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Solitary Waves with Prescribed Speed on Infinite Lattices

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Using a variant of the mountain pass theorem, we prove the existence of solitary waves with prescribed speed on infinite lattices of particles with nearest neighbor interaction. The problem is to solve a second-order forward–backward differential-difference equation. © 1997 Academic Press

1. INTRODUCTION

We consider an infinite lattice of particles with nearest neighbor interaction:

$$\ddot{q}_k = V'(q_{k+1} - q_k) - V'(q_k - q_{k-1}), \qquad k \in \mathbb{Z}. \tag{1}$$

After the pioneering work of Fermi, Pasta, and Ulam [6] on finite lattices, Toda, [10] discovered an integrable infinite lattice with exponential interaction potential.

Recently the calculus of variations was applied to the existence of periodic motions on lattices under general assumptions on the potential [1, 2, 3, 9].

A solitary wave is a solution of (1) of the form

$$q_k(t) = u(k - ct), \qquad k \in \mathbb{Z}.$$

Substituting in (1), we obtain the second order forward–backward differential–difference equation

$$c^2 u''(t) = V'(u(t+1) - u(t)) - V'(u(t) - u(t-1)). \tag{2}$$

When

$$V(u) = ab^{-1}(e^{-bu} + bu - 1),$$

Toda found explicit formula's for the solitary waves. Until now, the only general existence theorem is due to Friesecke and Wattis [7]. Under some

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assumptions, they prove the existence of solitary waves with prescribed average potential energy

$$\int_{\mathbb{R}} V(u(t-1)-u(t))\,dt = K. \tag{3}$$

The speed c of the wave is given by an *unknown* Lagrange multiplier. The approach is to minimize the average kinetic energy

$$\frac{1}{2}\int_{\mathbb{R}} [u'(t)]^2\,dt$$

subject to the constraint (3).

We consider in this paper the existence of solitary waves with *prescribed* speed. It seems impossible to solve this problem by constrained minimization, so that we use the mountain pass theorem. Since the Palais–Smale condition is not satisfied by the natural functional

$$\varphi(u) := \int_{\mathbb{R}} \left[\frac{c^2}{2}(u'(t))^2 - V(u(t+1)-u(t))\right] dt$$

on the space

$$X := \{u \in H^1_{loc}(\mathbb{R}) : u' \in L^2(\mathbb{R}),\, u(0)=0\},$$

we use a weak convergence argument inspired by chapter 7 of [11] together with Lieb's lemma [8]. Like in the case of the Kadomtsev–Petviashvili equation, minimax method and weak convergence are simpler to use than constrained minimization and concentration–compactness.

Monotonicity of solitary waves follows directly from minimization. In our setting, we use a refinement of the mountain pass theorem due to Brezis and Nirenberg [5] in order to prove monotonicity.

Our assumptions are not strictly comparable to the assumptions in [7]. For increasing waves, it is assumed in [7] that $V \in \mathscr{C}^2(\mathbb{R})$, $V \geqslant 0$ on $]-\delta, \delta[$, $V(0)=0$ and $V(u)/u^2$ increases strictly on $]0, \infty[$.

Section 2 is devoted to the functional setting, Section 3 to monotone waves and Section 4 to nonmonotone waves. In section 5 we give some examples of potentials satisfying our assumptions.

2. FUNCTIONAL SETTING

On

$$X := \{u \in H^1_{loc}(\mathbb{R}) : u' \in L^2(\mathbb{R}),\, u(0)=0\},$$

we define the inner product

$$(u, v) := \int_{\mathbb{R}} u'v'$$

and the corresponding norm

$$\|u\| := \left[\int_{\mathbb{R}} (u')^2 \right]^{1/2}.$$

It is easy to see that X is a Hilbert space. On X, we define also the linear operator

$$Au(t) := u(t+1) - u(t).$$

PROPOSITION 1. *The operator A is continuous from X to $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $|Au|_\infty \leqslant \|u\|$, $|Au|_2 \leqslant \|u\|$.*

Proof. By Cauchy–Schwarz inequality we have

$$|Au(t)| = |u(t+1) - u(t)| = \left| \int_t^{t+1} u'(s)\, ds \right|$$

$$\leqslant \left(\int_t^{t+1} |u'(s)|^2\, ds \right)^{1/2} \leqslant \|u\|$$

and

$$\int_{\mathbb{R}} |Au(t)|^2\, dt \leqslant \int_{\mathbb{R}} \int_t^{t+1} |u'(s)|^2\, ds\, dt = \|u\|^2. \quad \blacksquare$$

PROPOSITION 2. *If $V \in \mathscr{C}^1(\mathbb{R}, \mathbb{R})$, $V(0) = V'(0) = 0$ and $V''(0)$ exists, then*

$$\varphi(u) := \int_{\mathbb{R}} \left[\frac{c^2}{2} u'^2 - V(Au) \right]$$

is well defined on X; $\varphi \in \mathscr{C}^1(X, \mathbb{R})$ and

$$\langle \varphi'(u), h \rangle = \int_{\mathbb{R}} [c^2 u'h' - V'(Au)\, Ah].$$

Proof. By assumption we have, for every $R>0$,

$$\sup_{|u| \leqslant R} \left| \frac{V'(u)}{u} \right| < \infty.$$

It is then easy to prove Proposition 2 using Proposition 1. ∎

PROPOSITION 3. *Under the assumptions of Proposition* 2, *if* u *is critical point of* φ, *then* u *is a solution of* (2).

Proof. If u is a critical point of φ, then, for every $h \in \mathscr{D}(\mathbb{R})$, we have

$$0 = \int_{\mathbb{R}} [c^2 u'(t)\, h'(t) - V'(u(t+1) - u(t))(h(t+1) - h(t))]\, dt$$

$$= \int_{\mathbb{R}} [c^2 u'(t)\, h'(t) - [V'(u(t) - u(t-1)) - V'(u(t+1) - u(t))]\, h(t)]\, dt.$$

Hence u is a weak solution of (2). Since V' and u are continuous, $u \in \mathscr{C}^2(\mathbb{R})$. ∎

3. MONOTONE SOLITARY WAVES

We need the following version of the mountain pass theorem.

THEOREM 4. *Let* X *be a Banach space*, $\varphi \in \mathscr{C}^1(X, \mathbb{R})$, $e \in X$ *and* $r>0$ *be such that* $\|e\| > r$ *and*

$$b := \inf_{\|u\| = r} \varphi(u) > \varphi(0) \geqslant \varphi(e).$$

Let $P: X \to X$ *be a continuous mapping such that*

$$\forall u \in X,\ \varphi(Pu) \leqslant \varphi(u),\ P(0) = 0 \qquad \text{and} \qquad P(e) = e. \tag{4}$$

Then for every $\varepsilon > 0$ *and* $\delta > 0$ *there exists* $u \in X$ *such that*

(a) $d - 2\varepsilon \leqslant \varphi(u) \leqslant d + 2\varepsilon$,

(b) $dist(u, P(X)) \leqslant 2\delta$,

(c) $\|\varphi'(u)\| \leqslant 8\varepsilon/\delta$, *where*

$$d := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \varphi(\gamma(t))$$
$$\Gamma := \{\gamma \in \mathscr{C}([0,1], X) : \gamma(0) = 0, \gamma(1) = e\}. \tag{5}$$

Proof. Suppose the thesis is false. By the quantitative deformation lemma, (Lemma 2.3 in [11]), there exists a deformation $\eta \in \mathscr{C}([0,1] \times X, X)$ such that

(i) $\eta(t, u) = u$ if $t = 0$ or if $u \notin \varphi^{-1}([d - 2\varepsilon, d + 2\varepsilon) \cap P(X)_{2\delta}$,
(ii) $\eta(1, \varphi^{d+\varepsilon} \cap P(X)) \subset \varphi^{d-\varepsilon}$.

Assume that $c - \varphi(0) > 2\varepsilon$ and let $\gamma \in \Gamma$ be such that

$$\max_{t \in [0,1]} \varphi(\gamma(t)) \leqslant d + \varepsilon.$$

Define $\beta(t) := \eta(1, P\gamma(t))$. It is easy to verity that $\beta \in \Gamma$ and $\max_{t \in [0,1]} \varphi(\beta(t)) \leqslant d - \varepsilon$. This contradicts the definition of d. ∎

Remark. Assumption (4) is due to Brezis and Nirenberg ([5] see also [4]). Under this assumption, if φ satisfies the Palais–Smale condition, there exists a critical point $u \in \overline{P(X)}$ such that $\varphi(u) = d$.

We assume that

(V_1) $V(u) = c_0^2(u^2/2) + W(u)$, $c_0 \geqslant 0$, $W \in \mathscr{C}^1(\mathbb{R}, \mathbb{R})$ $W(0) = 0$, $W'(u) = o(|u|)$, $u \to 0$, and

(V_2^+) there exists $u > 0$ such that $W(u) > 0$ and $\alpha > 2$ such that, for $u \geqslant 0$,

$$0 \leqslant \alpha W(u) \leqslant u W'(u)$$

or

(V_2^-) there exists $u < 0$ such that $W(u) > 0$ and $\alpha > 2$ such that, for $u \leqslant 0$,

$$0 \leqslant \alpha W(u) \leqslant u W'(u).$$

Since we are interested in monotone waves, we assume that $W(u) = 0$ for $u \leqslant 0$ (resp. $u \geqslant 0$) if (V_2^+) (resp. (V_2^-)) is satisfied. We fix $c > c_0$ and we define on X

$$\varphi(u) := \int_{\mathbb{R}} \left[\frac{c^2}{2} u'^2 - V(Au) \right].$$

We define also $P\colon X \to X$ by

$$Pu(t) := \int_0^t |\dot{u}(s)| \, ds.$$

LEMMA 5. *Under assumptions* (V_1) *and* (V_2^+) *there exists* $e \in P(X)$ *and* $r > 0$ *such that* $\|e\| > r$ *and*

$$b := \inf_{\|u\| = r} \varphi(u) > \varphi(0) \geqslant \varphi(e).$$

Proof. After integrating, we obtain from (V_2^+) the existence of $a_0 \geqslant 0$ such that

$$|u| \leqslant 1 \Rightarrow W(u) \leqslant a_0 \, |u|^\alpha.$$

If $\|u\| \leqslant 1$ then, by Proposition 1, $|Au|_\infty \leqslant 1$ and

$$\varphi(u) \geqslant \int_{\mathbb{R}} \left[\frac{c^2}{2} u'^2 - \frac{c_0^2}{2} |Au|^2 - a_0 \, |Au|^\alpha \right]$$

$$\geqslant \frac{c^2 - c_0^2}{2} \|u\|^2 - a_0 \, |Au|_\alpha^\alpha.$$

Since $A\colon X \to L^\alpha$ is continuous, there exists $r > 0$ such that $\inf_{\|u\| = r} \varphi(u) > 0 = \varphi(0)$. We obtain also from (V_1) and (V_2^+) the existence of $a_1 > 0$ such that, for $u \geqslant 0$

$$a_1(u^\alpha - u^2) \leqslant W(u).$$

Choosing $v \in P(X) \backslash \{0\}$, we have

$$\varphi(\lambda v) \leqslant \lambda^2 \frac{c^2}{2} \|v\|^2 + \lambda^2 a_1 \, |Av|_2^2 - \lambda^\alpha a_1 \, |Av|_\alpha^\alpha.$$

Since $\alpha > 2$, there exists $e := \lambda v \in P(X)$ such that $\|e\| > r$ and $\varphi(e) \leqslant 0$. ∎

LEMMA 6. *Under the assumptions of Lemma* 5, *there exists a sequence* $(u_n) \subset X$ *such that*

$$\varphi(u_n) \to d, \qquad \varphi'(u_n) \to 0, \qquad \operatorname{dist}(u_n, P(X)) \to 0,$$

where $d > 0$ *its defined by* (5).

Proof. In order to apply Theorem 4, it suffices to verify that, for every $u \in X$, $\varphi(Pu) \leqslant \varphi(u)$. It is clear that $\|Pu\| = \|u\|$ and

$$Au(t) = \int_t^{t+1} u'(s)\, ds \leqslant \int_t^{t+1} |u'(s)|\, ds = APu(t).$$

Since, by assumption (V_2^+), V is nondecreasing, the proof is complete. ∎

THEOREM 7. (a) *Under assumptions* (V_1) *and* (V_2^+), *for every* $c > c_0$, *Eq.* (2) *has a nontrivial nondecreasing solution* $u \in X$.

(b) *Under assumptions* (V_1) *and* (V_2^-), *for every* $c > c_0$, *Eq.* (2) *has a nontrivial nonincreasing solution* $u \in X$.

Proof. (1) We prove the first statement of the theorem. The proof of the second one is similar.

(2) Let (u_n) be given by the preceding lemma. For n sufficiently large

$$\begin{aligned} d + 1 + \|u_n\| &\geqslant \varphi(u_n) - \frac{1}{\alpha} \langle \varphi'(u_n), u_n \rangle \\ &= \left(\frac{1}{2} - \frac{1}{\alpha}\right)(c^2 \|u_n\|^2 - c_0^2 |Au_n|_2^2) \\ &\quad + \int [\alpha^{-1} Au_n W'(Au_n) - W(Au_n)] \\ &\geqslant \left(\frac{1}{2} - \frac{1}{\alpha}\right)(c^2 - c_0^2) \|u_n\|^2. \end{aligned}$$

Thus (u_n) is bounded in X.

(3) By Proposition 1,

$$\sup |Au_n|_\infty, \qquad \sup |Au_n|_2 \leqslant a_2 := \sup \|u_n\|.$$

In particular, (Au_n) in bounded in $H^1(\mathbb{R})$. By assumption

$$\tfrac{1}{2} W'(u)\, u - W(u) = o(u^2), \qquad u \to 0,$$

and

$$a_3 := \sup_{|u| \leqslant a_2} [\tfrac{1}{2} W(u)\, u - W(u)]/u^2 < \infty.$$

Let $\varepsilon > 0$. There exists $\delta > 0$ such that

$$|u| \leqslant \delta \Rightarrow |\tfrac{1}{2} W'(u)\, u - W(u)| \leqslant \varepsilon u^2.$$

It follows that

$$\int_{\mathbb{R}} \left[\tfrac{1}{2} W'(Au_n)\, Au_n - W(Au_n)\right] \leqslant \operatorname{mes}\{|Au_n| > \delta\}\ a_3\, |Au_n|_\infty^2 + \varepsilon\, |Au_n|_2^2$$
$$\leqslant \operatorname{mes}\{|Au_n| > \delta\}\ a_3 a_2^2 + \varepsilon a_2^2.$$

If $Au_n \to 0$ in measure on $\mathbb{R}$, we obtain

$$0 < d = \varphi(u_n) - \tfrac{1}{2}\langle \varphi'(u_n), u_n\rangle + o(1)$$
$$= \int_{\mathbb{R}} \left[\tfrac{1}{2} W'(Au_n)\, Au_n - W(Au_n)\right] + o(1) = o(1).$$

This is a contradiction.

(4) Since $Au_n \nrightarrow 0$ in measure on $\mathbb{R}$, by Lieb's lemma [8], there exists a sequence $(x_n) \subset \mathbb{R}$ and a subsequence (v_n) of (u_n) such that

$$Av_n(\cdot + x_n) \rightharpoonup f \neq 0$$

in $H^1(\mathbb{R})$. Going if necessary to a subsequence, we can assume that

$$w_n := v_n(\cdot + x_n) - v_n(x_n) \rightharpoonup w$$

in X. Since

$$Aw_n = Av_n(\cdot + x_n) \rightharpoonup f \neq 0$$

it follows that $w \neq 0$. It is clear that

$$\|\varphi'(w_n)\| = \|\varphi'(v_n)\| \to 0.$$

For every $h \in \mathscr{D}(\mathbb{R})$, we have

$$\langle \varphi'(w), h\rangle = \lim_{n\to\infty} \langle \varphi'(w_n), h\rangle = 0.$$

Hence $\varphi'(w) = 0$ and, by Proposition 2, w is a nontrivial solution of (2).

(5) It remains only to prove that $w \in P(X)$. Since, by the preceding lemma,

$$\operatorname{dist}(w_n, P(X)) = \operatorname{dist}(u_n, P(X)) \to 0,$$

there exists a sequence (h_n) in $P(X)$ such that $h_n \rightharpoonup w$. But $P(X)$ is closed and convex, so that $w \in P(X)$. ∎

4. NONMONOTONE SOLITARY WAVES

In this section we assume that

(V_1') $V(u) = \lambda(u^2/2) + W(u)$, $W \in \mathscr{C}^1(\mathbb{R}, \mathbb{R})$ $W(0) = 0$, $W'(u) = o(|u|)$, $u \to 0$, and

(V_2) $\sup_{\mathbb{R}} W > 0$ and there exists $\alpha > 2$ such that for all $u \in \mathbb{R}$,

$$\alpha W(u) \leqslant u W'(u).$$

Remark. Assumption (V_1') allows a negative quadratic part at 0.

THEOREM 8. *Under assumptions* (V_1') *and* (V_2), *for every* c *such that* $c^2 > \max(0, \lambda)$, *Eq.* (2) *has a nontrivial solution* $u \in X$.

Proof. The proof is Similar to the one of preceding section; just use the mountain pass lemma instead of its variant. ∎

5. EXAMPLES

(a) In the case of the Toda lattice

$$V(u) := ab^{-1}(e^{-bu} + bu - 1), \qquad ab > 0,$$

if $b > 0$, for every $c > ab$, Eq. (2) has a nontrivial nonincreasing solution and, if $b < 0$, for every $c > ab$ Eq. (2) has a nontrivial nondecreasing solution.

(b) In the case of the potential

$$V(u) := c_0^2 \frac{u^2}{2} + \frac{u^{2k+1}}{2k+1},$$

for every $c > c_0$, Eq. (2) has a nontrivial nondecreasing solution.

(c) In the case of the potential

$$V(u) := c_0^2 \frac{u^2}{2} + \frac{u^{2k}}{2k},$$

for every $c > c_0$, Eq. (2) has a pair of opposit nontrivial solutions, one nondecreasing and the other nonincreasing.

(d) In the case of the potentials

$$V(u) := \lambda \frac{u^2}{2} + \sum_{i=3}^{k} a_i |u|^i + \sum_{i=k+1}^{n} b_i |u|^i$$

with $a_i \leqslant 0$, $b_i \geqslant 0$, $b_n > 0$, for every $c^2 > \max(0, \lambda)$, Eq. (2) has a nontrivial not necessarily monotone solution.

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