Quasigroups, Nets, and Nomograms

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Introduction

Nomograms are graphic aids to complicated numerical computations. Systematically introduced at the end of the last century they continue to be useful today, particularly in applied mathematics (see, e.g., the comprehensive book by d'Ocagne [1]*). The geometric theory of "nets" was introduced in the 1920's.† Although it originated in connection with topological questions of differential geometry, it can also be considered as part of the theoretical background of nomograms. As late as 1955, Blaschke [1] recommended the theory of nets to those who prefer more

* Numbers in brackets refer to the references at the end of our paper. These papers can serve for the purpose of further studies in this field, but familiarity with them is not necessary to understand the present paper.

† Its founders, Blaschke, Bol [1] and Thomsen [1] (see Blaschke and Bol [1] and Pickert [2]) have called this theory "Geometrie der Gewebe," i.e., "Geometry of Webs." Later Blaschke himself—as he writes, because of an invitation he received with reference to the title of the work by Blaschke and Bol [1] to a meeting of textile specialists—changed its name to "Geometrie der Waben" (Blaschke [1]), "geometry of cells," but this new terminology was not accepted by many authors. Maybe this confusion of terminology contributed to the fact that in English the term "nets" (see, e.g., Bruck [1]) is preferred to the two prior ones. This would be a very good name, but alas, it interferes with the topological concept of "nets."

classical and intuitive geometric facts to algebraic abstractions: "If modern mathematics has fallen sick with abstractionism, then we may claim a great deal of health (or lack of modernism?) for our branch of geometry, since we deal mostly with very intuitive problems." Nevertheless, a number of algebraic theories of nets were soon developed (see, e.g., Aczél, Pickert, and Radó [1], Bruck [1], and Pickert [2]) which clarified the abstract algebraic content of this seemingly geometric theory and, at the same time, aided in the further development of the purely geometric aspects. The close connection here of algebra, geometry, and applied mathematics is surely remarkable. In addition, analysis and topology will be seen to play a role in the applications of the algebraic theory to geometric nets. The different algebraic theories mentioned above are equivalent in their essential content, but they vary greatly in their constructions.

In this article, we will develop another, again equivalent, algebraic theory of construction somewhat different from the previous ones, originating in nomography and closely connected with the theory of quasigroups. With the aid of nets we will be able to prove in a new, intuitive way quite a few theorems on quasigroups, some of which are rather involved. "Intuitive" is not meant here to imply "heuristic": our proofs will be quite exact and the figures will serve only for illustration. We will then apply the theorems of our algebraic theory to concrete geometric nets, giving also a new elementary proof of an important theorem on a class of topological quasigroups on real numbers. Finally, we will solve two general functional equations on quasigroups, that also have applications to algebraic and geometric nets, and formulate some open questions which we think to be important for the further development of these theories.

1. Nomograms

Nomograms serve as representations of functions of several variables in the plane in a manner which makes them also apt for numerical calculations (for details see, e.g., D'Ocagne [1]). We will consider here only the representation of a class of functions of two variables.

The so-called contour-line representation can be chosen as the point of departure which gives the simplest form of *nomograms*. One draws in the XY-plane curves through those points to which the same value of the function is assigned. These are the contour lines. For

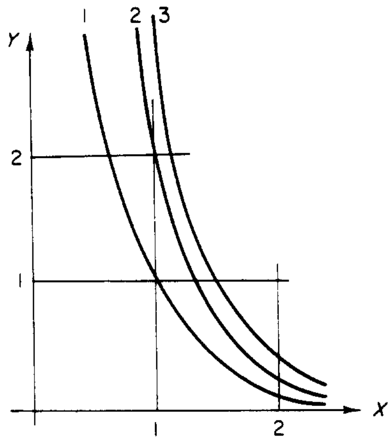

FIG. 1.

example, Fig. 1 shows this kind of nomogram of the function associating x^3y to x and y (the numbers with which the curves are marked show the common value of the function belonging to the points of the respective curves).* With the aid of this figure we can determine for any given pair x, y the value of $z = x^3y$, exactly or approximately by interpolation between the curves drawn in the figure; similarly, we can determine y, given x and z and also x, given y and z. The difficulty here lies partly in the great amount of work which is required to draw all these curves, and partly in the fact that reading even approximate values is more difficult between curves than between straight lines. Therefore, replacing curves by straight lines is of great importance. Sometimes this can be done simply by interchanging the roles of the variables and the function values. So in the case of $z = x^3y$, if we represent the values of x and z on the two axes, then the lines representing different constant y-values will again be curves (Fig. 2); but if we choose Y and Z as axes, then the lines representing constant x-values will already be straight (Fig. 3). Of course, this ***nomogram containing only straight lines*** also allows the calculation of one of the values x, y, and z if the two others are given. On the other hand, replacing the linear division of the axes by other scales sometimes makes the straightening ("anamorphosis")

* Here xy is the ordinary product of real numbers, but later on the same symbol will represent in general the result of multiplication in a quasigroup. We will use throughout the notations xy and $x \cdot y$ as equivalent.

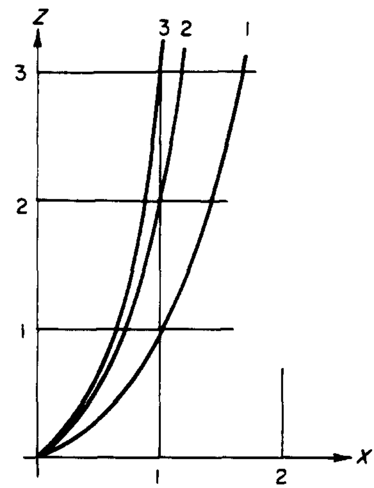

FIG. 2.

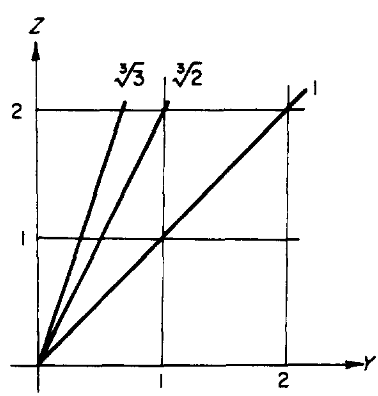

FIG. 3.

of the nomogram possible. For instance, $z = x^3y$ can also be written as $\log z = 3 \log x + \log y$ and therefore, if we choose logarithmic scales on the X- and Y-axes, then the points (x, y) to which the same value of $z = x^3y$ is attached lie on straight lines (Fig. 4); the situation is similar if we choose X and Z or Y and Z as axes.

This *problem of anamorphosis* (straightening) is one of the fundamental ones in nomography. Observe that there is more than one family of curves or straight lines on our figures: the coordinate lines ($x =$ const.,

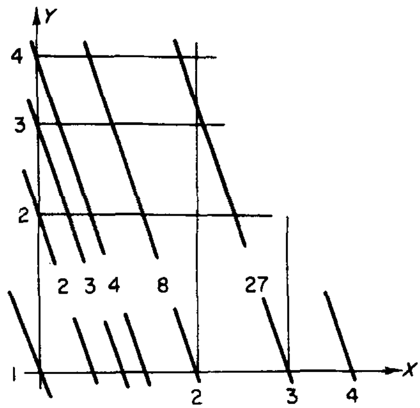

FIG. 4.

$y =$ const.; resp., $x =$ const., $z =$ const.; resp., $y =$ const., $z =$ const.) form also two bundles of parallel straight lines. The ***general straight nomogram*** is formed of three families of straight lines, one attached to the values of x, the other to those of y, and the third to those of z, and they are marked with these numbers. The straight lines of any one family need not be parallel. For instance, in case of a functional relation represented by such nomograms, one can find z corresponding to given x and y. We check the point of intersection of the respective x- and y-lines (if they are not on the figure, then by interpolating between

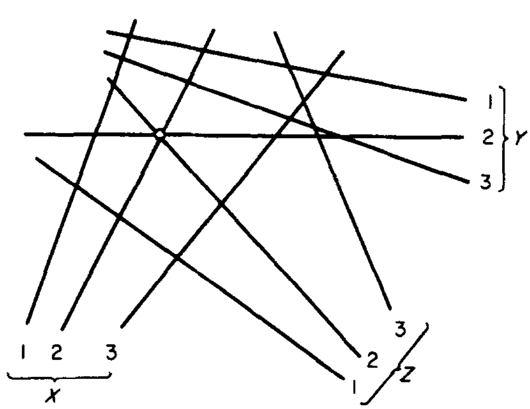

FIG. 5.

straight lines drawn in the figure) and the number attached to the z-line (whether or not it is drawn in the figure) going through this point is the respective z-value (Fig. 5).

In the case of *parallel nomograms* these three families of straight lines all consist of parallel ones. One can easily arrange that the straight lines of two of these bundles be orthogonal to each other, and so we get a configuration with which we are already familiar (Fig. 6, cf. Fig. 4).

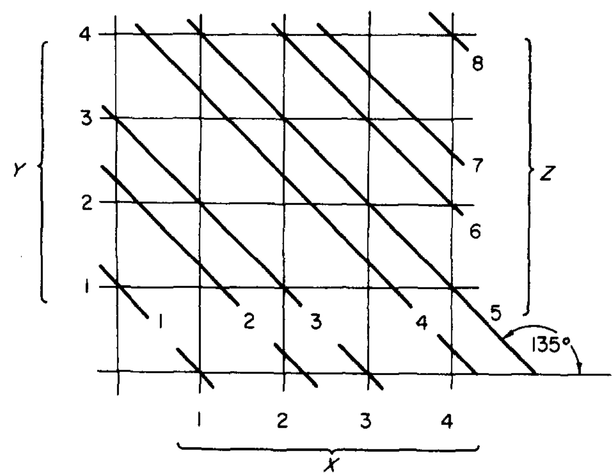

FIG. 6.

We can even arrange (in the case of Fig. 4, e.g., by stretching it horizontally in proportion 3 : 1) that the straight lines of the third bundle should intersect those of the first and second in an angle equal to 135°.

We call the transformation of a curvilinear nomogram into a straight one *anamorphosis*, that into a parallel one *parallel anamorphosis*. We shall consider only the latter in this article.

This means that we want to introduce, instead of x, y, and z, such new variables u, v, and w that the functional connection between x, y, and z should go over into

$$w = u + v$$

(the $u =$ const. and $v =$ const. lines of this latter connection are the vertical and horizontal lines of Fig. 6, while $w =$ const. is represented there by those straight lines intersecting the positive u-axis at an angle of 135°).

Here

$$u = f(x), \qquad v = g(y), \qquad w = h(z)$$

Here, $f(x)$ and $g(y)$, respectively, give the real distances from the Y- and X-axes of the straight lines marked by the respective x- and y-values, while $w = h(z)$ yields the real distance from the origin to the intersection of the straight line marked by this z-value with the X- (or with the Y-) axis.

$$h(z) = f(x) + g(y)$$

and this has to be equivalent with

$$z = xy$$

[here xy is no longer a product but the result of a general functional connection (see footnote on p. 385)], i.e.,

$$h(xy) = f(x) + g(y). \tag{1.1}$$

One usually supposes f, g, and h to be continuous and strictly monotonic functions. In most practical cases this holds at least locally.

We mention yet another kind of nomogram, the *alignment charts*. These are the projective geometric (plane) duals of general straight nomograms. In the sense of projective duality, points correspond to straight lines, points lying on the same straight line to straight lines intersecting in one point, and vice versa. Now, the three bundles of straight lines of a straight nomogram represented the variables x, y, z and the intersection of an x-, y-, and z-line in one point corresponded to the functional relation of the respective x-, y-, and z-values. Therefore, in the alignment charts, sets of points (curves) correspond to the x-, y-, and z-values, and these x-, y-, and z-values belong together if the respective x-, y-, and z-points are on one straight line (Fig. 7). The duals of parallel nomograms or of nomograms consisting of three bundles of straight lines each going through one point, are the *alignment charts with three straight scales* (Fig. 8). It can be shown that all functional connections which can be represented by such alignment charts can also be represented by alignment charts with three *parallel straight scales* which if desired can be chosen equidistant (Fig. 9).

All this shows that one can represent by alignment charts with three (parallel) straight scales exactly those functional connections $z = xy$

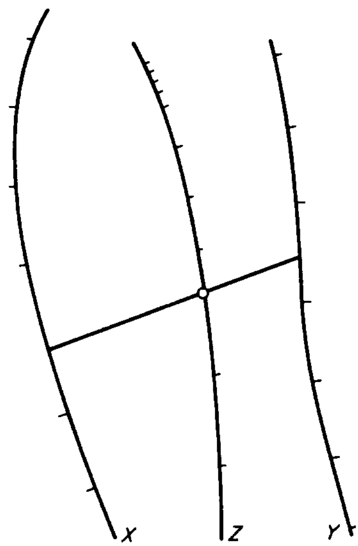

FIG. 7.

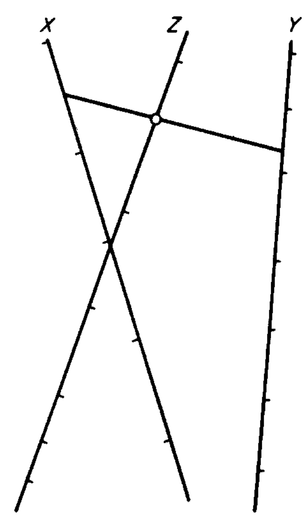

FIG. 8.

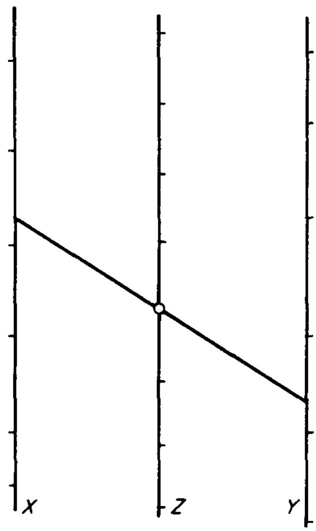

FIG. 9.

for which there exist such f, g, and h that (1.1) is fulfilled. This emphasizes still more the importance of this class of functions.

2. Geometric Nets

We will call three families of (continuous) curves in a domain of the plane a *geometric net* if exactly one curve of each family passes through each point of the domain (therefore curves of the same family cannot intersect) and two curves of different families intersect in exactly one point (Fig. 10).* A geometric net is *regular* if this configuration is homeomorphic to three bundles of parallel straight lines (i.e., if there exists an invertibly continuous 1-1 map of the first configuration onto the second).

As there can always be found an invertibly continuous 1-1 map carrying over two families of curves of a geometric net into two bundles of parallel straight lines which can even be choosen as orthogonal to

* We should call them geometric plane nets with three families of curves (3-nets), but except for a remark we will not treat spatial nets in this paper and k-nets with $k > 3$ at all.

each other, we can take these as coordinate lines of a Cartesian system and the homeomorphic images of the curves of the third family as contour lines of a functional relation $z = xy$; we have thus arrived back at the contour-line representation. The regularity of a geometric net means in this sense, that it is homeomorphic with the net (Fig. 11)

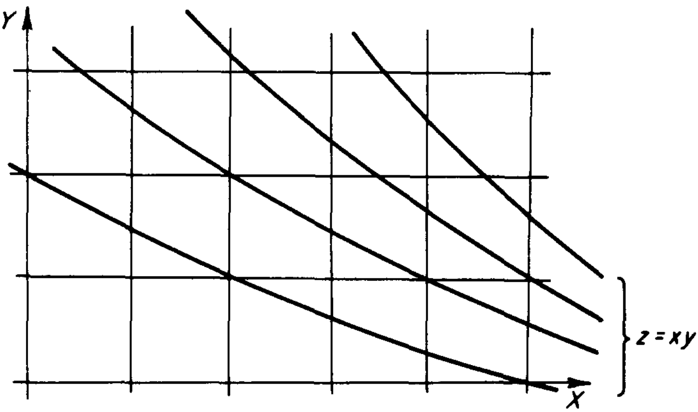

FIG. 10.

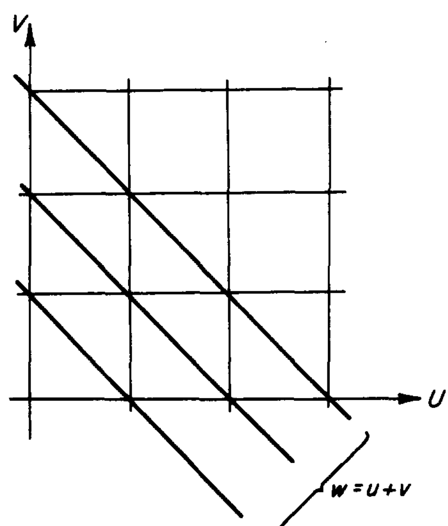

FIG. 11.

consisting of the coordinate lines and of the contour lines of the (linear) function $w = u + v$ (the 135° lines). It can be shown that this means the existence of three continuous and strictly monotonic functions f, g, and h with the set of all real numbers as common range such that

$$h(xy) = f(x) + g(y). \tag{2.1}$$

The net built of the coordinate lines and of the contour lines of $z = x^3y$ $(x > 0,\ y > 0)$ is an example of a regular net $[f(x) = 3 \log x$, $g(y) = \log y$, $h(z) = \log z)$; the net belonging to $z = (x + y)^3 + x$ is an example of a net which is not regular. The connection with nomograms and parallel anamorphosis is evident.

It is rather characteristic of the geometric theory of nets, as compared with nomography, that it seeks mainly for geometric conditions for the regularity of nets. These are so-called *closure conditions* (see, e.g., Bol [1], Thomsen [1], and Blaschke and Bol [1]).

One of these conditions is called the *Thomsen condition*, in short T, represented by Fig. 12 and described in the terminology of contour lines as follows: If x_1, x_2, and x_3 are three arbitrary x-values and y_1, y_2, and y_3 three arbitrary y-values, and if (x_1, y_2) and (x_2, y_1) lie on the same z-curve (contour line) and so also do the points (x_1, y_3) and (x_3, y_1), then

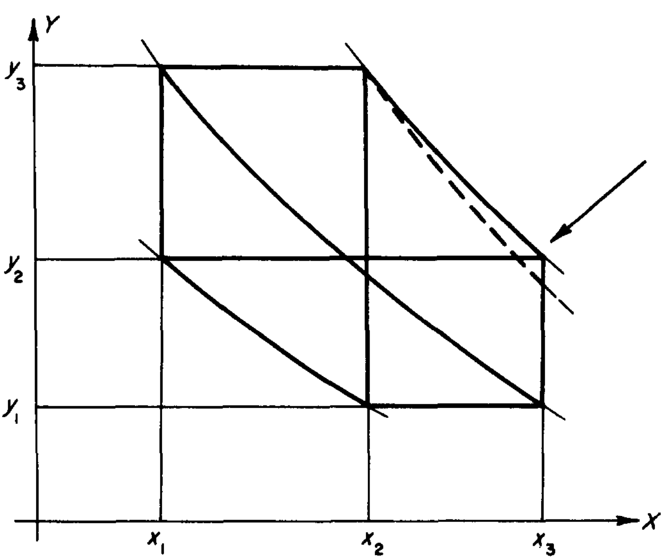

FIG. 12.

the points (x_2, y_3) and (x_3, y_2) also lie on the same contour line. Since at the points of the same z-curve the values (xy) of the function are the same, the Thomsen condition can be expressed in the following way*:

$$T: \quad (x_1y_2 = x_2y_1 \;\&\; x_1y_3 = x_3y_1) \Rightarrow x_2y_3 = x_3y_2 \,.$$

Similarly, the so-called *Reidemeister condition*—in short R—(Fig. 13) can be written:

$$R: \quad (x_1y_2 = x_2y_1 \;\&\; x_1y_4 = x_2y_3 \;\&\; x_3y_2 = x_4y_1) \Rightarrow x_3y_4 = x_4y_3 \,.$$

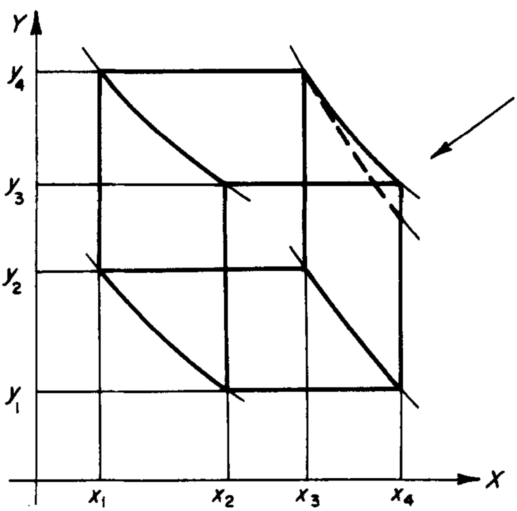

FIG. 13.

Fig. 13 might remind us of the perspective representation of a parallelepiped. The further closure conditions can be attained by specialization of R.

The three so-called *Bol conditions* come from R by equating x_2 and x_3, resp., y_2 and y_3, resp., x_2y_3 and x_3y_2 (i.e., the two x-, y-, or z-lines in the middle fall together, respectively). This is expressed by Figs. 14–16 and by the following formulas:

$$B_1: \quad (x_1y_2 = x_2y_1 \;\&\; x_1y_4 = x_2y_3 \;\&\; x_2y_2 = x_4y_1) \Rightarrow x_2y_4 = x_4y_3 \,,$$

$$B_2: \quad (x_1y_2 = x_2y_1 \;\&\; x_1y_4 = x_2y_2 \;\&\; x_3y_2 = x_4y_1) \Rightarrow x_3y_4 = x_4y_2 \,,$$

$$B_3: \quad (x_1y_2 = x_2y_1 \;\&\; x_1y_4 = x_2y_3 = x_3y_2 = x_4y_1) \Rightarrow x_3y_4 = x_4y_3 \,.$$

* Throughout this paper we use the symbols & for "and," $\Rightarrow$ for "implies," $\Leftrightarrow$ for "if and only if" ("iff"), and $\in$ for "element of."

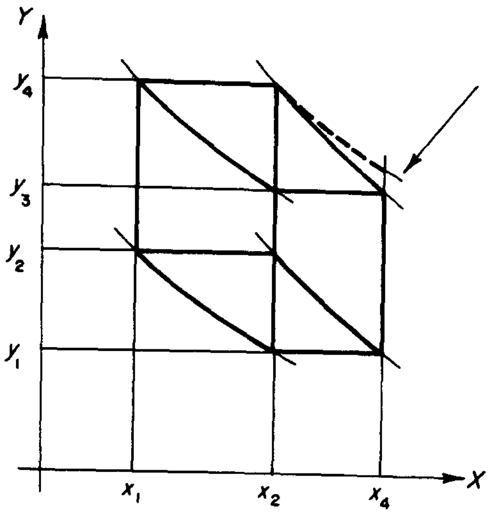

FIG. 14.

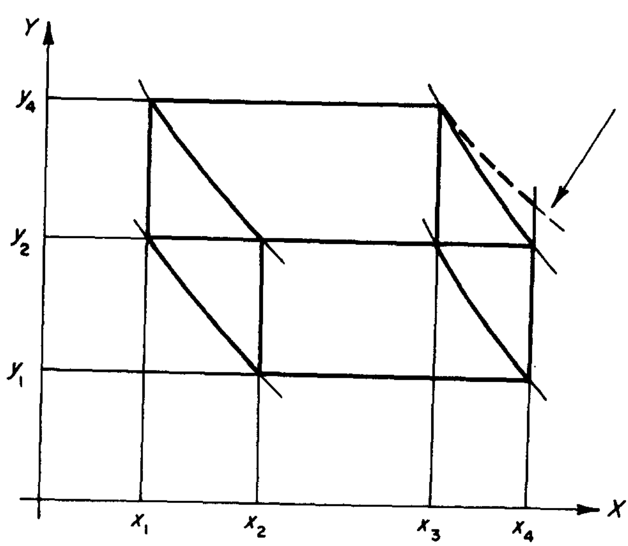

FIG. 15.

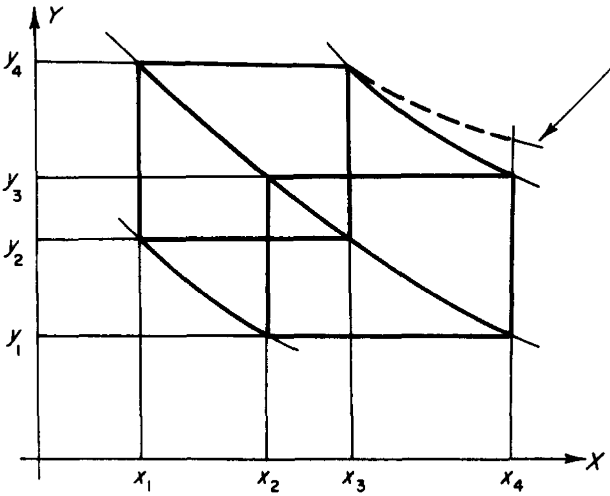

FIG. 16.

If for a net both B_1 *and* B_2 *are satisfied (each for arbitrary* x_j, y_j*), then we say that condition B is fulfilled for this net.* We will show in Theorem 2.1 that then B_3 is also satisfied for all x_j, y_j.

Finally, the so-called *hexagonal condition*—in short H—is a special case of all previous closure conditions, deriving from R (and from B_1, B_2, or B_3) by the specialization $y_2 = y_3$, $x_2 = x_3$, or from T by adjunction of $x_2y_2 = x_1y_3 = x_3y_1$ (see Fig. 17; the partly curvilinear hexagon to be seen on this figure explains the name of the condition):

$$H\colon \quad (x_1y_2 = x_2y_1 \,\&\, x_1y_3 = x_2y_2 = x_3y_1) \Rightarrow x_2y_3 = x_3y_2 .$$

These formulas are called *closure* conditions because they assert that the Figs. 12–17 "close" at the place marked by an arrow (the broken z-lines show the cases where the respective closure conditions are *not* fulfilled). The statement that *the closure conditions must be satisfied for arbitrary* x_j, y_j's (in the domain) means that we did not suppose anything about the order of the x_j's and y_j's; it is not at all necessary that $x_1 < x_2 < x_3 < x_4$, $y_1 < y_2 < y_3 < y_4$ should hold. For this reason the roles of the z-lines in T, R, B_1, and B_2 can be interchanged, whereas in B_3 and H only those of the two external z-lines.

As to the strength of the different closure theorems we prove the following:

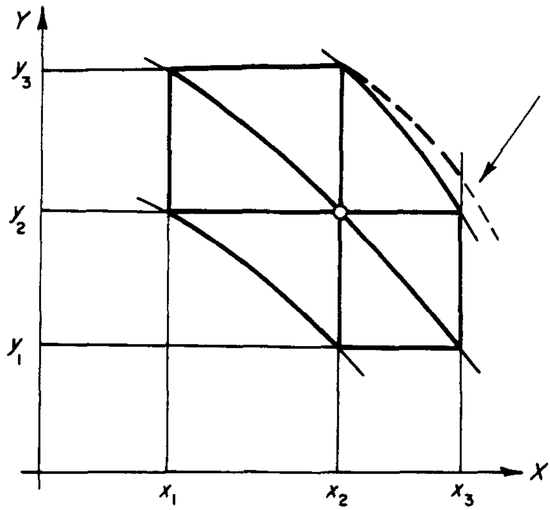

FIG. 17.

THEOREM 2.1.

$$T \Rightarrow R \Rightarrow B \Rightarrow \begin{matrix} \nearrow B_1 \searrow \\ B_2 \\ \searrow B_3 \nearrow \end{matrix} \Rightarrow H$$

Proof. As B_1 and B_2 are special cases of R, and H a special case of each of the B_k's ($k = 1, 2, 3$), and as $B \Leftrightarrow (B_1 \,\&\, B_2)$, we need only to prove

$$\text{(I)}\quad T \Rightarrow R \qquad \text{and} \qquad \text{(II)}\quad (B_1 \,\&\, B_2) \Rightarrow B_3$$

(I) We suppose that T is fulfilled for any six-tuples of x and y values. We have to prove that for arbitrary x_1, x_2, x_3, x_4, y_1, y_2, y_3, and y_4 (Fig. 18) the validity of

$$x_1y_2 = x_2y_1 \,\&\, x_1y_4 = x_2y_3 \,\&\, x_3y_2 = x_4y_1 \tag{2.2}$$

implies that of

$$x_3y_4 = x_4y_3\,. \tag{2.3}$$

We introduce the new notations

$$X_1 = x_2\,,\quad X_2 = x_1\,,\quad Y_1 = y_4\,,\quad Y_2 = y_3\,,\quad Y_3 = y_2\,; \tag{2.4}$$

then because of (2.2)

$$X_1Y_2 = X_2Y_1\,. \tag{2.5}$$

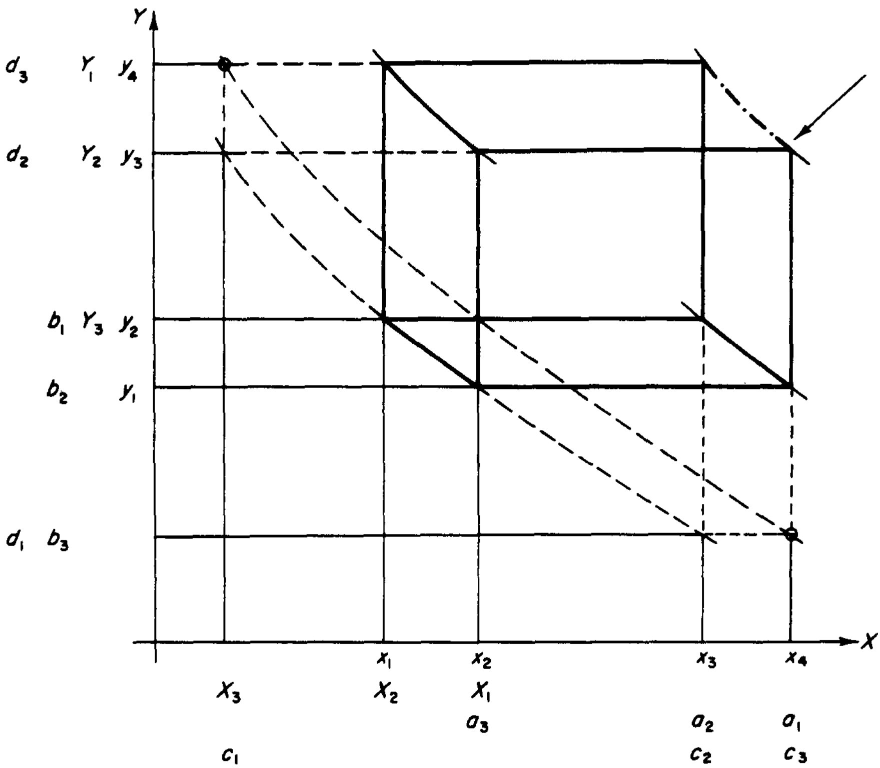

FIG. 18.

Let us define X_3 as the x-value corresponding to the y-value Y_1 on the z-curve (countour line) on which the point (X_1, Y_3) lies (such an x-value exists, since by the definition of nets the z-curve with the label $z = X_1Y_3$ and the y-line with $y = Y_1$ have a point of intersection); in symbols

(2.6) $$X_1Y_3 = X_3Y_1 .$$

But (2.5) and (2.6) constitute just the first part (the hypotheses) of T for X_1, X_2, X_3, Y_1, Y_2, andY_3 and as T is by supposition valid for all x's and y's, therefore (2.5) and (2.6) imply

(2.7) $$X_2Y_3 = X_3Y_2 .$$

With the further new notation

(2.8) $$a_1 = x_4 , \quad a_2 = x_3 , \quad a_3 = X_1 = x_2 , \quad b_1 = Y_3 = y_2 , \quad b_2 = y_1$$

we have from (2.2)

$$a_1b_2 = a_2b_1$$

and we define b_3 by

$$a_1b_3 = a_3b_1; \tag{2.9}$$

then the hypotheses of T are again fulfilled for a_1, a_2, a_3, b_1, b_2, b_3, and so its conclusion

$$a_2b_3 = a_3b_2 \tag{2.10}$$

has to hold too.

Finally, with the notation

$$\begin{aligned} c_1 &= X_3, & c_2 &= a_2 = x_3, & c_3 &= a_1 = x_4, \\ d_1 &= b_3, & d_2 &= Y_2 = y_3, & d_3 &= Y_1 = y_4, \end{aligned} \tag{2.11}$$

we get from (2.11), (2.7), (2.4), (2.2), (2.8), and (2.10) and again (2.11)

$$c_1d_2 = X_3Y_2 = X_2Y_3 = x_1y_2 = x_2y_1 = a_3b_2 = a_2b_3 = c_2d_1 \tag{2.12}$$

and again from (2.11), (2.6), (2.8), (2.9), and (2.11)

$$c_1d_3 = X_3Y_1 = X_1Y_3 = a_3b_1 = a_1b_3 = c_3d_1. \tag{2.13}$$

If we consider the first and last members of the chains of equalities (2.12) and (2.13), we have again the hypothesis of T, this time for c_1, c_2, c_3, d_1, d_2, and d_3, and so the conclusion of T is valid too:

$$c_2d_3 = c_3d_2$$

or, according to (2.11)

$$x_3y_4 = x_4y_3.$$

This is just the equation (2.3) to be proven, the second part of R. Thus, assertion (I) has been proved.

(II) In order to prove this assertion, we suppose that B_1 and B_2 are true for arbitrary x's and y's and prove that B_3 holds too for arbitrary x_1, x_2, x_3, x_4, y_1, y_2, y_3, and y_4, i.e.,

$$x_1y_2 = x_2y_1 \,\&\, x_1y_4 = x_2y_3 = x_3y_2 = x_4y_1 \tag{2.14}$$

imply

$$x_3y_4 = x_4y_3. \tag{2.15}$$

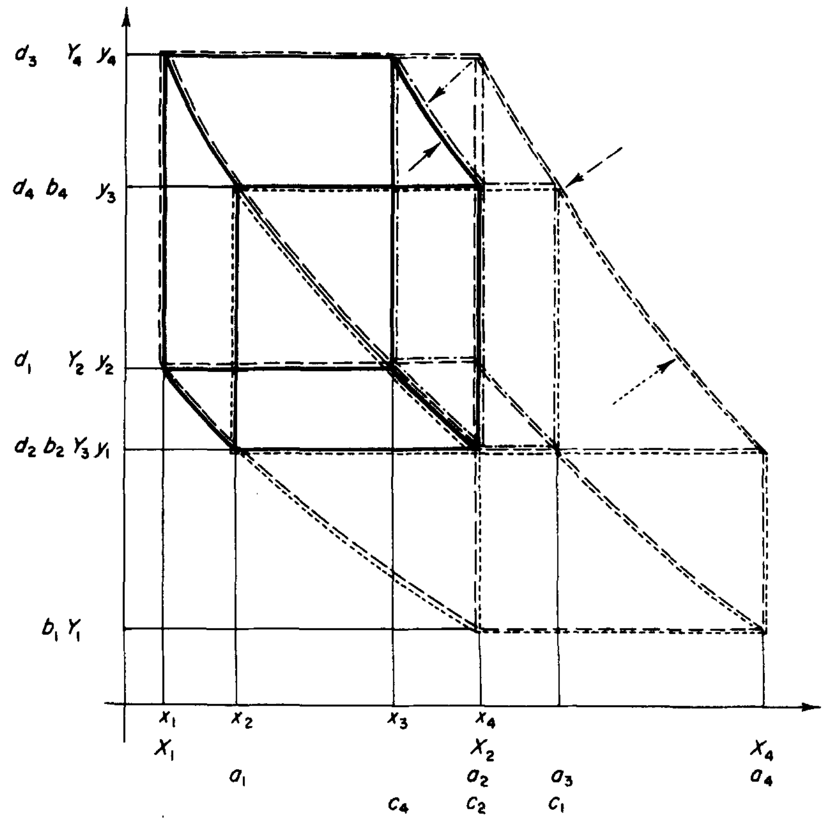

FIG. 19.

Let us introduce (Fig. 19) the notations

(2.16) $\quad X_1 = x_1, \qquad X_2 = x_4, \qquad Y_2 = y_2, \qquad Y_3 = y_1, \qquad Y_4 = y_4;$

then

(2.17) $$X_1Y_4 = X_2Y_3$$

follows from (2.14) and we define Y_1 and X_4 by

(2.18) $$X_1Y_2 = X_2Y_1 \quad \text{and} \quad X_2Y_2 = X_4Y_1,$$

respectively. Equations (2.17) and (2.18) are just the hypotheses of B_1 for X_1, X_2, X_4, Y_1, Y_2, Y_3, and Y_4, thus by supposition B_1's conclusion holds too:

(2.19) $$X_2Y_4 = X_4Y_3.$$

Again with new notations

$$a_1 = x_2 , \quad a_2 = X_2 = x_4 , \quad a_4 = X_4 ,$$
$$b_1 = Y_1 , \quad b_2 = Y_3 = y_1 , \quad b_4 = y_3 . \tag{2.20}$$

we have by (2.14), (2.16), (2.18), and (2.14)

$$a_1b_2 = x_2y_1 = x_1y_2 = X_1Y_2 = X_2Y_1 = a_2b_1$$

and

$$a_1b_4 = x_2y_3 = x_4y_1 = a_2b_2 .$$

We define a_3 by

$$a_3b_2 = a_4b_1 . \tag{2.21}$$

This time the hypotheses of B_2 are fulfilled for a_1, a_2, a_3, a_4, b_1, b_2, b_4 and so also is its conclusion:

$$a_3b_4 = a_4b_2 . \tag{2.22}$$

Finally, with the notations

$$c_1 = a_3 , \quad c_2 = a_2 = X_2 = x_4 , \quad c_4 = x_3 ,$$
$$d_1 = Y_2 = y_2 , \quad d_2 = b_2 = Y_3 = y_1 , \quad d_3 = Y_4 = y_4 , \quad d_4 = b_4 = y_3 \tag{2.23}$$

we get from (2.21), (2.20), and (2.18)

$$c_1d_2 = a_3b_2 = a_4b_1 = X_4Y_1 = X_2Y_2 = c_2d_1 ,$$

from (2.22), (2.20), and (2.19)

$$c_1d_4 = a_3b_4 = a_4b_2 = X_4Y_3 = X_2Y_4 = c_2d_3$$

and from (2.14)

$$c_2d_2 = x_4y_1 = x_3y_2 = c_4d_1 .$$

So the hypotheses of B_1 hold again, this time for c_1, c_2, c_4, d_1, d_2, d_3, and d_4 and so the conclusion is also valid:

$$c_2d_4 = c_4d_3$$

and by (2.23) this just means that (2.15) is satisfied. Thus (II) is proved also, and the proof of Theorem 2.1 is finished.

Both proofs can be followed on Figs. 18 and 19, but they can be understood also independently of the figures through the above sequences of equalities. The introduction of many new notations (2.4), (2.8), (2.11), (2.16), (2.20), and (2.23) is not indispensable; these illuminated only the repeated applications of T, B_1, and B_2.

Although, owing to Theorem 2.1, T is the "strongest" and H the "weakest" closure condition, still the following is true:

THEOREM 2.2. *For geometric nets each of the conditions T, R, B, B_1, B_2, B_3, and H is necessary and sufficient for the net to be regular.*

We will see that for so-called algebraic nets these conditions are no longer equivalent. We will nevertheless deduce the above Theorem 2.2 from theorems to be proven on algebraic nets.

3. Algebraic Nets

If we reconsider the formulas T, R, B_1, B_2, B_3, and H, we see that these "geometric" closure conditions are of purely algebraic nature. This suggests the idea of giving algebraic definitions to the notions of nets and regular nets too.

We call a set **G** together with an operation on it whose result is $z = xy (= x \cdot y)$ a *quasigroup* (cf. Bruck [1], Kertész [1]) and denote it by $(\mathbf{G}, \cdot)$ or simply by **G** if, for any given elements a, b, c, and d of **G**, equations of the form $xb = c$ and $ay = d$ have unique solutions in **G**, x, and y, respectively. We will call ordered pairs (x, y) of elements of **G** "points". We define 1-*curves* to be sets of points of the form (c, y) with fixed c and variable y, similarly 2-*curves* are sets of points of the form (x, c) and 3-*curves* are such sets of points (x, y) for which $xy = c$. c is an arbitrary element of **G**, but in case of the same curve it is always the same.

The k-curves ($k = 1$ or $k = 2$ or $k = 3$) formed with all possible elements c in **G** form together a *family of curves.*

We define an *algebraic net* as the set of 1-, 2-, and 3-curves in the quasigroup **G** for all possible c's in **G**, i.e., as all three families of curves together. These curves satisfy, on the set of pairs of elements of **G**, all conditions stated in Section 1 in the definition of (geometric) nets [each "point" (x, y) belongs to exactly one "1-curve," one "2-curve", and one "3-curve"; two "curves" of the same family have no common "point," two "curves" of different families have exactly one common "point"] except the condition of continuity which has no sense in this generality.

The formulas T, R, B_1, B_2, B_3, and H all make sense in this interpretation too. (In what follows, by saying that T, R, B_1, B_2, B_3, or H hold, we mean that they are valid for *all* x_j and y_j in $\mathbf{G}$ if the contrary is not explicitly stated.) Moreover *Theorem* 2.1 *and its proof also remain valid for algebraic nets.*

In order to get nearer to the concept of regular algebraic nets, let us consider Eq. (2.1). The continuity and strict monotony of f, g, and h here have no sense. Instead, we will consider 1-1 maps. We call two quasigroups $(\mathbf{G}, \cdot)$ and $(\mathbf{H}, \circ)$ *isotopic* (cf. Bruck [1], Kertész [1]), if there exist three 1-1 maps f, g, and h of $\mathbf{G}$ onto $\mathbf{H}$, such that for all elements x, y, and z in $\mathbf{G}$

$$h(xy) = f(x) \circ g(y) \tag{3.1}$$

is fulfilled.

To a 1-curve $x = c_1$ of $(\mathbf{G}, \cdot)$ there corresponds a 1-curve $x' = f(x) = c_1' = f(c_1)$ in a quasigroup $(\mathbf{H}, \circ)$ isotopic to $(\mathbf{G}, \cdot)$; similarly, $y' = g(y) = c_2'$ corresponds to $y = c_2$ and $x' \circ y' = f(x) \circ g(y) = h(xy) = c_3'$ corresponds to $xy = c_3$. Moreover, *the validity of a closure condition* T, R, B_1, B_2, B_3, *or* H *in a quasigroup implies its validity in all isotopic quasigroups.* Therefore, *we relate an algebraic net* not just to one quasigroup but *to a whole class of isotopic quasigroups* (cf. also Section 4 for a further motivation). Sometimes we identify these two concepts and understand by an algebraic net just this class.

As we will see, in these classes there are always *loops*, i.e., quasigroups with *unit elements* e for which

$$xe = ex = x \tag{3.2}$$

for any element x of $\mathbf{G}$. Among the loops the following are particularly important: the *power-associative loops* in which

$$xx \cdot x = x \cdot xx \tag{3.3}$$

holds for every $x \in \mathbf{G}$; the M_1*-loops* in which

$$y(z \cdot yx) = (y \cdot zy)x, \tag{3.4}$$

the M_2*-loops* in which

$$(xy \cdot z)y = x(yz \cdot y) \tag{3.5}$$

and the *Moufang loops* in which

$$(xy \cdot z)y = x(y \cdot zy) \tag{3.6}$$

are valid for all $x, y, z \in \mathbf{G}$, finally, the *groups* with the associativity

$$xy \cdot z = x \cdot yz \tag{3.7}$$

for all $x, y, z \in \mathbf{G}$, and the *Abelian groups* for which besides (3.7) also the commutativity

$$xy = yx$$

is satisfied for all $x, y \in \mathbf{G}$. The following statements are evident.

Lemma 3.1. *Every Abelian group is a group, every group is a Moufang loop, and all M_1-loops, all M_2-loops and all Moufang loops are power-associative.*

The first two statements need no proofs. For the proof of the last three statements it is enough to put $z = e$, $y = x$ into (3.4), (3.5), or (3.6) in order to get (3.3) with the help of (3.2).

We further prove the following:

Lemma 3.2. *All Moufang-loops are M_1-loops and also M_2-loops. Conversely, if a loop is at the same time an M_1-loop and an M_2-loop, then it is also a Moufang loop.*

Proof. We have two statements to prove:

(I) *If for all $x, y, z \in \mathbf{G}$* (3.6) *holds and for all $x \in \mathbf{G}$* (3.2), *then also* (3.4) *and* (3.5) *hold for all $x, y, z \in \mathbf{G}$*, and

(II) *If* (3.2), (3.4), *and* (3.5) *hold for all elements of* $\mathbf{G}$, *then also* (3.6) *does.*

(I) By putting into (3.6) $x = e$, we get because of (3.2)

$$yz \cdot y = y \cdot zy$$

and thus (3.6) goes over into (3.5) immediately. So we have already derived (3.5) from (3.6) and (3.2).

On the other hand, let us define the *left inverse* y^{-1} of y by

$$y^{-1}y = e \tag{3.8}$$

and put into (3.6) $x = y^{-1}$, $zy = w$. Then we get because of (3.2)

$$w = y^{-1} \cdot yw \tag{3.9}$$

for all $y, w \in \mathbf{G}$. If we now make the replacements $y = z^{-1}$, $xy = xz^{-1} = r$ in (3.5) [which we have already derived from (3.6) and (3.2)], and make use of (3.8) and (3.2) we get

$$rz \cdot z^{-1} = r \tag{3.10}$$

for all $r, z \in \mathbf{G}$. If we multiply (3.9) from the right by $(yw)^{-1}$, then (3.10) with $r = y^{-1}$, $z = yw$ yields

$$w(yw)^{-1} = y^{-1}.$$

If we multiply the latter equation by w^{-1} on the left and use (3.9) we get the important relation

$$(yw)^{-1} = w^{-1}y^{-1}. \tag{3.11}$$

Now we put into (3.5) x^{-1}, y^{-1}, and z^{-1} for x, y, and z:

$$(x^{-1}y^{-1} \cdot z^{-1})y^{-1} = x^{-1}(y^{-1}z^{-1} \cdot y^{-1})$$

from which we get by repeated use of (3.11) first

$$((yx)^{-1}z^{-1})y^{-1} = x^{-1}((zy)^{-1}y^{-1})$$

then

$$(z \cdot yx)^{-1}y^{-1} = x^{-1}(y \cdot zy)^{-1}$$

and, finally,

$$(y(z \cdot yx))^{-1} = ((y \cdot zy)x)^{-1}$$

which by the uniqueness of the left inverses, i.e., of the solutions of equations of the form (3.8) in loops, just yields (3.4). Thus the statement (I) is proved.

(II) We put into (3.4) $z = y^{-1}$, $yx = w$; then (3.8) and (3.2) imply

$$y \cdot y^{-1}w = w \tag{3.12}$$

for all $y, w \in \mathbf{G}$. On the other hand, by (3.5) with $x = y^{-1}$ and again because of (3.8) and (3.2)

$$zy = y^{-1}(yz \cdot y)$$

and by multiplying both sides of this equation by y and taking (3.12) with $yz \cdot y = w$ into consideration, we get again

$$y \cdot zy = yz \cdot y$$

So (3.5) goes over into (3.6) by which (II) and hence Lemma 3.2 is proved.

After these preliminary remarks we can define several grades of regularity of algebraic nets, differing in their strength from stronger to successively weaker ones.

An algebraic net is a-regular, if the corresponding class of isotopic quasigroups contains an Abelian group, g-regular if the class contains a group, and m-regular, m_1-regular, or m_2-regular if the class contains a Moufang loop, a M_1-loop, or a M_2-loop, respectively.

In these cases at the same time *all loops of the class are Abelian groups, groups, Moufang loops, M_1-loops, or M_2-loops, respectively* (cf. Bruck [1], Kertész [1], and see Corollary 5 below). In a similar way we define an algebraic net to be *p-regular, if all loops of the corresponding class are power-associative* (3.3). We will abbreviate the statement that a net is h-regular ($h = a, g, m, m_1, m_2, p$) by h.

The above definitions and Lemmas 3.1 and 3.2 immediately imply the following:

Corollary 1.

$$a \Rightarrow g \Rightarrow m \begin{matrix} \nearrow m_1 \searrow \\ \searrow m_2 \nearrow \end{matrix} p$$

This is very similar to Theorem 2.1. There is a significant reason: the different grades of regularity are equivalent to different closure conditions. In particular, we make the following statements of equivalence between properties of algebraic nets and those of the corresponding classes of isotopic quasigroups:

THEOREM 3.1. $a \Leftrightarrow T$.

THEOREM 3.2. $g \Leftrightarrow R$.

THEOREM 3.3. $m_1 \Leftrightarrow B_1$.

THEOREM 3.4. $m_2 \Leftrightarrow B_2$.

THEOREM 3.5. $p \Leftrightarrow H$.

The definition of B together with Theorems 3.2 and 3.3 and with Lemma 3.2 imply also

Corollary 2. $m \Leftrightarrow B$.

The proofs of Theorems 3.1–3.5 can be followed in Figs. 21–25, which serve, however, only as illustrations (as they replace **G** by the real axis or by a subinterval of it).

In these proofs the so-called *LP*(loop-principal)-isotopes of quasigroups will be fundamental aids. We define an *LP*-isotope (**G**, $\circ$) of (**G**, $\cdot$) by

$$sv \circ ut = st \tag{3.13}$$

where u and v are, for the time being, fixed elements of **G**. As for given u, v any two arbitrary elements x, y of **G** can be represented in a unique way in the form

$$x = sv, \qquad y = ut,$$

so (3.13) defines $x \circ y$ unambiguously for all $x, y \in \mathbf{G}$ and (**G**, $\circ$) is really an isotope of (**G**, $\cdot$): here $f(s) = sv$, $g(t) = ut$, and $h(z) = z$. Those isotopes where $h(z) = z$ are called *principal isotopes.* On the other hand (**G**, $\circ$) is a *loop* and $e = uv$ is its unit element. In fact, if we put $s = u$, resp. $t = v$, into (3.13) then we get

$$e \circ ut = ut, \qquad sv \circ e = sv,$$

respectively, and so the identities corresponding to (3.2) are valid. Reciprocally, *all principal isotopes of* (**G**, $\cdot$), *which are loops are given by* (3.13). In fact, if (**G**, $\circ$) is a principal isotope of (**G**, $\cdot$):

$$st = f(s) \circ g(t) \tag{3.14}$$

and if (**G**, $\circ$) is a loop with the unit element e, then $-f$ and g being 1-1 maps of **G** onto itself $-$ there exist elements u and v in **G** such that

$$f(u) = e, \qquad g(v) = e.$$

From (3.14) by substitution of $t = v$, resp., $s = u$ we get

$$f(s) = sv \qquad \text{and} \qquad g(t) = ut,$$

respectively, so that (3.14) goes over into (3.13) as we have asserted. This motivates the name loop-principal(*LP*)-isotope, but we will not need these properties until later. In this connection, however, we mention that all isotopes (**H**, $*$) of a quasigroup (**G**, $\cdot$)

$$h(xy) = f(x) * g(y) \tag{3.15}$$

are isomorphic

$$h(x \circ y) = h(x) * h(y) \tag{3.16}$$

to a principal isotope, namely to that defined by

$$uv = h^{-1}f(u) \circ h^{-1}g(v), \qquad x \circ y = f^{-1}h(x) \cdot g^{-1}h(y) \tag{3.17}$$

(f^{-1}, g^{-1}, and h^{-1} are the inverse maps of f, g, and h, respectively, mapping **H** onto **G** in a 1-1 way). In fact, (3.15) and (3.17) imply (3.16):

$$h(x \circ y) = h(f^{-1}h(x) \cdot g^{-1}h(y)) = ff^{-1}h(x) * gg^{-1}h(y) = h(x) * h(y).$$

Thus familiarity with the loop-principal-isotopes puts the key to all loop isotopes in our hands, and this is the deeper cause of the importance of the *LP*-isotopes in our following proofs.

Figure 20 illustrates the result $x \circ y$ of the operation $\circ$. Here u, s

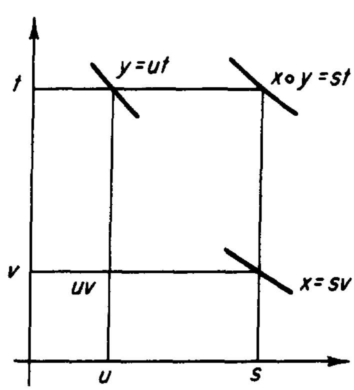

FIG. 20.

are on the horizontal, v, t on the vertical axis, while the contour lines on the figure have the marks x, y and $x \circ y$. In what follows, x or y can also denote z-values, but x_j and y_j will always denote x-values and y-values, respectively.

Proof of Theorem 3.1.

(I) $T \Rightarrow a$.

T implies the commutativity of all LP-isotopes of (**G**, $\cdot$). In fact, let $x_1 = u$, $y_1 = v$ be given. (**G**, $\cdot$) being a quasigroup, for arbitrary given x, $y \in$ **G**, there exist elements x_2, x_3, y_2, and y_3 (Fig. 21), such that

$$x = uy_2 = x_2v \quad \text{and} \quad y = uy_3 = x_3v\,; \tag{3.18}$$

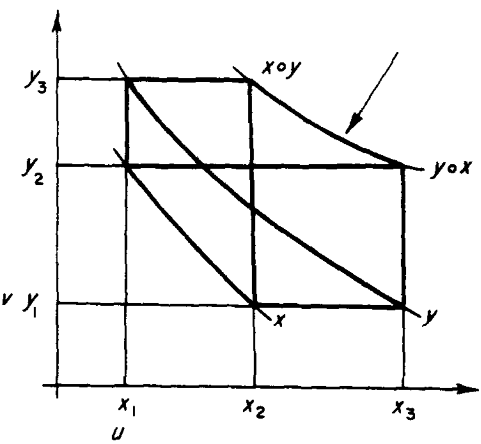

FIG. 21.

i.e.,

$$x_1y_2 = x_2y_1 \;\&\; x_1y_3 = x_3y_1$$

hold. But these are the hypotheses of T, so also its conclusion is valid

$$x_2y_3 = x_3y_2\,. \tag{3.19}$$

Now by (3.18) and (3.13)

$$x \circ y = x_2v \circ uy_3 = x_2y_3 \quad \text{and} \quad y \circ x = x_3v \circ uy_2 = x_3y_2\,,$$

so that (3.19) really states that all LP-isotopes (**G**, $\circ$) of (**G**, $\cdot$) are commutative:

$$x \circ y = y \circ x. \tag{3.20}$$

On the other hand, *if all LP-isotopes of* (**G**, $\cdot$) *are commutative,* then *they are all associative too* (while from the commutativity of a single

LP-isotope its associativity *does not* follow). In fact (cf. Pickert [2]), if also the operation $*$, defined with the aid of w instead of u ($v = y_1$ remains unchanged)

$$sv * wt = st,$$

is commutative

$$x * z = z * x, \tag{3.21}$$

then for arbitrary $x, y \in \mathbf{G}$ and given u, v, there are s, t, such that

$$x = sv, \qquad y = ut\,;$$

i.e.,

$$x \circ y = sv \circ ut = st = sv * wt = x * (wv \circ ut) = x * (q \circ y). \tag{3.22}$$

Here w can be chosen arbitrarily; thus also $q = wv$. More precisely, for every q one can define such an operation $*$ that (3.22) remains valid for all $x, y \in \mathbf{G}$. Now for arbitrary $p, q, r \in \mathbf{G}$—after having chosen $*$ so that (3.22) holds—we have

$$\begin{aligned}(p \circ q) \circ r = (q \circ p) \circ r &= (q \circ p) * (q \circ r) = (q \circ r) * (q \circ p)\\ &= (q \circ r) \circ p = p \circ (q \circ r),\end{aligned}$$

[by application of (3.20), (3.22), (3.21), (3.22), and (3.20)], so that every *LP*-isotope $(\mathbf{G}, \circ)$ of $(\mathbf{G}, \cdot)$ is not only commutative, but also associative. But the commutative and associative quasigroups are the Abelian groups and as all loop isotopes are isomorphic to an *LP*-isotope, therefore T implies that *all loops isotopic to* $(\mathbf{G}, \cdot)$ *are Abelian groups too* and so we have proved the statement (I), [even more, as in (I) only the Abelian group properties of *one* *LP*-isotope were stated].

Conversely,

$$\text{(II)} \quad a \Rightarrow T.$$

In fact, for that member of the class of isotopic quasigroups associated with the net, which by a is an Abelian group, the condition T is satisfied as

$$\begin{aligned}(x_1y_2 = x_2y_1 \;\&\; x_1y_3 = x_3y_1) &\Rightarrow x_1y_3(x_1y_2)^{-1} = x_3y_1(x_2y_1)^{-1}\\ &\Rightarrow y_3y_2^{-1} = x_3x_2^{-1} \Rightarrow x_2y_3 = x_3y_2\end{aligned}$$

(for an Abelian group $xy \cdot z = x \cdot yz = xyz$, $xy = yx$, $(xy)^{-1} = y^{-1}x^{-1} = x^{-1}y^{-1}$, etc.) and, as we have already remarked, if T holds for a quasigroup, then it holds for all its isotopes too, and so for all quasigroups of the class related to the net. Thus we have proved assertion (II) and finished the proof of Theorem 3.1.

From the above proof we can even derive the following:

Corollary 3. *If T is satisfied for one $y_1 \in \mathbf{G}$ and for all x_1, x_2, x_3, y_2, $y_3 \in \mathbf{G}$, then it holds as well for all x_1, x_2, x_3, y_1, y_2, $y_3 \in \mathbf{G}$.*

Proof of Theorem 3.2.

(I) $R \Rightarrow g$.

Let us fix $x_1 = u$ and $y_1 = v$ (Fig. 22). Then, by the quasigroup properties, for arbitrary x, y, and z there are x_2, x_3, x_4, y_2, y_3, and y_4 such that

(3.23) $$y = uy_2 = x_2v$$

(this determines x_2 and y_2),

(3.24) $$x = x_3v, \qquad z = uy_3$$

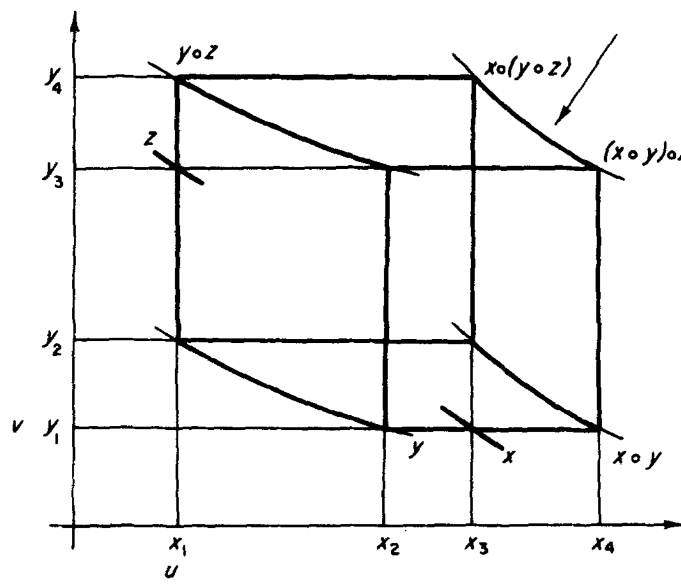

FIG. 22.

(these determine x_3 and y_3) and

$$x_2y_3 = uy_4\,, \qquad x_3y_2 = x_4v \tag{3.25}$$

(these determine x_4 and y_4). But (3.23) and (3.25) contain the hypotheses of R so that its conclusion holds too:

$$x_3y_4 = x_4y_3\,. \tag{3.26}$$

By applying consecutively (3.24), (3.23), (3.13), (3.25), (3.24), and (3.13) we get

$$(x \circ y) \circ z = (x_3v \circ uy_2) \circ z = x_3y_2 \circ z = x_4v \circ uy_3 = x_4y_3\,,$$

$$x \circ (y \circ z) = x \circ (x_2v \circ uy_3) = x \circ x_2y_3 = x_3v \circ uy_4 = x_3y_4\,,$$

so that (3.26) really yields the associativity of the operation $\circ$. As $(\mathbf{G}, \circ)$ is a quasigroup and associative quasigroups are groups, we have proved (I). If we now temporarily abandon the fixing of u and v, we get that all LP-isotopes of $(\mathbf{G}, \cdot)$ are groups and, since any isomorphic image of a group is a group, so *R implies that all loops isotopic to* $(\mathbf{G}, \cdot)$ *are groups*.

From the statement just proved part (I) of Theorem 3.1 follows, for by combining the above assertion with Theorem 2.1 we get $T \Rightarrow R \Rightarrow g$. But then T has to be fulfilled also for that member of the class of isotopic quasigroups, which is a group (a loop):

$$(x_1y_2 = x_2y_1 \,\&\, x_1y_3 = x_3y_1) \Rightarrow x_2y_3 = x_3y_2\,.$$

Let us choose $x_1 = y_1 = e$ (unit element), then

$$(y_2 = x_2 \,\&\, y_3 = x_3) \Rightarrow x_2y_3 = x_3y_2$$

or, equivalently,

$$x_2x_3 = x_3x_2$$

for all x_2, x_3, so that this group is Abelian and a is valid (moreover, we have derived again that all loop isotopes are commutative).

(II) $y \Rightarrow R$.

In fact, for that member of the class of isotopic quasigroups associated with the net, which by g is a group, the condition R is satisfied, as

$$(x_1y_2 = x_2y_1 \,\&\, x_1y_4 = x_2y_3 \,\&\, x_3y_2 = x_4y_1)$$
$$\Rightarrow x_3y_2(x_1y_2)^{-1}x_1y_4 = x_4y_1(x_2y_1)^{-1}x_2y_3 \Rightarrow x_3y_4 = x_4y_3$$

(since $xy \cdot z = x \cdot yz = xyz$, $(xy)^{-1} = y^{-1}x^{-1}$, etc., in groups). And, if R holds for a quasigroup, then it holds for all its isotopes too, so also for all quasigroups of the class associated with the net. Thus we have proved assertion (II) and finished the proof of Theorem 3.2.

From the above proof we can also derive the following:

Corollary 4. *If R is satisfied for one x_1, one y_1, and all x_2, x_3, x_4, y_2, y_3, and y_4 in G, then it holds as well for all x_1, x_2, x_3, x_4, y_1, y_2, y_3, and y_4.*

Proof of Theorems 3.3 and 3.4.

$$\text{(I)} \quad B_1 \Rightarrow m_1 .$$

Let $x_2 = u$, $Y_1 = v$, (Fig. 23) then for arbitrarily given x, y, and z there are x_1, x_4, y_1, y_2, y_3, y_4, X_2, X_4, and Y_3 such that

$$x = uY_3, \qquad z = x_4v \tag{3.27}$$

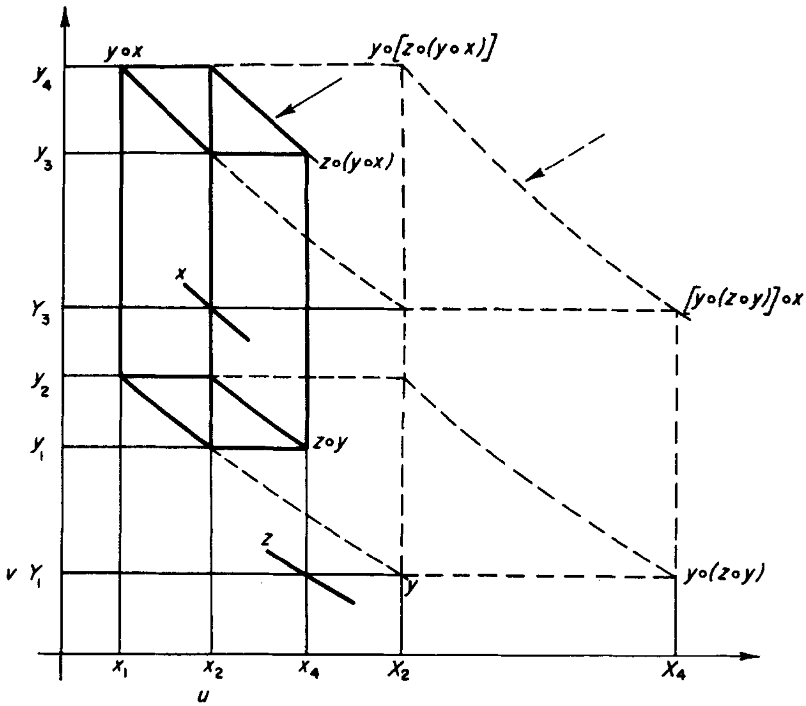

FIG. 23.

(these determine Y_3, x_4)

(3.28) $$y = uy_1 = x_2y_1 = X_2Y_1 = X_2v$$

(this determines y_1, X_2); further, taking (3.27), (3.28), and (3.13) into account

(3.29) $$z \circ y = x_4v \circ uy_1 = x_4y_1 = x_2y_2 = uy_2$$

(this determines y_2) and

(3.30) $$y \circ (z \circ y) = X_2v \circ uy_2 = X_2y_2 = X_4Y_1 = X_4v$$

(this determines X_4); also in complementing (3.28) we set

(3.31) $$y = x_2y_1 = X_2Y_1 = x_1y_2$$

(this determines x_1) and with reference to (3.13), (3.27), and (3.28)

(3.32) $$y \circ x = X_2v \circ uY_3 = X_2Y_3 = x_1y_4 = x_2y_3 = uy_3$$

(this finally determines y_4 and y_3). By (3.31), (3.32), and (3.29) the hypotheses of B_1 are fulfilled:

$$x_1y_2 = x_2y_1 \;\&\; x_1y_4 = x_2y_3 \;\&\; x_2y_2 = x_4y_1,$$

so its conclusion has to be valid too:

$$x_2y_4 = x_4y_3.$$

But then, taking also (3.27), (3.32), and (3.13) into account

$$z \circ (y \circ x) = x_4v \circ uy_3 = x_4y_3 = x_2y_4 = uy_4$$

and with (3.28) and (3.13)

(3.33) $$y \circ (z \circ (y \circ x)) = X_2v \circ uy_4 = X_2y_4.$$

On the other hand, (3.31), (3.32), and (3.30) show that

$$x_1y_2 = X_2Y_1 \;\&\; x_1y_4 = X_2Y_3 \;\&\; X_2y_2 = X_4Y_1$$

i.e., the hypotheses of B_1 are satisfied this time for x_1, X_2, X_4, Y_1, y_2, Y_3, and y_4, so that its conclusion holds too:

$$X_2y_4 = X_4Y_3.$$

But then (3.30), (3.27), (3.13), and (3.33) imply

$$(y \circ (z \circ y)) \circ x = X_4 v \circ u Y_3 = X_4 Y_3 = X_2 y_4 = y \circ (z \circ (y \circ x)),$$

so that the loop (**G**, $\circ$) has the property (3.4), i.e., it is an M_1-loop. This proves (I).

B_1 can be transformed into B_2 by interchanging the roles of the x's and the y's (geometrically— see Figs. 14 and 15—this means the interchanging of the horizontal and vertical axes). A similar change transforms

$$x \circ y = sv \circ ut = st$$

into

$$y \circ x = t'v \circ us' = t's'$$

(cf. with Fig. 24, where again the horizontal and vertical axes were

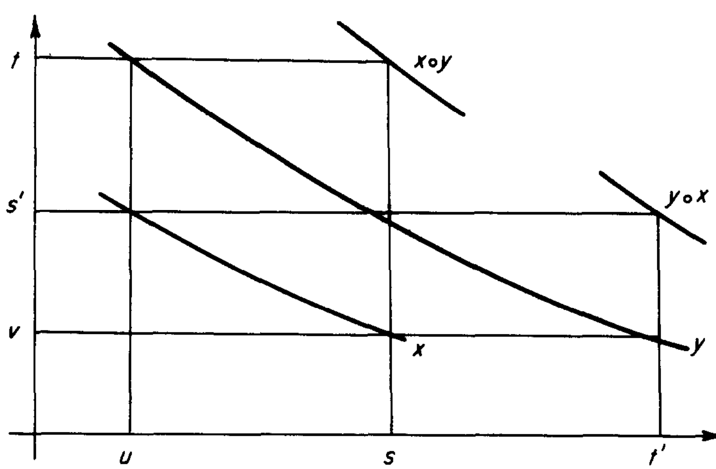

FIG. 24.

interchanged when going over from $x \circ y$ to $y \circ x$), so that it also transforms the property

$$y \circ (z \circ (y \circ x)) = (y \circ (z \circ y)) \circ x$$

which we have just proved and which corresponds in (**G**, $\circ$) to (3.4) belonging to m_1, into

$$((x \circ y) \circ z) \circ y = x \circ ((y \circ z) \circ y)$$

which corresponds in (G, ∘) to (3.5) belonging to m_2. So (I) also has

$$B_2 \Rightarrow m_2,$$

as a consequence and owing to the definition of B and to Lemma 3.2

$$B \Rightarrow m$$

as stated in Theorem 3.4 and in Corollary 2, respectively.

If u, v run through **G**, then we see that (3.4), (3.5), or (3.6), respectively, are fulfilled in all LP-isotopes of (**G**, ·) and as these identities carry over to isomorphic loops, therefore B_1, B_2, *or* B *imply that all loops isotopic to* (**G**, ·) *are* M_1*-loops,* M_2*-loops, or Moufang loops, respectively.*

For the same reason as above it is enough to prove the first one of the converse statements

$$m_1 \Rightarrow B_1, \qquad m_2 \Rightarrow B_2, \qquad m \Rightarrow B.$$

$$\text{(II)} \quad m_1 \Rightarrow B_1.$$

To prove this, we recall that in part (II) of the proof of Lemma 3.2 we have derived

$$y \cdot y^{-1}w = w \tag{3.12}$$

from (3.4). On the other hand $zy = e$ (i.e., $z = y^{-1}$) and $yx = e$ in (3.4) gives

$$yz = e$$

(the left inverse is at the same time also a right inverse), so that we have $y = z^{-1}$ which transforms (3.12) into

$$z^{-1} \cdot zw = w \tag{3.34}$$

[cf. (3.9)]. Thus in that quasigroup of the net, which is an M_1-loop, (3.34) holds too. But then also B_1 is satisfied in this quasigroup:

$$x_1y_2 = x_2y_1, \tag{3.35}$$

$$x_1y_4 = x_2y_3, \tag{3.36}$$

and

$$x_2y_2 = x_4y_1 \tag{3.37}$$

together imply

$$x_2y_4 = x_4y_3 . \tag{3.38}$$

In fact, by multiplying (3.35) from the left by x_1^{-1}, (3.34) gives

$$y_2 = x_1^{-1} \cdot x_2y_1$$

and from (3.37), by taking (3.4) into account, we get

$$x_4y_1 = x_2y_2 = x_2(x_1^{-1} \cdot x_2y_1) = (x_2 \cdot x_1^{-1}x_2)y_1 .$$

Now equations of the form

$$xy_1 = z$$

have unique solutions x, and so

$$x_4 = x_2 \cdot x_1^{-1}x_2 ,$$

so that (3.4), (3.36), and (3.34) imply

$$x_4y_3 = (x_2 \cdot x_1^{-1}x_2)y_3 = x_2(x_1^{-1} \cdot x_2y_3) = x_2(x_1^{-1} \cdot x_1y_4) = x_2y_4$$

and this is (3.38) which shows that B_1 really holds in this quasigroup. But then B_1 holds also in all isotopic quasigroups of the class associated with the net. This proves (II) and also

$$m_2 \Rightarrow B_1 , \qquad m \Rightarrow B$$

as mentioned above, so that we have finished the proof of Theorems 3.3 and 3.4 and of corollary 2.

By comparing the statements (II) in the proofs of the Theorems 3.1–3.4 with the statements in italics at the ends of the parts (I) of the same proofs, we see that *one* quasigroup of the class associated with the net being an Abelian group or group or Moufang loop or M_1-loop or M_2-loop already establishes that T, R, B, B_1 , and B_2 are satisfied, respectively, and this implies that *all* loops of the class of isotopic quasigroups associated with the nets are, respectively, Abelian groups, groups, Moufang loops, M_1-loops, or M_2-loops too. Thus (cf. Bruck [1] and Kertész [1]) we have the following:

Corollary 5. *Every loop isotopic to an Abelian group or to a group or to a Moufang loop or M_1- or M_2-loop, respectively, is also itself an Abelian group; resp., a group; resp., a Moufang loop; an M_1- or M_2-loop.*

It might be interesting to observe that while the identities (3.4) and (3.5) corresponded to the closure conditions B_1 and B_2, *there is no simple identity known which is equivalent to* B_3.

Proof of Theorem 3.5.

$$\text{(I)} \quad H \Rightarrow p.$$

For arbitrary x,

$$x_1 = u, \qquad y_1 = v, \tag{3.39}$$

we can choose x_2, x_3, y_2, y_3 so that

$$x = uy_2 = x_2v \tag{3.40}$$

(this determines x_2, y_2) and then [cf. (3.13)]

$$x \circ x = x_2v \circ uy_2 = x_2y_2 = uy_3 = x_3v \tag{3.41}$$

(which determines x_3, y_3). By (3.39), (3.40), (3.41) the hypotheses of H are fulfilled (cf. Fig. 25):

$$x_1y_2 = x_2y_1 \,\&\, x_1y_3 = x_2y_2 = x_3y_1\,, \tag{3.42}$$

so that its conclusion holds too:

$$x_2y_3 = x_3y_2\,. \tag{3.43}$$

From (3.40) and (3.41) we get with reference to (3.13)

$$(x \circ x) \circ x = x_3v \circ uy_2 = x_3y_2 \tag{3.44}$$

and

$$x \circ (x \circ x) = x_2v \circ uy_3 = x_2y_3\,, \tag{3.45}$$

therefore (3.43) gives

$$(x \circ x) \circ x = x \circ (x \circ x). \tag{3.46}$$

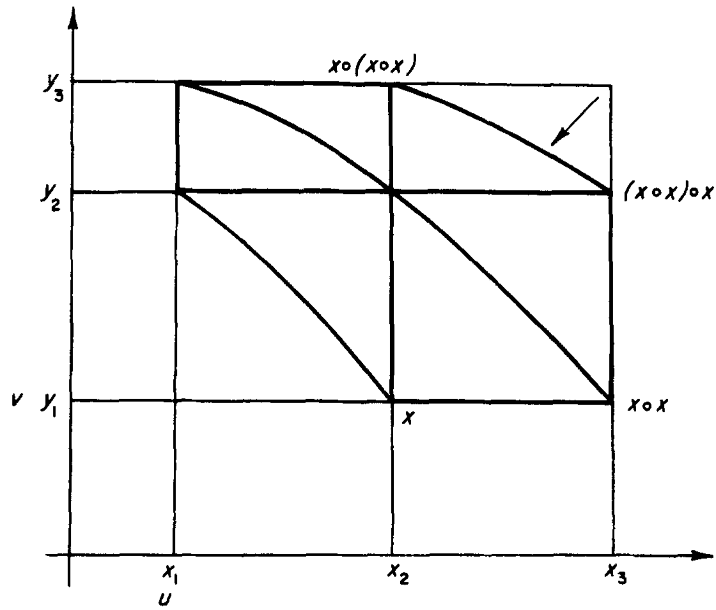

FIG. 25.

So that—u and v being arbitrary—all LP-isotopes are power-associative and as (3.46) remains true for isomorphic loops, all loops isotopic to (**G**, $\cdot$) will be power-associative, which proves (I).

(II) $p \Rightarrow H$.

It is enough to suppose, that (3.46) holds for all LP-isotopes (**G**, $\circ$) and show that then H is valid for all quasigroups of the class. We prove this assertion with the same formulas (3.39)–(3.46) with which we have proved (I). But now for given x_1, x_2, x_3, y_1, y_2, and y_3, which fulfill the hypotheses of H, (3.39) serves to present those u and v with which we construct (3.13), i.e., with which we select an LP-isotope of the given quasigroup; (3.40) defines x [$uy_2 = x_2v$ holds by (3.39) and by (3.42), i.e., by a hypothesis of H], and (3.41) follows from (3.13) and (3.42). Finally, reading (3.44) and (3.45) from the right to the left, they are consequences of (3.13), (3.40), and (3.41), and so (3.46) implies (3.43), so that the conclusion of H is valid too. This finishes the proof of (II) and at the same time that of Theorem 3.5.

If we define the "powers" of an element x of a loop in a natural way by

$$x^0 = e, \quad x^1 = x, \quad x^2 = xx, \quad x^3 = xx^2, \ldots,$$

in general by

$$x^{n+1} = xx^n \qquad (n = 0, 1, 2, \ldots)$$

for nonnegative integral exponents and by

$$x^n = (x^{-1})^{-n} \qquad (n < 0)$$

for negative integer exponents, where x^{-1} is the left inverse of x

$$x^{-1}x = e,$$

then the power-associativity

$$xx \cdot x = x \cdot xx \tag{3.3}$$

can be written as

$$x^2x^1 = x^3 \tag{3.47}$$

and the fact, proved in part (II) of Theorem 3.3 for M_1-loops, that *the left inverse is also a right inverse*

$$xx^{-1} = e$$

can be written as

$$x^1x^{-1} = x^0. \tag{3.48}$$

Both (3.47) and (3.48) are special cases of the *strong power-associativity*

$$x^mx^n = x^{m+n} \qquad \text{for all integers } m, n. \tag{3.49}$$

We will prove in Theorem 3.7 that also (3.49) follows for all loops isotopic to a quasigroup fulfilling H. But first we will prove the interesting special case that the *property*

$$x^{-1}x = e \Rightarrow xx^{-1} = e \tag{3.50}$$

follows not only from (3.4) or (3.6) but also from H. If in all loops of the class associated with a net the property (3.50) is valid, then we will say that the net has the *property i*. So we prove the following

THEOREM 3.6. $i \Leftrightarrow H$.

Proof.

(I) $H \Rightarrow i$.

For arbitrary $x_2 = u$, $y_2 = v$ (Fig. 26) we can choose x_1, y_1, x_3, and y_3 such that

$$x = x_1 v = u y_1 \tag{3.51}$$

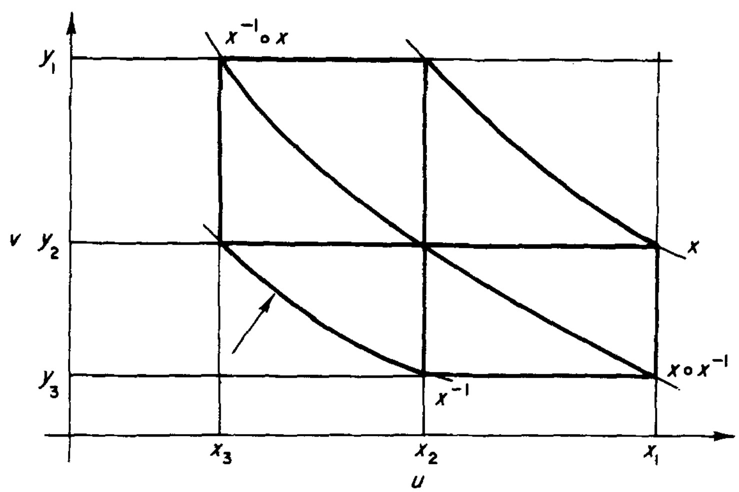

FIG. 26.

(this determines x_1, y_1) and

$$uv = x_1 y_3 = x_3 y_1 \tag{3.52}$$

(this determines x_3, y_3). Equations (3.51) and (3.52) show that the hypotheses of H are fulfilled, so its conclusion has to hold too:

$$u y_3 = x_3 v. \tag{3.53}$$

As we have seen before the proof of Theorem 3.1,

$$uv = e \tag{3.54}$$

is a unit element of the LP-isotope $(\mathbf{G}, \circ)$ defined by (3.13). We define x^{-1} which is to be inverse element of x in $(\mathbf{G}, \circ)$ [and not in $(\mathbf{G}, \cdot)$, which need not even be a loop] by

(3.55) $$x^{-1} = x_3 v = u y_3 ,$$

which we can do after having established (3.53). For this x^{-1} we really have

$$x^{-1} \circ x = x_3 v \circ u y_1 = x_3 y_1 = uv = e$$

and

$$x \circ x^{-1} = x_1 v \circ u y_3 = x_1 y_3 = uv = e$$

by (3.51), (3.55), (3.13), (3.52), and (3.54). The equation $y \circ x = e$ having a unique solution y, this is just the x^{-1} defined by (3.55) and we have $x \circ x^{-1} = e$ for this x^{-1}, so that—as u, v are arbitrary—the property (3.50) is valid for every LP-isotope of $(\mathbf{G}, \cdot)$. As this property carries over to isomorphic loops, so it remains valid in all loops of the class associated with the net, which proves (I).

(II) $i \Rightarrow H$.

H already follows from the fact that the left inverse of every element of each LP-isotope is equal to the right inverse. In fact for given x_1, x_2, x_3, y_1, y_2, and y_3 for which the hypotheses

(3.56) $$x_1 y_2 = x_2 y_1 \;\&\; x_1 y_3 = x_2 y_2 = x_3 y_1$$

of H are fulfilled, let

(3.57) $$u = x_2 , \qquad v = y_2 , \qquad x = x_1 v = u y_1$$

and define

(3.58) $$y = x_3 v, \qquad z = u y_3 .$$

Then (3.54), (3.57), and (3.56) imply

$$e = uv = x_1 y_3 = x_3 y_1$$

and because of (3.58), (3.57), and (3.13)

$$y \circ x = x_3 v \circ u y_1 = x_3 y_1 = e,$$

$$x \circ z = x_1 v \circ u y_3 = x_1 y_3 = e,$$

so that y is the left inverse, z the right inverse of x in $(\mathbf{G}, \circ)$. i states that $z = y$, *i.e.*, by (3.58) and (3.57)

$$x_2 y_3 = x_3 y_2 ,$$

so that the conclusion of H holds too, which finishes the proof of (II) and at the same time that of Theorem 3.6.

From Theorems 3.5 and 3.6 it follows that

Corollary 6. $p \Leftrightarrow i$.

Now we go over to the strong power-associativity

$$x^m \circ x^n = x^{m+n} \qquad \text{for all integers } m, n \tag{3.59}$$

relating to the powers defined by

$$\begin{aligned} x^{n+1} &= x \circ x^n \qquad (n = 0, 1, 2, \ldots), \qquad x^0 = e = x^{-1} \circ x, \\ x^n &= (x^{-1})^{-n} \qquad (n = -1, -2, \ldots). \end{aligned} \tag{3.60}$$

If for all elements of every loop of the class associated with a net, (3.59) holds under the definition (3.60), then we say, that the net has *property s*.

THEOREM 3.7. $s \Leftrightarrow H$.

Since owing to Theorems 3.5 and 3.6 H follows from p or i, which are special cases of s, we need not prove $s \Rightarrow H$. We prove $H \Rightarrow s$ in four steps:

Step (A). *From H it follows for all elements of every LP-isotope $(\mathbf{G}, \circ)$ of $(\mathbf{G}, \cdot)$ that*

$$x^m \circ x^n = x^{m+n} \qquad \text{for all} \quad m \geqslant 0, n \geqslant 0. \tag{3.61}$$

We prove this by a somewhat unusual induction with respect to m and n: (3.61) holds for $m = 0$, n arbitrary and for $n = 0$, m arbitrary because of the definition (3.60) and of the loop property analogous to (3.2) in $(\mathbf{G}, \circ)$:

$$x^0 \circ x^n = e \circ x^n = x^n = x^{0+n}, \qquad x^m \circ x^0 = x^m \circ e = x^{m+0};$$

(3.61) is also valid for $m = 1$, n arbitrary because of (3.60):

$$x^1 \circ x^n = x \circ x^n = x^{n+1}.$$

We prove that the validity of (3.61) for $m = M - 1$, n arbitrary, for $m = M$, n arbitrary, and for $m = M + 1$, $n = N - 1$ implies also its validity for

$$m = M + 1, \qquad n = N.$$

From this, (3.61) really follows for all nonnegative m, n (Fig. 27). It is true as we have just seen for $m = 0$, n arbitrary and for $m = 1$,

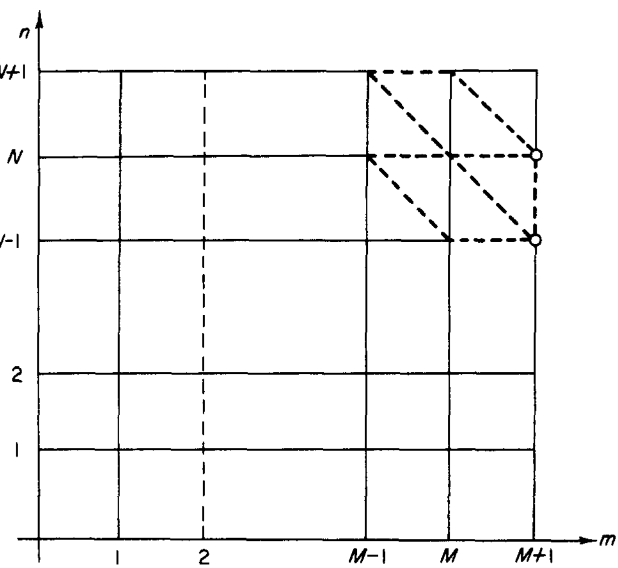

FIG. 27.

n arbitrary; it holds for $m = 2$, $n = 0$, thus ($M = 1$, $N = 1$) it holds for $m = 2$, $n = 1$ too, then for $m = 2$, $n = 2$ ($M = 1$, $N = 2$), etc., and if it is true for all $m \leqslant M$ and all n, then—being valid also for $m = M + 1, n = 0$—it remains true for $m = M + 1, n = 1$ ($N = 1$), from this for $m = M + 1$, $n = 2$ ($N = 2$), etc., and from the validity for $m = M + 1$ and for all $n \leqslant N - 1$ its validity for $m = M + 1$, $n = N$ follows too, and so on.

In our case we work out this reasoning by induction as follows: u, v, x^{M-1}, x^M, x^{M+1}, x^{N-1}, x^N, and x^{N+1} are given; by the hypotheses of induction

$$\begin{aligned} &x^{M-1} \circ x^N = x^{M+N-1} = x^M \circ x^{N-1}, \\ (3.62)\qquad &x^{M-1} \circ x^{N+1} = x^{M+N} = x^M \circ x^N = x^{M+1} \circ x^{N-1} \end{aligned}$$

and

$$x^M \circ x^{N+1} = x^{M+N+1} \tag{3.63}$$

x_1, x_2, x_3, y_1, y_2, and y_3 can be determined so that (Fig. 28)

$$x^{M-1} = x_1 v, \qquad x^M = x_2 v, \qquad x^{M+1} = x_3 v \tag{3.64}$$

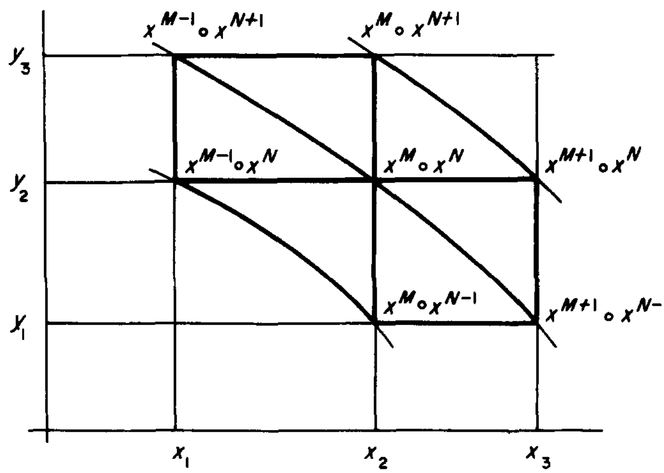

FIG. 28.

(which determine x_1, x_2, and x_3)

$$x^{N-1} = uy_1, \qquad x^N = uy_2, \qquad x^{N+1} = uy_3 \tag{3.65}$$

(which determine y_1, y_2, y_3). Then (3.13) and (3.62) imply

$$x_1 y_2 = x_1 v \circ uy_2 = x^{M-1} \circ x^N = x^M \circ x^{N-1} = x_2 v \circ uy_1 = x_2 y_1,$$

$$x_1 y_3 = x_1 v \circ uy_3 = x^{M-1} \circ x^{N+1} = x^M \circ x^N = x_2 v \circ uy_2 = x_2 y_2$$
$$= x^{M+1} \circ x^{N-1} = x_3 v \circ uy_1 = x_3 y_1,$$

so that the hypotheses of H are fulfilled and also its conclusion has to be valid:

$$x_2 y_3 = x_3 y_2 .$$

Thus (3.64), (3.65), (3.13), and (3.63) imply

$$x^{M+1} \circ x^N = x_3 v \circ uy_2 = x_3 y_2 = x_2 y_3 = x_2 v \circ uy_3 = x^M \circ x^{N+1} = x^{M+N+1}$$

and this was the assertion to be proved. Therefore, (3.61) is valid for $m = M + 1$, $n = N$. Since by varying u and v we can form all LP-isotopes, we have proved step (A).

Step (B). *From H it follows for all elements of every LP-isotope that*

$$x^m \circ x^n = x^{m+n} \qquad \text{for all} \quad m < 0, n < 0.$$

Because of (3.60) this follows immediately from step (A) if we replace x by x^{-1}.

Step (C). *From H it follows for all elements of every LP-isotope that*

$$x^m \circ x^n = x^{m+n} \qquad \text{for all} \quad m < 0, n > 0.$$

In order to establish this, we will need a new form of condition H (cf. Fig. 29, which shows that it states the closure of the hexagonal figure at its "medial 3-curve"):

Lemma 3.3. *H is equivalent to the validity of*

$$H': \quad (x_1 y_2 = x_2 y_1 \;\&\; x_2 y_3 = x_3 y_2 \;\&\; x_2 y_2 = x_3 y_1) \Rightarrow x_1 y_3 = x_2 y_2$$

for all $x_1, x_2, x_3, y_1, y_2, y_3$

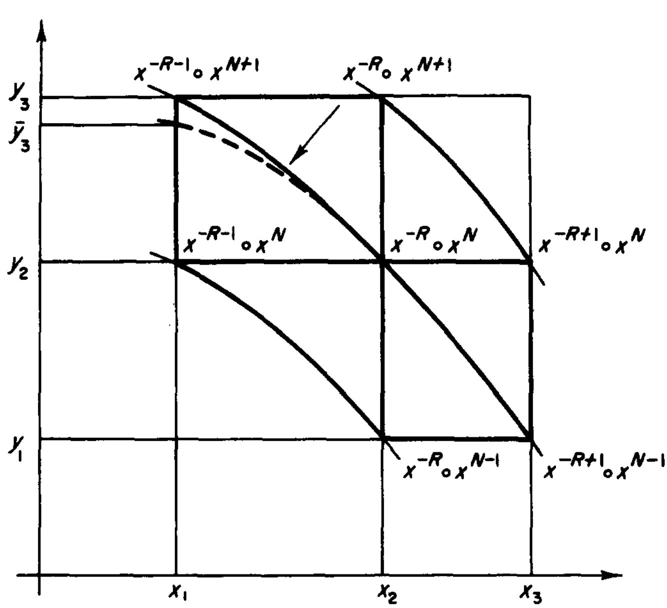

FIG. 29.

Proof of Lemma 3.3.

(I) $H \Rightarrow H'$

Let the hypotheses of H' be valid:

$$x_1y_2 = x_2y_1\,, \qquad x_2y_2 = x_3y_1,$$

(3.66) $$x_2y_3 = x_3y_2$$

and define $\bar{y}_3$ so that

(3.67) $$x_1\bar{y}_3 = x_2y_2\,.$$

But this states that for x_1, x_2, x_3, y_1, y_2, $\bar{y}_3$ the hypotheses of H are valid:

$$x_1y_2 = x_2y_1 \;\&\; x_1\bar{y}_3 = x_2y_2 = x_3y_1\,,$$

then its conclusion holds too:

$$x_2\bar{y}_3 = x_3y_2\,.$$

If we compare this with (3.66) we get $\bar{y}_3 = y_3$, so that (3.67) goes over into

$$x_1y_3 = x_2y_2$$

which is the conclusion of H' proving (I).

(II) The proof of $H' \Rightarrow H$ can be established similarly, which finishes the proof of Lemma 3.3.

Continuation of the proof of Theorem 9: *proof of Step* (*C*). We have to prove that

(3.68) $$x^{-r} \circ x^n = x^{-r+n} \qquad (r > 0, n \geqslant 0)$$

holds for all elements of every LP-isotope. This we will prove again by an induction with respect to r and n: (3.68) is true for $r = -1$, n arbitrary, and $r = 0$, n arbitrary by Step (A), further it is true for $n = 0$, r arbitrary because of $x^0 = e$. We prove that

(3.69) $$x^{-R+1} \circ x^n = x^{-R+n+1}$$
(3.70) $$x^{-R} \circ x^n = x^{-R+n} \qquad (n = 0, 1, 2, \ldots)$$

and

$$x^{-R-1} \circ x^{N} = x^{-R+N-1} \tag{3.71}$$

imply

$$x^{-R-1} \circ x^{N+1} = x^{-R+N}. \tag{3.72}$$

This (Fig. 30) will prove (3.68) for all positive (nonnegative) r, n.

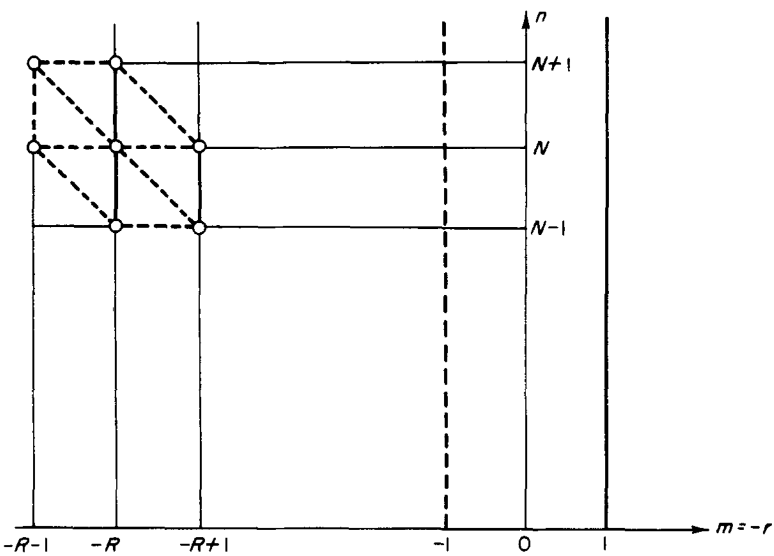

FIG. 30.

In order to carry out our project we choose x_1, x_2, x_3, y_1, y_2, and y_3 with given u, v, x^{-R-1}, x^{-R}, x^{-R+1}, x^{N-1}, x^{N}, and x^{N+1} so that (Fig. 29)

$$x^{-R-1} = x_1 v, \qquad x^{-R} = x_2 v, \qquad x^{-R+1} = x_3 v,$$

$$x^{N-1} = u y_1, \qquad x^{N} = u y_2, \qquad x^{N+1} = u y_3,$$

then (3.13) and (3.71), (3.70), and (3.69) imply

$$x_1 y_2 = x_1 v \circ u y_2 = x^{-R-1} \circ x^{N} = x^{-R+N-1} = x^{-R} \circ x^{N-1} = x_2 v \circ u y_1 = x_2 y_1,$$

$$x_2 y_3 = x_2 v \circ u y_3 = x^{-R} \circ x^{N+1} = x^{-R+N+1} = x^{-R+1} \circ x^{N} = x_3 v \circ u y_2 = x_3 y_2,$$

$$x_2 y_2 = x_2 v \circ u y_2 = x^{-R} \circ x^{N} = x^{-R+N} = x^{-R+1} \circ x^{N-1} = x_3 v \circ u y_1 = x_3 y_1,$$

so that the hypotheses of H' are fulfilled and its conclusion also holds:

$$x_1 y_3 = x_2 y_2$$

and so

$$x^{-R-1} \circ x^{N+1} = x_1 v \circ u y_3 = x_1 y_3 = x_2 y_2 = x_2 v \circ u y_2 = x^{-R} \circ x^{N} = x^{-R+N}$$

This is (3.72), which was to be proven.

[Our reasoning remains valid for $r = -1$, because (3.69) and (3.70) are trivial for $R = 0$, while for $R = 0$, $N = 1$ we get (3.71) from (3.60); but then the above considerations yield (3.72) so that (3.71) remains true for $N = 2$ and again because of (3.72) also for $N = 3$, etc.]. This proves Step (C).

Step (D). *From H it follows for all elements of every LP-isotope that*

$$x^m \circ x^n = x^{m+n} \qquad \text{for all} \quad m \geqslant 0, n < 0.$$

This statement follows from step (C) by changing x into x^{-1} since by Theorem 3.6 H implies i, i.e., the validity of (3.50) for all LP-isotopes and so $(x^{-1})^{-1} = x$.

Now we have proved (3.59) for all LP-isotopes and for all integers m, n and, since obviously this property carries over to isomorphic loops, (3.59) remains true for all loops of the class associated with the net, which finishes the proof of Theorem 3.7. (For Theorems 3.1–3.7 cf. Pickert [2] and Aczél, Pickert and Radó [1], for theorems 3.5–3.7 also Pickert [1] and Radó [1].) From Theorems 3.5 and 3.7 we can derive a stronger statement than corollary 6:

Corollary 7. $p \Leftrightarrow s$.

That s implies p is obvious. Much the more surprising is that the validity of the weaker condition (3.3) for all loops of the class implies for the same loops the much more general condition (3.59). Of course the validity of (3.3) in a single loop does not imply that of (3.59) in the same loop.

4. Other Definitions of Algebraic Nets

There are also other possible definitions of algebraic nets. For example, in Aczél, Pickert, and Radó [1] we have associated only a single quasigroup with the net instead of the whole class of isotopic quasigroups.

On the other hand, it is possible to relate only loops with a net (a class of all isotopic loops or just one single loop with each; see, e.g., Pickert [2] and Bruck [1]; the theorems in Section 3 and the definitions of the different kinds of regularity also refer just to the loops of the class). This is equivalent to the following: Instead of associating (**G**, $\cdot$) with a net, we associate with it the *LP*-isotope (**G**, $\circ$) defined by (3.13). The intuitive meaning of this in Fig. 20 is that we translate the vertical axis by u and the horizontal one by v so that the origin is transferred into the point $e = uv$, i.e., into the unit element (Fig. 31). This relates

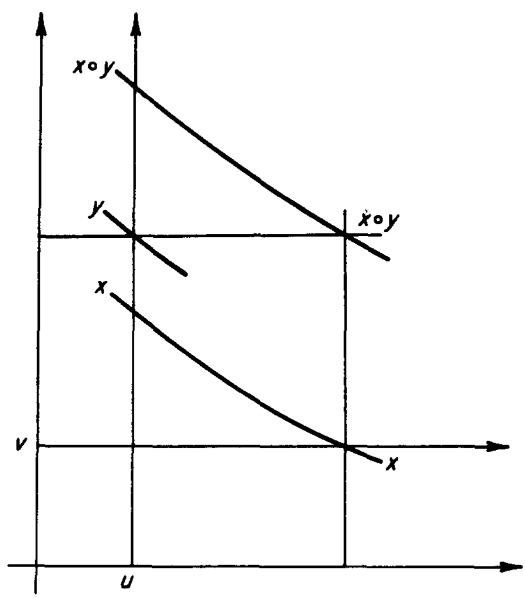

FIG. 31.

to another abstract treatment of nets, to which we actually referred at the beginning of Section 3 and which is a direct generalization of the definition of geometric nets without the continuity hypothesis. We define a 3-net to be a nonvoid set **H**—the elements of which are called points — together with three classes of its subsets; we call these subsets *curves*, in particular the three classes are the 1-curves, 2-curves, and 3-curves, if the following conditions are fulfilled. Every point of **H** is contained in exactly one i-curve ($i = 1, 2, 3$) and an i-curve has exactly one common point with a j-curve if $i \neq j$. (Instead of elements and "subsets" one can begin by introducing two kinds of elements, "points" and "curves," and then instead of "containing" one has to introduce a new relation of "incidence" between "points" and "curves.")

These conditions imply the existence of 1-1 maps of the set of i-curves onto that of j-curves (i, $j = 1, 2, 3$). Therefore, each of the three sets can be mapped in a 1-1 way onto the same set **G**. In this set **G** we can define an operation by calling $z = xy$ the element $z \in \mathbf{G}$ associated with the 3-curve containing the common point of the 1-curve associated with $x \in G$ and of the 2-curve associated with $y \in \mathbf{G}$. **G** is a quasigroup with respect to this operation. The above choice of the set **G** and thus also of the operation in it is not unique, but all quasigroups so determined are isotopic. And this actually leads us back to the representation in Section 3. (This is essentially the process followed in Bruck [1].)

But we can even directly associate a loop with the net, also defining the operation in it in a more "geometrical" way. We can prescribe which element of **G** is to be the unit element of the loop and which point is to be the unit point (i.e., the common point of the 1- and 2-curves belonging to the unit element). If the prescribed point of **H** was contained originally in the 1-curve belonging to u and in the 2-curve belonging to v, then we distinguish these two curves. By a further 1-1 mapping of the set of 1-, 2-, and 3-curves (i.e., by isotopy) we can arrange that these distinguished 1- and 2-curves and the 3-curve containing the prescribed point be associated with the same element of **G**. For this purpose it is sufficient to make the set of 3-curves correspond in such a manner to **G** that the 3-curve containing the prescribed point be associated with the element of **G** prescribed to be the unit and then to make the 1- and 2-curves correspond in such a manner to the elements of **G** that to every i-curve ($i = 1, 2$) that element of **G** be associated which was related to a 3-curve containing the common point of that i-curve and of the distinguished j-curve ($j \neq i$, i.e., $j = 2$ if $i = 1$ and $j = 1$ if $i = 2$). All this is represented in Fig. 31 and the above description just expresses abstractly the earlier mentioned "translation" of the horizontal and vertical axes. If we define now the result $x \circ y$ of the operation $\circ$ as the element of **G** associated in this new relationship with the 3-curve containing the common point of the 1-curve belonging to x and of the 2-curve belonging to y, then this operation can be geometrically represented by the following "addition of segments":

We identify the points of the distinguished 1-curve—in what follows, we call it the "axis"— with the elements of **G** newly associated with the

2- or 3-curves containing them. We take the x- and y-element on the axis and $x \circ y$ will be the common point of the axis and of the 3-curve passing through the intersection of the 2-curve belonging to y and thus actually containing y with the 1-curve containing the common point of the 3-curve associated with x and of the distinguished 2-curve (cf. Fig. 31; if the 3-curves are parallel straight lines there, then $x \circ y = x + y$ and this explains the terminology "addition of segments"). One sees immediately that e is a unit element of the quasigroup (**G**, $\circ$) thus constructed so that (**G**, $\circ$) is a loop. The comparison of Figs. 20 and 31 also indicates that (**G**, $\circ$) actually is the *LP*-isotope defined by (3.13) which fact can be proved algebraically without difficulty (see Aczél, Pickert, and Radó [1]). It is instructive to draw into the figures illustrating Theorems 3.1–3.6 these "additions of segments" which make their relation to (**G**, $\circ$) still more intuitive.

Essentially, the above construction is given in Bol [1], Thomsen [1], and Bruck [1], further in Blaschke and Bol [1] and Pickert [2], but in Bol [1] and Thomsen [1] the orders of x and y are reversed, while in Blaschke and Bol [1] and Pickert [2] the quasigroup (**G**, $\backslash$) figures instead of (**G**, $\cdot$), where the operation $\backslash$ is defined by the unique solution $y = x \backslash z$ of the equation $z = xy$. These suggest further definitions of algebraic nets: those in which (**G**, $\cdot$) is replaced by (**G**, $\backslash$) or by (**G**, $/$) where $/$ is defined by the unique solution $x = z/y$ of $z = xy$ and those built with the further three quasigroups arise by interchanging the factors in these three operations. These quasigroups are called "*parastrophic*" to the original (**G**, $\cdot$). [A further possibility would be to relate with a net not only the quasigroups isotopic to (**G**, $\cdot$), but also all those isotopic with all six parastrophic quasigroups.] The transition to parastrophic quasigroups corresponds to the interchange of the 1-, 2-, and 3-curves of the net (six possibilities).

The closure conditions are hereditary under these changes, only B_i goes over into B_j while interchanging the i- and j-curves ($i, j = 1, 2, 3$). As an example we show how H carries over from (**G**, $\cdot$) into (**G**, $\backslash$) (cf. Fig. 32). More exactly our result is expressed by:

THEOREM 4.1. *If H is fulfilled in* (**G**, $\cdot$) *for all x_j, y_j, then it holds in* (**G**, $\backslash$) *too*:

$$(x_1 \backslash y_2 = x_2 \backslash y_1 \,\&\, x_1 \backslash y_3 = x_2 \backslash y_2 = x_3 \backslash y_1) \Rightarrow x_2 \backslash y_3 = x_3 \backslash y_2$$

for arbitrary x_1, x_2, x_3, y_1, y_2, and y_3.

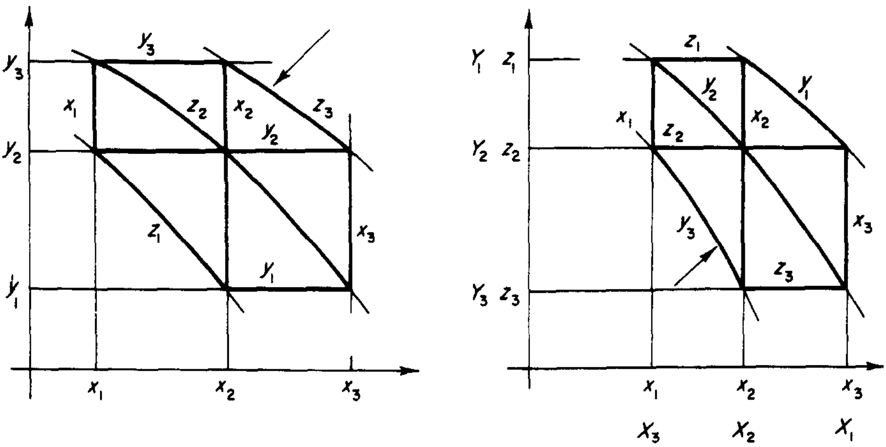

FIG. 32.

In fact, let us suppose that H is fulfilled in $(\mathbf{G}, \cdot)$ for all x_j, y_j and let us consider arbitrary $x_1, x_2, x_3, y_1, y_2, y_3$ satisfying

(4.1) $$x_1 \backslash y_2 = x_2 \backslash y_1$$

(4.2) $$x_1 \backslash y_3 = x_2 \backslash y_2 = x_3 \backslash y_1 .$$

We have to prove that then

(4.3) $$x_2 \backslash y_3 = x_3 \backslash y_2$$

holds too. Let us denote

$$z_1 = x_1 \backslash y_2 = x_2 \backslash y_1 , \qquad z_2 = x_1 \backslash y_3 = x_2 \backslash y_2 = x_3 \backslash y_1 .$$

and

(4.4) $$z_3 = x_3 \backslash y_2 .$$

Then the definition of the operation $\backslash$ implies

$$y_1 = x_2 z_1 = x_3 z_2 , \qquad y_2 = x_1 z_1 = x_2 z_2 = x_3 z_3$$

and

(4.5) $$y_3 = x_1 z_2 .$$

With the notations

$$(4.6) \qquad X_1 = x_3, \quad X_2 = x_2, \quad X_3 = x_1, \quad Y_1 = z_1, \quad Y_2 = z_2, \quad Y_3 = z_3$$

we thus have

$$X_1Y_2 = X_2Y_1 \,\&\, X_1Y_3 = X_2Y_2 = X_3Y_1,$$

and these are just the hypotheses of H for X_1, X_2, X_3, Y_1, Y_2, and Y_3. Therefore the conclusion

$$X_2Y_3 = X_3Y_2$$

of H holds too. With (4.5) and (4.6) this gives

$$x_2z_3 = x_1z_2 = y_3,$$

so with the operation $\backslash$

$$z_3 = x_2 \backslash y_3.$$

If we compare this with (4.4) we get the equation (4.3) which was to be proved.

It is still easier to prove that the other closure conditions also carry over to (**G**, $\backslash$) since in these the roles of the 3-curves are more symmetric and it is easy to show the carrying over of the closure conditions to other parastrophics.

It is an interesting and not yet entirely solved problem as to when the adjunction of the parastrophics means an extension of the class, or conversely, *when a quasigroup* (**G**, $\cdot$) *is isotopic to one of its parastrophics, e.g., to* (**G**, $\backslash$).

5. Regularity of Geometric Nets. Continuous Loops Isomorphic to the Additive Group of Real Numbers

We return now to the proof of Theorem 2.2 asserting that each closure condition alone is necessary and sufficient for the regularity of a geometric net, defined by (2.1.). Owing to Theorem 2.1 it is enough to prove that even T the "strongest" condition is necessary and even H the "weakest" one is sufficient for the regularity. In other words T follows from the regularity which in its turn follows from H.

By Theorem 3.1, T is equivalent to the statement that (**G**, $\cdot$) has an isotope which is an Abelian group. On the other hand, the definition (2.1) of a regular geometric net can be rephrased by stating that a real interval equipped with the operation resulting in xy for x and y is isotopic to the real axis furnished with the ordinary addition as operation. As the latter is of course an Abelian group, we have already proved that T is necessary.

By Theorem 3.7, H is equivalent to the statement that every loop isotope of $(\mathbf{G}, \cdot)$ is strongly power associative. As for geometric nets the operation $(x, y) \rightarrow xy$ is continuous, therefore the operation $\circ$ in the LP-isotope defined by (3.13) is continuous too. From the strong power associativity of such an LP-isotope we will be able to deduce (2.1). More exactly we prove the following:

THEOREM 5.1. *All strongly power-associative loops on the finite or infinite real interval (a, b) with the continuous operation $(x, y) \rightarrow xy$ are isomorphic to the additive group of real numbers (and are thus Abelian groups) and there exist also continuous mappings establishing these isomorphisms. (Hereby a loop is strongly power associative, if*

$$x^m x^n = x^{m+n} \quad \text{for all integers } m, n \tag{3.49}$$

holds for the powers defined by

$$x^{n+1} = xx^n \ (n=0,1,2,\ldots), \quad x^0 = e = x^{-1}x, \quad x^n = (x^{-1})^{-n} \ (n=-1,-2\ldots) \tag{5.1}$$

where e is the unit element of the loop.)

The fundamental idea of the *proof* is to generalize the powers defined by (5.1) for integer exponents to the case of arbitrary exponents and the dependence of these powers on the exponents will yield the isomorphism. We prove the assertion of Theorem 5.1 after decomposing it into several minor statements.

(A) *$z = xy$ strictly increases if x or y increase.* In fact for arbitrary

$$y < z \tag{5.2}$$

we have

$$ey = y < z = ez$$

and if there existed an x such that

$$xy \geqslant xz \tag{5.3}$$

then, because of the continuity of the operation $(x, y) \to xy$, there would be an x_0 between e and x such that

$$x_0 y = x_0 z$$

[if in (5.3) equality holds, then of course $x_0 = x$]. By the quasigroup property, however, this implies

$$y = z$$

in contradiction to (5.2). The strict increasing in x can be proved similarly.

(B) *If n runs through the integers and $x > e$, then x^n increases with n, while x^n decreases with increasing n if $x < e$.* Let us take, e.g., $x > e$, then (5.1) and (A) imply

$$x^{n+1} = xx^n > ex^n = x^n \qquad (n = 0, 1, 2, \ldots)$$

and

$$ex = x > e = x^{-1}x,$$

so that again by (A)

$$x^{-1} < e$$

and for $n = -m < 0$, also because of (5.1) and (A)

$$x^n = (x^{-1})^m = x^{-1}(x^{-1})^{m-1} < e(x^{-1})^{m-1} = x^{n+1}$$

and similarly for $x < e$.

It follows from (B) that $x^n > x$ *if* $x > e$, $n > 1$ *and* $x^n < x$ *if* $x < e$, $n > 1$, while $x^n < x^{-1} < e$ *if* $x > e$, $n < -1$ *and* $x^n > x^{-1} > e$ *if* $x < e$, $n < -1$

(C) x^n *for fixed* $n > 0$ *is a strictly increasing and continuous function of* x. In fact, $x^2 = xx$ strictly increases and is continuous, by the supposed continuity and by the strict monotony of the operation $(x, y) \to xy$ proved in (A), and if we have already x^n continuous and strictly increasing in x then so is $x^{n+1} = xx^n$ too.

(D) *For fixed* $n > 0$, x^n *takes every value in the interval* (a, b) *as* x *runs through* (a, b). In fact, let y be an arbitrarily prescribed real number in (a, b), e.g., $y < e$; then by (B)

$$y^n < y$$

while [cf. (5.1)] obviously

$$e^n = e > y.$$

Owing to the continuity of x^n proved in (C) there exists an x (between e and y) such that

$$x^n = y. \tag{5.4}$$

Because of the strict monotony proved in (C) there is *exactly* one value x which satisfies (5.4), this we call the nth root*:

$$x = \sqrt[n]{y}.$$

By resubstituting this into (5.4) we have of course

$$(\sqrt[n]{y})^n = y \tag{5.5}$$

(E) *If* $c > e$, *then* $\sqrt[n]{c} > e$. Because if $\sqrt[n]{c} \leqslant e$ would hold, then (B) would imply $c = (\sqrt[n]{c})^n \leqslant e$ contrary to the supposition.

(F) *For all* $x \in (a, b)$ *and for all integers* m, k

$$(x^m)^k = x^{mk}$$

holds. For $k = 0$, 1 this statement is evident. If it holds for k then (5.1) and (3.49) imply

$$(x^m)^{k+1} = x^m(x^m)^k = x^m x^{mk} = x^{m+mk} = x^{m(k+1)}$$

This proves (F) for nonnegative k's. For negative k's the proof is easy too, but this we won't need here.

(G) *For each* $c \in (a, b)$, *for every integer* m *and for all positive integers* n, k

$$(\sqrt[n]{c})^m = (\sqrt[nk]{c})^{mk}$$

* We call attention again (cf. footnote on p. 385) to the fact that here xy is no longer an ordinary product and x^n, $\sqrt[n]{y}$ are no longer ordinary powers, resp., roots, but xy is the result of the operation in the loop and x^n, $\sqrt[n]{y}$ are the quantities defined by (5.1) and (5.4). On the other hand km, kn will denote ordinary products of integers.

holds. First we prove

$$\sqrt[nk]{c} = \sqrt[k]{\sqrt[n]{c}}. \tag{5.6}$$

In fact, if

$$\sqrt[k]{\sqrt[n]{c}} = x$$

then by the definition of the root

$$\sqrt[n]{c} = x^k \quad \text{and} \quad c = (x^k)^n = x^{kn},$$

taking also (F) into account, i.e.,

$$x = \sqrt[nk]{c}$$

which proves (5.6). But then (5.6), (5.5), and (F) imply

$$(\sqrt[nk]{c})^{mk} = ((\sqrt[k]{\sqrt[n]{c}})^k)^m = (\sqrt[n]{c})^m$$

and this was the assertion. [We might observe that the proofs of (F) and (G) are similar to those of the analogous properties for ordinary powers and roots.]

Now we can begin to define the function F establishing the announced isomorphism.

(H) *The definition*

$$F\left(\frac{m}{n}\right) = (\sqrt[n]{c})^m \qquad (n > 0) \tag{5.7}$$

of the function F for rational variables, where $c > e$ is a constant element of (a, b), is unique and satisfies

$$F\left(\frac{k}{n} + \frac{m}{n}\right) = F\left(\frac{k}{n}\right) F\left(\frac{m}{n}\right). \tag{5.8}$$

F is strictly increasing for rational values of the variable.

The uniqueness follows from (G) while (5.8) follows from (5.7) and from (3.49). Finally, (B) and (E) imply

$$F\left(\frac{m+1}{n}\right) = (\sqrt[n]{c})^{m+1} > (\sqrt[n]{c})^m = F\left(\frac{m}{n}\right),$$

so that F actually increases in the rational domain.

(I) $\lim_{m\to\infty} F(m) = b$ and $\lim_{m\to-\infty} F(m) = a$. The proofs of these two statements are so similar that it is enough to prove the first one. As $F(m)$ increases by (B) with m, it has a finite or infinite limit and since $e < F(m) \in (a, b)$ therefore

$$b' = \lim_{m\to\infty} F(m)$$

is on the right of e too, in the interval (a, b) or at its end point. If however b' were to belong to (a, b) then we would get from the particular case

$$F(2m) = F(m)F(m)$$

of (5.8) by passing to the limits

$$b' = b'b'$$

in contradiction with

$$b'b' > b'$$

owing to (B). So $b' = \lim_{m\to\infty} F(m) = b$ and in the same manner $\lim_{m\to-\infty} F(m) = a$ lie at end points of the interval (a, b) and do not belong to it, so in addition to (I) we have proved that *the interval* (a, b) figuring in Theorem 5.1 *is open from both sides.*

(J) $\lim_{n\to\infty} F(1/n) = e$ *holds.* For the sequence $F(1/n)$ is decreasing because of (H) and all its members are greater than e by (E) so

$$e' = \lim_{n\to\infty} F\left(\frac{1}{n}\right) \geqslant e.$$

If however $e' > e$ were to hold, then we would get from the particular case

$$F\left(\frac{1}{n}\right) = F\left(\frac{1}{2n}\right) F\left(\frac{1}{2n}\right)$$

of (5.8) by passing to the limit, because of (B)

$$e' = e'e' > e'.$$

This contradiction proves (J).

Now *we define F* also *for irrational values* of the variable. Let r_n and R_n be lower and upper approximating fractions of p such that

$$r_n \leqslant r_{n+1} < p < R_{n+1} \leqslant R_n \qquad \text{and} \qquad R_n - r_n = \frac{1}{n} \qquad (n = 1, 2, \ldots).$$

Then by (H) the sequence $F(r_n)$ is nondecreasing, while the sequence $F(R_n)$ is nonincreasing and

$$F(r_n) < F(R_n) = F\left(r_n + \frac{1}{n}\right) = F(r_n)F\left(\frac{1}{n}\right)$$

so that $F(r_n)$ has a limit $F(r_n) \to c$ and the limit of $F(R_n)$ is $ce = c$ by the continuity of the operation $(x, y) \to xy$ and by $f(1/n) \to e$ (J). We define $F(p)$ as this common limit c of $F(r_n)$ and of $F(R_n)$. Again because of (J), this definition is unambiguous.

(K) *The above defined F remains strictly increasing and continuous.* In fact, for arbitrary real $p_1 < p_2$, there exist an upper approximating fraction R of p_1 and a lower approximating fraction r of p_2 such that $R < r$ and thus

$$F(p_1) < F(R) < F(r) < F(p_2)$$

so that F actually is strictly increasing. But it is also continuous, for if $p \to q$ and, e.g., $p > q$, then by the definition of $F(q)$ we can find an upper approximating fraction R_n of q such that

$$0 < F(R_n) - F(q) < \epsilon$$

[if q happens to be rational, then we choose $R_n = q + (1/n)$ so that

$$0 < F(R_n) - F(q) = F(q)F\left(\frac{1}{n}\right) - F(q) < \epsilon$$

holds]; if $p < R_n$ then

$$0 < F(p) - F(q) < F(R_n) - F(q) < \epsilon$$

and this actually means that F is continuous.

(L) *F satisfies*

$$F(p + q) = F(p)F(q) \tag{5.9}$$

for all real p, q and in (a, b) the inverse function F^{-1} of F exists and satisfies

$$xy = F(F^{-1}(x) + F^{-1}(y)) \tag{5.10}$$

for all $x, y \in (a, b)$. In fact, (5.9) follows by passing to the limits in (5.8) because of the continuity of F, further F takes every value in (a, b)

exactly once by (I) and (K) so that there exists exactly one real p such that

$$F(p) = x.$$

This defines the inverse of F:

$$p = F^{-1}(x).$$

Finally, (5.9) goes over into (5.10) with $F(p) = x$, $F(q) = y$.

(L) actually expresses the fact that the loop given by the interval (a, b) and by the operation $(x, y) \to xy$ is isomorphic to the additive group of real numbers, so that Theorem 5.1 is proved. (Theorem 5.1 follows also from a more involved theorem of Salzmann [1]. The above elementary proof was shaped after reasonings in Aczél [1].)

Finally, for geometric nets, as we have seen at the beginning of Section 5, H implies that the LP-isotope defined by (3.13) is a strongly power-associative loop and so by Theorem 5.1

$$x \circ y = F(F^{-1}(x) + F^{-1}(y))$$

i.e., from (3.13)

$$xy = xv \circ uy = F(F^{-1}(xv) + F^{-1}(uy))$$

which goes over into (2.1) with the functions defined by $h(z) = F^{-1}(z)$, $f(x) = F^{-1}(xv)$, $g(y) = F^{-1}(uy)$ (which obviously are again continuous and strictly monotonic):

$$h(xy) = f(x) + g(y).$$

This shows that the geometric net satisfying H is regular and finishes the proof of Theorem 2.2.

As obviously every group, and so much the more every Abelian group is strongly power-associative, therefore Theorem 5.1 implies the following:

Corollary 8. *Every continuous group (and so much the more every continuous Abelian group) on an interval (a, b) is isomorphic to the additive group of real numbers and among the mappings establishing the isomorphism there are also continuous ones.*

The situation is not quite so simple with continuous Moufang loops. We have seen at the beginning of Section 3 (Lemma 3.1) that every

Moufang loop (even every M_1- or M_2-loop) is power-associative, but from the power-associativity of a loop its strong power-associativity does not follow. (Corollary 7 stated only that the power-associativity of *all LP*-isotopes implies their strong power-associativity.) And Theorem 5.1 does not refer to power-associative but to strongly power-associative loops. The Moufang properties themselves, however, imply also the strong power-associativity, (cf. Moufang [1] for another proof of a part of the following Theorem 5.2).

We prove even more:

THEOREM 5.2. *Every M_1-loop, every M_2-loop, and so much the more every Moufang-loop is strongly power-associative.*

This follows from our previous theorems and proofs. In fact, in part (II) of the proof of Theorem 3.3 we have shown that B_1 is valid in each M_1-loop. As B_1 implies H, (Theorem 2.1), so in the same loop H holds too. But by Theorem 3.7, H implies the strong power-associativity in all isotopic loops, so also in the original one. This proves the first statement of Theorem 5.2. The proof of the second statement is quite analogous, whereas the third one follows from either of the previous two.

Theorems 5.1 and 5.2 imply the following:

Corollary 9. *Every continuous M_1- (or M_2- or Moufang) loop defined on a real interval is isomorphic to the additive group of real numbers (and is thus itself an Abelian group) and there exist continuous mappings establishing this isomorphism.*

6. Functional Equations on Quasigroups. Applications on Nets. Further Problems

Equations (3.4), (3.5), (3.6), etc., just as the equation

$$xy \cdot z = x \cdot yz \tag{3.7}$$

of associativity can be considered as functional equations with the operation $(x, y) \to xy$ as the unknown function. The fact that an operation satisfies such a functional equation still admits a great diversity of possible operations. There are however functional equations, some of which are even similar to the previous ones, which determine rather closely the operations figuring in them. Moreover, one equation can serve to determine several functions (operations) occurring in it.

We begin with the example of the following generalization of the Eq. (3.7) of associativity:

$$(x \mathbf{1} y) \mathbf{2} z = x \mathbf{3} (y \mathbf{4} z), \tag{6.1}$$

where $\mathbf{1}$, $\mathbf{2}$, $\mathbf{3}$, and $\mathbf{4}$ denote four operations. More exactly we suppose that a set $\mathbf{G}$ forms quasigroups with respect to all four operations and (6.1) is satisfied for all $x, y, z \in \mathbf{G}$. We prove that each of these four quasigroups is isotopic with the same group (i.e., they all belong to the class associated with the same g-regular net) and with the aid of this group the explicit form of $x \mathbf{i} y$ ($\mathbf{i} = \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}$) can be given.

Let us introduce the right and left translations

$$xR_{\mathbf{i}} = x \mathbf{i} a, \qquad xL_{\mathbf{i}} = a \mathbf{i} x \qquad (\mathbf{i} = \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4})$$

(cf. Kertész [1]), where a is a constant (fixed element of $\mathbf{G}$). As all equations of the forms

$$x \mathbf{i} a = z \qquad \text{or} \qquad a \mathbf{i} y = w$$

have unique solutions x, resp., y in the quasigroup $(\mathbf{G}, \mathbf{i})$ ($\mathbf{i} = \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}$), therefore the functions $R_{\mathbf{i}}$ and $L_{\mathbf{i}}$ have inverses $R_{\mathbf{i}}^{-1}$, $L_{\mathbf{i}}^{-1}$ and also functions composed of the former ones have inverses. Of course,

$$(tL_{\mathbf{i}}^{-1})L_{\mathbf{i}} = t = (tR_{\mathbf{i}}^{-1})R_{\mathbf{i}} \qquad (\mathbf{i} = \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}).$$

If we put $x = z = a$ into (6.1) we get

$$yL_{\mathbf{1}}R_{\mathbf{2}} = yR_{\mathbf{4}}L_{\mathbf{3}}. \tag{6.2}$$

We substitute now $x = a$, $y = uR_{\mathbf{2}}^{-1}L_{\mathbf{1}}^{-1}$, $z = vL_{\mathbf{3}}^{-1}L_{\mathbf{4}}^{-1}$ into (6.1):

$$uR_{\mathbf{2}}^{-1} \mathbf{2}\, vL_{\mathbf{3}}^{-1}L_{\mathbf{4}}^{-1} = (uR_{\mathbf{2}}^{-1}L_{\mathbf{1}}^{-1} \mathbf{4}\, vL_{\mathbf{3}}^{-1}L_{\mathbf{4}}^{-1})L_{\mathbf{3}}; \tag{6.3}$$

then we put again into (6.1) $x = uR_{\mathbf{2}}^{-1}R_{\mathbf{1}}^{-1}$, $y = a$, $z = vL_{\mathbf{3}}^{-1}L_{\mathbf{4}}^{-1}$ and get

$$uR_{\mathbf{2}}^{-1} \mathbf{2}\, vL_{\mathbf{3}}^{-1}L_{\mathbf{4}}^{-1} = uR_{\mathbf{2}}^{-1}R_{\mathbf{1}}^{-1} \mathbf{3}\, vL_{\mathbf{3}}^{-1}; \tag{6.4}$$

and, finally, we substitute into the same equation $x = uR_{\mathbf{2}}^{-1}R_{\mathbf{1}}^{-1}$, $y = vL_{\mathbf{3}}^{-1}R_{\mathbf{4}}^{-1}$, $z = a$:

$$(uR_{\mathbf{2}}^{-1}R_{\mathbf{1}}^{-1} \mathbf{1}\, vL_{\mathbf{3}}^{-1}R_{\mathbf{4}}^{-1})R_{\mathbf{2}} = uR_{\mathbf{2}}^{-1}R_{\mathbf{1}}^{-1} \mathbf{3}\, vL_{\mathbf{3}}^{-1}. \tag{6.5}$$

Since the left-hand sides of (6.3) and (6.4) and also the right-hand sides of (6.4) and (6.5) are equal, all terms figuring in these equations are equal. Let us designate this common value by $u \circ v$. If we now introduce the notations*

$$f(x) = xR_1R_2, \qquad G(z) = zL_4L_3, \qquad h(s) = sR_2, \qquad H(t) = tL_3$$

and taking also (6.2) into account

$$g(y) = yL_1R_2 = yR_4L_3,$$

then from (6.3), (6.4), and (6.5) we have

$$h^{-1}(u)\,\mathbf{2}\,G^{-1}(v) = H(g^{-1}(u)\,\mathbf{4}\,G^{-1}(v)) = f^{-1}(u)\,\mathbf{3}\,H^{-1}(v)$$
$$= h(f^{-1}(u)\,\mathbf{1}\,g^{-1}(v)) = u \circ v,$$

i.e.,

$$\text{(6.6)} \qquad \begin{aligned} x\,\mathbf{1}\,y &= h^{-1}(f(x) \circ g(y)), & s\,\mathbf{2}\,z &= h(s) \circ G(z), \\ x\,\mathbf{3}\,t &= f(x) \circ H(t), & y\,\mathbf{4}\,z &= H^{-1}(g(y) \circ G(z)). \end{aligned}$$

These latter formulas (6.6) express the fact that the quasigroups $(\mathbf{G}, \mathbf{1})$, $(\mathbf{G}, \mathbf{2})$, $(\mathbf{G}, \mathbf{3})$, $(\mathbf{G}, \mathbf{4})$ are all isotopic with the quasigroup $(\mathbf{G}, \circ)$. If we resubstitute (6.6) into (6.1), we get

$$(f(x) \circ g(y)) \circ G(z) = f(x) \circ (g(y) \circ G(z)),$$

so that the operation $\circ$ is *associative*, i.e., $(\mathbf{G}, \circ)$ is a group. On the other hand (6.6) always satisfies (6.1) if $\circ$ is associative. This proves the following

THEOREM 6.1. *If* $(\mathbf{G}, \mathbf{1})$, $(\mathbf{G}, \mathbf{2})$, $(\mathbf{G}, \mathbf{3})$, $(\mathbf{G}, \mathbf{4})$ *are all quasigroups and*

$$\text{(6.1)} \qquad (x\,\mathbf{1}\,y)\,\mathbf{2}\,z = x\,\mathbf{3}\,(y\,\mathbf{4}\,z)$$

holds for all $x, y, z \in \mathbf{G}$, *then each* $(\mathbf{G}, \mathbf{i})$ $(\mathbf{i} = \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4})$ *is isotopic to the same group* $(\mathbf{G}, \circ)$. *In particular there exist five* 1-1-*maps* f, g, G, h, H *such that* (6.6) *holds.*

* We evidently form the inverses of the functions $f, g, \ldots$ in the following way: $f^{-1}(u) = uR_2^{-1}R_1^{-1}$, $g^{-1}(v) = vR_2^{-1}L_1^{-1} = vL_3^{-1}R_4^{-1}$, etc.

As a second example we consider the equation

$$(6.7)\qquad (x \mathbf{1} y) \mathbf{2} (w \mathbf{3} z) = (x \mathbf{4} w) \mathbf{5} (y \mathbf{6} z)$$

for all $x, y, w, z \in G$, where $(\mathbf{G}, \mathbf{i})$ $(\mathbf{i} = \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6})$ are all quasigroups. Let us substitute $w = a$ into (6.7) and observe that this carries (6.7) over into (6.1) with the notations

$$x \mathbf{1}' y = x \mathbf{1} y, \qquad s \mathbf{2}' z = s \mathbf{2} (a \mathbf{3} z),$$
$$x \mathbf{3}' t = (x \mathbf{4} a) \mathbf{5} t, \qquad y \mathbf{4}' z = y \mathbf{6} z.$$

The $(\mathbf{G}, \mathbf{i})$'s $(\mathbf{i} = \mathbf{1}', \mathbf{2}', \mathbf{3}', \mathbf{4}')$ are also quasigroups, therefore owing to Theorem 6.1

$$(6.8)\qquad \begin{aligned} x \mathbf{1} y &= h^{-1}(f(x) \circ g(y)), & s \mathbf{2} p &= h(s) \circ J(p), \\ r \mathbf{5} t &= K(r) \circ H(t), & y \mathbf{6} z &= H^{-1}(g(y) \circ G(z)), \end{aligned}$$

where $\circ$ is an associative operation: $(u \circ v) \circ w = u \circ (v \circ w) = u \circ v \circ w$.

Let us resubstitute all this into (6.7); then we get

$$(6.9)\qquad f(x) \circ g(y) \circ J(w \mathbf{3} z) = K(x \mathbf{4} w) \circ g(y) \circ G(z)$$

or with $x = y = a$

$$J(w \mathbf{3} z) = g(a)^{-1} \circ f(a)^{-1} \circ K(a \mathbf{4} w) \circ g(a) \circ G(z)$$

and so

$$(6.10)\qquad w \mathbf{3} z = J^{-1}(L(w) \circ G(z))$$

or after substituting (6.10) back into (6.9)

$$(6.11)\qquad f(x) \circ g(y) \circ L(w) \circ G(z) = K(x \mathbf{4} w) \circ g(y) \circ G(z).$$

If we put $y = g^{-1}(e)$ in (6.11) [e is the unit element of $(\mathbf{G}, \circ)$], we get

$$(6.12)\qquad x \mathbf{4} w = K^{-1}(f(x) \circ L(w)).$$

Putting back (6.12) into (6.11) we have

$$f(x) \circ g(y) \circ L(w) \circ G(z) = f(x) \circ L(w) \circ g(y) \circ G(z)$$

or

$$g(y) \circ L(w) = L(w) \circ g(y).$$

So the operation is also *commutative*. On the other hand (6.8), (6.10), and (6.12) always satisfy (6.7) if $\circ$ is commutative and associative. These formulas state that the quasigroups (**G**, 1), (**G**, 2), (**G**, 3), (**G**, 4), (**G**, 5), and (**G**, 6) are all isotopic to the Abelian group (**G**, $\circ$), so they all belong to the class associated with the same *a*-regular net. We thus have the following

THEOREM 6.2. *If* (**G**, 1), (**G**, 2), (**G**, 3), (**G**, 4), (**G**, 5), and (**G**, 6) *are all quasigroups and*

$$(x\,1\,y)\,2\,(w\,3\,z) = (x\,4\,w)\,5\,(y\,6\,z) \tag{6.7}$$

for all x, y, w, and z, then every (**G**, i) *is isotopic with the same abelian group* (**G**, $\circ$). *More exactly there exist eight* 1-1 *mappings f, g, G, h, H, J, K, and L, so that* (6.8), (6.10), *and* (6.12) *are valid.*

Several other equations can be solved in a similar way [cf. e.g., Aczél, Belousov, and Hosszú [1] also for (6.1) and (6.7)].

From Corollary 8 every continuous group (and so much the more every Abelian group) on an interval is isomorphic to the additive group of real numbers under a continuous isomorphism. If on the other hand, the operations **1**, **2**, **3**, **4**, resp., **1**, **2**, **3**, **4**, **5**, **6** in Theorems 6.1 and 6.2 are continuous, then by their definitions f, g, G, h, H, J, K, L and the operation $\circ$ are continuous too. So Theorems 6.1 and 6.2 imply the following:

Corollary 10. *If a real interval* **I** *forms quasigroups with respect to each continuous operation* **1**, **2**, **3**, **4** or **1**, **2**, **3**, **4**, **5**, **6** *and if* (6.1), *resp.*, (6.7) *are satisfied for all x, y, z, $w \in$* **I**, *then all the quasigroups* (**I**, **i**) (**i** = **1**, **2**, **3**, **4**) *and* (**I**, i) (**i** = 1, 2, 3, 4, 5, 6), *respectively, are isotopic to the additive group of real numbers and there exist continuous functions establishing these isotopisms. The general continuous real quasigroup solutions of* (6.1) *are*

$$x\,\mathbf{1}\,y = \bar{h}^{-1}(f(x) + \bar{g}(y)), \qquad s\,\mathbf{2}\,z = F(\bar{h}(s) + \bar{G}(z)),$$
$$x\,\mathbf{3}\,t = F(f(x) + \bar{H}(t)), \qquad y\,\mathbf{4}\,z = \bar{H}^{-1}(\bar{g}(y) + \bar{G}(z)),$$

and those of (6.7) *are*

$$\begin{gathered} x\,1\,y = \bar{h}^{-1}(f(x) + \bar{g}(y)), \qquad s\,2\,p = F(\bar{h}(s) + \bar{J}(p)), \\ w\,3\,z = \bar{J}^{-1}(\bar{L}(w) + \bar{G}(z)), \\ x\,4\,w = \bar{K}^{-1}(f(x) + \bar{L}(w)), \qquad r\,5\,t = F(\bar{K}(r) + \bar{H}(t)), \\ y\,6\,z = \bar{H}^{-1}(\bar{g}(y) + \bar{G}(z)), \end{gathered} \tag{6.13}$$

where $\bar{f}$, $\bar{g}$, $\bar{G}$, $\bar{h}$, $\bar{H}$, $\bar{J}$, $\bar{K}$, $\bar{L}$ *are continuous* 1-1 *maps of* **I** *onto* $(-\infty, \infty)$ *and* F *is a continuous* 1-1 *map of* $(-\infty, \infty)$ *onto* **I**.

We can also use Theorem 6.2 and its corollary to prove part (II) of Theorem 3.1: T implies that the class of isotopic quasigroups associated with the net contains an Abelian group. In order to show this, we transcribe

$$T: \quad (x_1y_2 = x_2y_1 \,\&\, x_1y_3 = x_3y_1) \Rightarrow x_2y_3 = x_3y_2$$

with the aid of the notations

$$x = y_1, \qquad y = x_1y_2 = x_2y_1, \qquad w = x_1y_3 = x_3y_1, \qquad z = x_1$$

and of the operations /, **1** and \, **3** defined by

$$s = r/t = t\,\mathbf{1}\,r \qquad \text{and} \qquad t = s\backslash r = r\,\mathbf{3}\,s$$

as equivalents to

$$st = r.$$

In fact, then $x_1y_2 = x_2y_1 = y$ can be written as

$$x_2 = y/y_1 = y/x = x\,\mathbf{1}\,y, \qquad y_2 = x_1\backslash y = z\backslash y = y\,\mathbf{3}\,z$$

and $x_1y_3 = x_3y_1 = w$ as

$$x_3 = w/y_1 = w/x = x\,\mathbf{1}\,w, \qquad y_3 = x_1\backslash w = z\backslash w = w\,\mathbf{3}\,z,$$

so that $x_2y_3 = x_3y_2$ is equivalent to

$$(x\,\mathbf{1}\,y)(w\,\mathbf{3}\,z) = (x\,\mathbf{1}\,w)(y\,\mathbf{3}\,z).$$

This is an equation of the form (6.7) with $x\,\mathbf{4}\,w = x\,\mathbf{1}\,w, y\,\mathbf{6}\,z = y\,\mathbf{3}\,z$ and with $r\,\mathbf{5}\,t = r\,\mathbf{2}\,t = rt$, so that by (6.8)

$$rt = K(r) \circ H(t) \tag{6.14}$$

where the operations $\circ$ is commutative and associative, i.e., (**G**, $\circ$) is again an Abelian group. (6.14) says that (**G**, $\cdot$) is actually isotopic to an Abelian group and so the net is *a*-regular. If we have a *geometric net*, then (6.13) gives

$$rt = F(\bar{K}(r) + \bar{H}(t))$$

which, disregarding the notation, is just (2.1), so that T implies the regularity of the geometric net and this is a part of Theorem 2.2.

These ideas might help in the solution of some further problems, which are presently unsolved. The applications in nomography, for instance, not only raise problems of specialization but also problems of generalization. In fact, there is a need for nomographic representation of functions for which $z = xy$ does not necessarily have unique solutions with respect to x and y. If none of the functions defined by $z = x_0y$ and $z = xy_0$ for all fixed x_0, y_0 assume any z-value twice (e.g., they are strictly monotonic), then the structure is called a *cancellation groupoid*; if even this does not necessarily hold, then we have a *groupoid* (cf. Bruck [1], Kertész [1]). If $z = xy$ makes a set correspond to x and y as function-value, then we have a *multigroupoid*. On the other hand, in the definition of nets the condition that two curves of different families should intersect in *exactly one* point is sometimes replaced by the weaker one that they should intersect in *at most one* point. This means that the corresponding operation need not even be defined for any two elements of $\mathbf{G}$, and then $(\mathbf{G}, \cdot)$ is a *half-groupoid* (cf. Bruck [1]). If finally not even the two domains of definition and the range of the values $z = xy$ of the function are the same, then we have to deal with general algebraic structures with *three fundamental sets* $\mathbf{G}_1$, $\mathbf{G}_2$, and $\mathbf{G}_3$, instead of one and with an operation associating a value $z \in \mathbf{G}_3$ to the pair $x \in \mathbf{G}_1$, $y \in \mathbf{G}_2$. It seems to be a difficult task, but interesting and important for nomography, to work out a net theory for such structures. The first steps in this direction were made by Hosszú [1] who succeeded in proving theorems about the functional equations (6.1) and (6.7) also in cases, where x, y, z, and w are elements of different sets.

Another way of treating the above problem is to consider geometric nets *locally* (in neighborhood of a point). Radó (see, e.g., [1, 2]) has worked intensively in this field and also considered (cf. [1, 3]) the *spatial nets* introduced by Blaschke (see, e.g., [1], Blaschke and Bol [1]), from this point of view. In the algebraic theory of these nets *ternary* operations figure instead of binary ones and here too different regularity and closure conditions, etc., can be formulated. The algebraic theory of two-dimensional nets dealt with above and Theorem 6.1 can also be useful in the investigation of spatial nets.

As an example we mention the following open problem: Given four bundles of planes in space so that the intersections of each plane in three bundles with the planes of the other bundles form regular (geometric) nets. Does this imply that the nets originating in a similar way on the planes of the fourth bundle are regular too? If we suppose the functions with the values $z = xy$, resp., $t = xyw$ to be differentiable, then the answer is positive (theorem of Dubordieu-Arf, see, e.g., Blaschke [1], Blaschke and Bol [1]), but the question persists, whether the answer remains true even if only continuity of these functions is supposed, i.e., for general geometric nets. Since regularity follows (by the above Theorem 2.2) from the closure condition H, which in the sense of Theorem 2.1 is the "weakest," while the regularity itself implies even the "strongest" closure condition T, therefore it would be enough to deduce the validity of H for the fourth bundle of nets from that of T for the three other bundles. But nobody has yet succeeded even in proving this.*

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