly, the second-order approximate angular frequency of Eq. (35) is given by

$$\Omega_2 = \frac{\pi \omega_2}{2K(k_2^2)}, \quad (60)$$

where  $k_2$  and  $\omega_2$  are determined from Eq. (46).



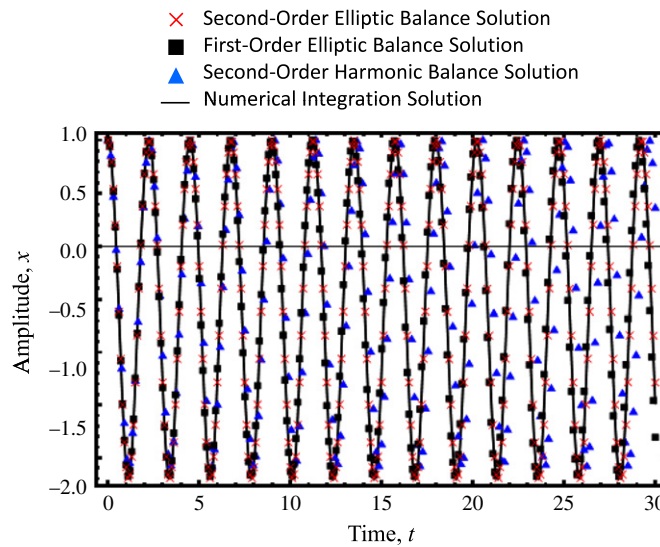


Fig. 1. Amplitude-time response curves of a Duffing oscillator for parameter values of  $A = 1, B = 0.5, C_1 = 5, \varepsilon = 10, x_{10} = 1$ .

Table 1

Comparison of the approximate and exact angular frequency values.

$x_{10}$	$A$	$B$	$C$	$\varepsilon$	Exact value $\omega_{ex}$	Harmonic balance second order solution $\omega_B$	Elliptic first order solution $\Omega$	Elliptic second order solution $\Omega_2$	MHPM $\Omega_H$	% Error harmonic balance second order solution	% Error elliptic first order solution	% Error elliptic second order solution	% Error MHPM
1	1	1	1	1	0.8978	0.8938	0.9006	0.8977	0.8979	0.4451	0.3161	0.0103	0.0111
1	1	0.5	5	10	2.7971	2.7533	2.8061	2.7925	2.799	1.5657	0.3208	0.1634	1.6598
5	1	1	3	3	1.6916	1.6931	1.6955	1.6953	1.6919	0.0880	0.2298	0.2161	0.0709
10	-1	-1	-1	-1	0.9888	0.9900	0.9903	0.9903	0.9889	0.1164	0.1450	0.1425	0.1111
25	0.01	1	3	1	8.9206	8.8371	8.9530	8.9204	8.9423	0.9342	0.3635	0.0020	1.1904
100	0.25	5	2	1	1.9962	1.9968	1.9968	1.9968	1.9962	0.0300	0.0300	0.0300	0.0000

### 5. Simulations

In this section, we compare the exact angular frequency values obtained by integrating Eq. (56) with the approximate angular frequencies  $\omega_B, \Omega, \Omega_2$  computed from our derived elliptic and harmonic solutions and with the angular frequency value  $\Omega_H$  obtained by following Hashim and Chowdhury Multistage Homotopy-Perturbation Method (MHPM) [25].

As we may see from Table 1 and for the assigned parameter values shown there, our expressions for the approximate angular frequencies computed by the elliptic balance procedure compare favorably to the exact value. In fact, under these conditions the percentage error of the first-order rational elliptic angular frequency solution  $\Omega$  when compared to the exact angular frequency values  $\omega_{ex}$  is less than 0.36%, while the second-order rational elliptic angular frequency  $\Omega_2$  solution has smaller percentage errors that do not exceed 0.22%, compared to the 1.56% error obtained from the second-order harmonic balance solution or from the 1.65% error predicted from the Multistage Homotopy-Perturbation approximate solution. Fig. 1 shows the comparison of the elliptic, harmonic and numerical solutions for parameter values of  $A = 1, B = 0.5, C_1 = 5, \varepsilon = 10, x_{10} = 1$ . We may see from Fig. 1 that the elliptic balance solutions follow very closely the numerical solution while the second-order harmonic balance solution starts to deviate from the numerical one at values of  $t \geq 10$ . In general, and for the parameter values show in Table 1, we may conclude that the derived elliptic balance solutions are more accurate than those of the second-order harmonic balance solution and compare favorably with the predicted values obtained from the Multistage Homotopy-Perturbation solution. Also, we may see from Table 1 that for larger values of  $x_{10}$  the elliptic, harmonic and MHPM approximate angular frequency values are almost the same. As expected, the condition  $|b| > |a| > |c|$  for the elliptic and harmonic solutions is satisfied in all cases studied here as shown in Table 2.

### 6. Discussion

In this paper, we have introduced a new approach based on rational elliptic forms to obtain analytical approximate solutions to strong nonlinear oscillators described by Eqs. (21) and (35). The main motivation for the use of rational elliptic forms to seek the solution of nonlinear oscillators comes from the fact that (a) the quadratic mixed-parity Helmholtz–Duffing

**Table 2**

Parameter values of the proposed rational elliptic form solutions.

$x_1^0$	A	B	C	$\varepsilon$	Second-order harmonic balance			First-order rational elliptic form			Second-order rational elliptic form		
					a	b	c	a	b	c	a	b	c
1	1	1	1	1	-1.0167	1.9257	-0.0909	-0.9653	2.0740	0.1086	-0.9554	1.7500	-0.2053
1	1	0.5	5	10	-0.4795	1.3643	-0.1152	-0.4692	1.5440	0.0748	-0.4515	1.2064	-0.2451
5	1	1	3	3	-0.9730	5.9095	-0.0126	-0.9787	5.9996	0.0041	-0.9783	5.8404	-0.0275
10	-1	-1	-1	-1	0.9870	8.9592	-0.0053	0.9887	9.0013	-0.0009	0.9887	8.9179	-0.0093
25	0.01	1	3	1	-0.0275	22.6179	-0.0963	-0.0278	25.0329	0.0001	-0.0266	19.3273	-0.2279
100	0.25	5	2	1	-0.4989	100.415	-0.0008	-0.4989	100.5	0.0000	-0.4989	100.389	-0.0010

oscillator [12] and (b) the undamped nonhomogeneous Duffing equation (6) have exact solutions described by rational elliptic forms. For instance, if one wants to seek the solution of a nonlinear oscillator of the form (1) that satisfies the initial conditions given by Eq. (2), a first-order rational expression of the form:

$$x(t) = \frac{a + bc n(\omega t + \phi, k^2)}{1 + cc n(\omega t + \phi, k^2)}, \quad (61)$$

could be used. To obtain the second-order approximate solution of Eq. (1), we can use the rational elliptic form

$$x(t) = \frac{a + bc n(\omega t + \phi, k^2)}{1 + cc n^2(\omega t + \phi, k^2)}, \quad (62)$$

or the following equation

$$x(t) = \frac{acn(\omega_{22}t + \phi, k_{22}^2)}{\{1 + bc n^2(\omega_{22}t + \phi, k_{22}^2)\}}, \quad (63)$$

to obtain the analytical solution of a singular oscillator described by Eq. (21). Of course, the condition  $|b| > |a| > |c|$  for Eqs. (61) and (62), and  $|a| > |b|$  in Eq. (63) must be satisfied to have periodic response solutions [6]. Also, notice from Eqs. (24), (37) and (46) that these rational elliptic forms provide information on all the harmonics since Jacobi elliptic functions are a generalization of the trigonometric ones. Furthermore, our derived first-order elliptic balance approximate solution (8) predicts well not only the solution of Eq. (35) but also it captures the exact solution in the case when  $A = 0$  since this Eq. (35) reduces to the non-homogeneous undamped, Duffing equation (8). In this case, the approximate second-order harmonic balance solution given by Eq. (51) fails to provide the exact solution of Eq. (8).

## 7. Conclusions

A rational elliptic balance method was adapted to obtain exact and approximate solutions of nonlinear oscillators. In reference to angular frequency values, when comparing our approximate rational elliptic balance solution results with both the numerical integration and the Multistage Homotopy Perturbation Method, we found good agreement in most cases.

It should be noted that the proposed rational elliptic balance method has some limitations. The inappropriate application of this method can lead to large errors in the solution. The correct form of the rational solution should be used, as described by Mickens in [26]. Future work is focused on applying rational elliptic forms to investigate approximate solutions of forced, damped nonlinear systems with one or more degrees of freedom with preliminary and encouraging results.

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## Appendix

The expressions of  $H_1$  through  $H_6$  to compute  $k$ ,  $\omega$ ,  $k_2$ , and  $\omega_2$  values in Eqs. (43) and (47) are given by:

$$\begin{aligned} H_1 = & A^2b^4(36c^2 + 25c^4 - 24)C + 2Ab^2B(72c^2 + 83c^4 + 25c^6 - 40)C \\ & + 32a^5A(5c^2 - 2)\varepsilon + Ab^5c(25c^2 - 6)\varepsilon + 8a^4A(A(24c^2 + 9c^4 - 8)C \\ & + bc(8 + 17c^2)\varepsilon) + 8a^3(4A^2bc(1 + 6c^2)C + 2Ab^2(3 + 5c^2)\varepsilon + B(10c^2 \\ & - 3c^4 - 8)\varepsilon) + B(B(272c^2 + 240c^4 + 10c^6 + 25d^8 - 64)C + b^3c(72 \\ & - 26c^2 + 25c^4)\varepsilon) + 2a^2(A(3Ab^2(8 - 5c^4) + B(168c^2 + 264c^4 + 45c^6 \\ & - 64))C + 3bc(48B + (5c^2 - 8)(3Bc^2 - Ab^2))\varepsilon) + 2ab(2Ac(Ab^2c^2 + B(72 \\ & - 48c^2 - 59c^4))C + b(Ab^2(23c^2 - 12) - 3B(16 - 56c^2 + 9c^4))\varepsilon), \end{aligned} \quad (A.1)$$

$$\begin{aligned}
 H_2 = & 2(2Ab^2B(5c^2 - 1)(8 + 5c^2 + c^4)C + A^2b^4(9c^2 + 5c^4 - 4)C \\
 & + 5Ab^5d^3\varepsilon + 8Aa^5(7c^2 - 2)\varepsilon + 8a^4A(9c^2 + 3c^4 - 2)C + bc(6 + 7c^2)\varepsilon) \\
 & + 2a^3(16A^2bd(1 + 3c^2)C + Ab^2(8 + 39c^2)\varepsilon - 2B(4 - 12c^2 + c^4)\varepsilon) \\
 & + B(B(5c^2(24 + 34c^2 + 3c^4 + c^6) - 16)C + b^3c(24 + 6c^2 + 5c^4)\varepsilon) \\
 & + ab(4Ac(Ab^2(3 + c^2) + B(24 + 15c^2 - 11c^4))C + b(13Ab^2c^2 \\
 & - 3B(8 - 46c^2 + 3c^4))\varepsilon) + 2a^2(A(Ab^2(8 + 21c^2 + c^4) + B(80c^2 \\
 & + 105c^4 + 13c^6 - 16))C + bc(Ab^2(18 + c^2) + 3B(16 + 2c^2 + 3c^4))\varepsilon)),
 \end{aligned} \tag{A.2}$$

$$\begin{aligned}
 H_3 = & (b - ac)(8a^4A^2c + 32a^3A^2bc^2 + 4aAb(Ab^2 + 13Bc^2) + 2a^2Ac(9Ab^2 \\
 & + B(8 + 21c^2))) + c(A^2b^4 + 2Ab^2B(7 + c^2) + B^2(8 + 54c^2 + c^4)).
 \end{aligned} \tag{A.3}$$

$$\begin{aligned}
 H_4 = & (256a^4A^2(128 + c(128 + c(-192 + c(-104 + c(15c - 2))))))C - (2Ab^2(64 + c(144 \\
 & + 5c(24 + 7c))) + B(2 + c)(128 + c(256 + c(352 + 7c(32 \\
 & + 9c)))))(6Ab^2(-16 + c(15c - 16)) + B(-128 + c(-384 + c(-160 + c(120 \\
 & + 77c))))))C + 64a^2A(3Ab^2(-128 + c(-640 + c(-904 + c(-352 + 5c(23 \\
 & + 17c)))))) + B(1024 + c(3328 + c(4864 + c(5088 - c(-2072 + c(826 \\
 & + c(1072 + 273c))))))C + 512a^5A(64 + c(32 + c(-32 + 9c(12 + 7c))))\varepsilon \\
 & + 32ab^2(Ab^2(384 + c(832 + c(968 + 5c(188 + 75c)))) \\
 & + 2B(768 + c(3456 + c(5856 + c(4720 + c(1658 + (57 - 70c)c))))))\varepsilon \\
 & - 32a^3(6Ab^2(4 + c)(32 + c(184 + c(244 + 85c))) - B(1024 + c(4096 \\
 & + c(12416 + c(20672 + c(17624 + c(7424 + 1269c))))))\varepsilon);
 \end{aligned} \tag{A.4}$$

$$\begin{aligned}
 H_5 = & (-2Ab^2(-8 + c(5c - 9)) + B(-32 + c(-80 + c(3 + c)(7c \\
 & - 10))))(2Ab^2(64 + c(144 + 5c(24 + 7c))) + B(2 + c)(128 + c(256 + c(352 \\
 & + 7c(32 + 9c))))C + 256a^4A^2(32 + c(32 + c(-24 + c(10 + c(34 \\
 & + 15c))))C + 16a^2A(16Ab^2(-32 + c(-138 + c(-193 - 97c + 10c^3))) \\
 & + B(1024 + c(3072 + c(4704 + c(5536 + c(3232 + c(168 - c(653 \\
 & + 196c))))))C + 128a^5A(64 + c^2(-32 + c(128 + 81c)))\varepsilon + 2ab^2(2Ab^2 \\
 & \times c(-448 + c(-48 + 5c(208 + 125c))) + B(6144 + c(23040 \\
 & + c(35456 + c(28288 + c(11312 + (1654 - 35c)c))))))\varepsilon - 16a^3(8Ab^2(1 \\
 & + c)(64 + c(228 + c(138 + 5c))) - B(512 + c(1024 + c(3488 + c(6496 + c(5364 \\
 & + c(5364 + c(2000 + 303c))))))\varepsilon)),
 \end{aligned} \tag{A.5}$$

$$\begin{aligned}
 H_6 = & ((2Ab^2(-8 + c(-9 + 5c)) + B(-32 + c(-80 + c(3 + c)(-10 + 7c))))(2Ab^2(64 \\
 & + c(144 + 5c(24 + 7c))) + B(2 + c)(128 + c(256 + c(352 + 7c(32 + 9c))))D1 \\
 & - 256a^4A^2(32 + c(32 + c(-24 + c(10 + c(34 + 15c))))D1 \\
 & - 16a^2A(16Ab^2(-32 + c(-138 + c(-193 - 97c + 10c^3))) \\
 & + B(1024 + c(3072 + c(4704 + c(5536 + c(3232 + c(168 - c(653 + 196c))))))D1 \\
 & - 128a^5A(64 + c^2(-32 + c(128 + 81c)))\varepsilon - 2ab^2(2Ab^2c(-448 + c(-48 + 5c(208 + 125c))) \\
 & + B(6144 + c(23040 + c(35456 + c(28288 + c(11312 + (1654 - 35c)c))))))\varepsilon \\
 & + 16a^3(8Ab^2(1 + c)(64 + c(228 + c(138 + 5c))) \\
 & - B(512 + c(1024 + c(3488 + c(6496 + c(5364 + c(2000 + 303c))))))\varepsilon),
 \end{aligned} \tag{A.6}$$

$$\begin{aligned}
 H_7 = & (a(4A^2(64a^4c(16 + c^2(57c - 34)) + 48a^2b^2c(-16 + c(-92 + 3c(5c - 31))) + b^4(512 \\
 & + c(1280 + c(208 + 5c(75c - 136)))))) + 4ABc(16a^2(128 + c(288 + c(180 + c(396 \\
 & + (76 - 9c)c)))) + b^2(2304 + c(5440 + c(2816 + c(208 + c(461 + 525c)))))) + B^2c(4096 \\
 & + c(18432 + c(22272 + c(17536 + c(15904 + c(10224 + 7c(430 + 53c)))))))).
 \end{aligned} \tag{A.7}$$

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