Strongly and Weakly Harmonizable Stochastic Processes of $H$-Valued Random Variables

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Let $H$ be a Hilbert space and $(\Omega, \mathcal{F}, \mu)$ be a probability measure space. Consider the Hilbert space $L_2^0(\Omega; H)$ consisting of all $H$-valued strong random variables on $\Omega$ with zero mean which are square integrable with respect to $\mu$. We study $L_2^0(\Omega, H)$-valued processes over the real line $\mathbb{R}$. Strong and weak harmonizabilities are defined for such processes. It is shown that, as in the scalar valued case, every weakly harmonizable process is approximated pointwisely on $\mathbb{R}$ by a sequence of strongly harmonizable processes. To prove this we obtain a series representation of a continuous process.

1. INTRODUCTION


In this paper we generalize the above result to the case of Hilbert space valued second order stochastic processes over $\mathbb{R}$. To this end let $H$ be a Hilbert space and $(\Omega, \mathcal{F}, \mu)$ be a probability measure space. $L_2^0(\Omega; H)$ denotes the Hilbert space of all $H$-valued strong random variables on $\Omega$ with zero mean which are square integrable with respect to (w.r.t.) $\mu$. Then we study $L_2^0(\Omega; H)$-valued processes over $\mathbb{R}$.

Stationary $L_2^0(\Omega; H)$-valued processes have been extensively studied by several authors such as Kallianpur and Mandrekar [4], Mandrekar and Salehi [6], and Rosenberg [11] (see also Salehi [13] and references
therein). These authors regarded the space $L^2_0(\Omega; H)$ as the space $S(L^2_0(\Omega), H)$ of all Hilbert–Schmidt class operators from $L^2_0(\Omega)$ into $H$ where $L^2_0(\Omega) = \{ f \in L^2(\Omega); \int_{\Omega} fd\mu = 0 \}$, and introduced the $T(H)$-valued Gramian structure and left $B(H)$-module structure in it, where $T(H)$ is the set of all trace class operators on $H$ and $B(H)$ is the algebra of all bounded linear operators on $H$. Spaces with such structures were termed (normal) Hilbert $B(H)$-modules (cf. Kakihara [1, 2] and Ozawa [8]). On the other hand nonstationary processes seem to have been less studied. Lately (weak) harmonizability was introduced in [2] by making use of the study of normal Hilbert $B(H)$-module valued measures which were considered in [1].

In Section 2, we give some preliminary definitions and results which will be used in Section 3. First we give some properties of $L^2_0(\Omega; H)$ as a normal Hilbert $B(H)$-module (cf. [2, 8]). $L^2_0(\Omega; H)$-valued measures on $\mathbb{R}$ and $T(H)$-valued bimeasures on $\mathbb{R} \times \mathbb{R}$ play an important role and we state fundamental properties of them (cf. [1, 2]). In Section 3, we obtain the (Gramian orthogonal) series representation of a continuous process. Strong and weak harmonizabilities are defined and it is shown that every weakly harmonizable process is approximated pointwisely on $\mathbb{R}$ by a sequence of strongly harmonizable processes.

2. PRELIMINARIES

Let $(\Omega, \mathcal{F}, \mu)$ be a probability measure space and $H$ be a Hilbert space with the inner product $(\cdot, \cdot)$ and the norm $\| \cdot \|$. $B(H)$ denotes the Banach space of all bounded linear operators on $H$ with the uniform norm $\| \cdot \|$, and $T(H)$ denotes the set of all trace class operators on $H$ with the trace $\text{Tr}(\cdot)$ and the trace norm $\| \cdot \|_T$. As in the Introduction let $L^2_0(\Omega; H)$ be the set of all $H$-valued strong random variables $x(\cdot)$ on $\Omega$ such that

$$\int_{\Omega} x(\omega) \mu(d\omega) = 0, \quad \int_{\Omega} \| x(\omega) \|^2 \mu(d\omega) < \infty.$$ 

For $x, y \in L^2_0(\Omega; H)$ define

$$(x, y)_2 = \int_{\Omega} (x(\omega), y(\omega)) \mu(d\omega), \quad \| x \|_2 = (x, x)^{1/2}.$$ 

Then $L^2_0(\Omega; H)$ becomes a Hilbert space with the inner product $(\cdot, \cdot)_2$. Moreover, there is a $T(H)$-valued inner product $[\cdot, \cdot]$, called a Gramian, on $L^2_0(\Omega; H)$ defined as follows: for $x, y \in L^2_0(\Omega; H)$

$$([x, y] \phi, \psi) = \int_{\Omega} \left( (x(\omega) \otimes y(\omega)) \phi, \psi \right) \mu(d\omega)$$

where $x(\cdot) \otimes y(\cdot)$ is the rank-one tensor product of $x(\cdot)$ and $y(\cdot)$. 

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for every $\phi, \psi \in H$, where the tensor product $\otimes$ is in the sense of Schatten [14], i.e., $\phi \otimes \psi : H \to H$ is an operator such that $(\phi \otimes \psi) \phi' = (\phi', \psi) \phi$, $\phi' \in H$. Then we have that $[x, y] \in T(H)$ and $\text{Tr}[x, y] = (x, y)$, for $x, y \in L_0^2(\Omega; H)$ (cf. Umegaki and Bharucha-Reid [15]). Hence $L_0^2(\Omega; H)$ is a normal Hilbert $B(H)$-module in the sense of [2, Definitions 2.1 and 2.2], where the terminology Hilbert $B(H)$-module was used in [8]. That is, $L_0^2(\Omega; H)$ is a left $B(H)$-module with the action $(ax)(\cdot) = ax(\cdot)$ for $a \in B(H)$ and $x \in L_0^2(\Omega; H)$, and $[ \cdot, \cdot ]$ satisfies that for $x, y, z \in L_0^2(\Omega; H)$, and $a \in B(H)$,

\[
\begin{align*}
(1^\circ) \ [x, x] &\geq 0, \text{ and } [x, x] = 0 \text{ iff } x = 0; \\
(2^\circ) \ [x + y, z] = [x, z] + [y, z]; \\
(3^\circ) \ [ax, y] = a[x, y]; \\
(4^\circ) \ [x, y]^* = [y, x].
\end{align*}
\]

The Gramian plays an important role. In the sequel we write $X = L_0^2(\Omega; H)$ for the sake of simplicity. We need the notion of modular bases for $X$ which was introduced in [8].

2.1. DEFINITION. A family $\{x_j\}$ of elements in $X$ is said to be modular orthonormal if

\[
\begin{align*}
(1) \ [x_j, x_k] &\geq 0 \text{ for } j \neq k; \\
(2) \ [x_j, x_j]^2 = [x_j, x_j] \text{ and } \|x_j\|_2 = 1 \text{ for each } j.
\end{align*}
\]

A maximal modular orthonormal family is called a modular basis.

As was proved in [8, Theorem 4.5] we can obtain the Fourier expansion of elements in $X$ w.r.t. any modular basis. More fully, the followings are equivalent for a modular orthonormal family $\{x_j\} \subset X$:

(a) $\{x_j\}$ is a modular basis for $X$;

(b) for each $x \in X$, $x = \sum_j [x, x_j] x_j$, where the series converges in the norm $\| \cdot \|_2$.

Note that $X = L_0^2(\Omega; H) = L_0^2(\Omega) \otimes H$, the tensor product of $L_0^2(\Omega)$ and $H$, where $L_0^2(\Omega) = \{ f \in L^2(\Omega); \int_\Omega f d\mu = 0 \}$. For an elementary tensor $f \otimes \phi$ the following identification is made:

\[(f \otimes \phi)(\cdot) = f(\cdot) \phi.\]

Hence, we simply denote by $f \phi$ instead of $f \otimes \phi$. If $\{f_j\}_{j \in I}$ and $\{\phi_\lambda\}_{\lambda \in \Lambda}$ are orthonormal bases for $L_0^2(\Omega)$ and $H$, respectively, then the family $\{f_j \phi_\lambda\}_{j \in I, \lambda \in \Lambda}$ forms an orthonormal basis for $L_0^2(\Omega) \otimes H = X$. Let us denote by $\langle \cdot, \cdot \rangle$ the inner product in $L_0^2(\Omega)$. A modular basis for $X$ is obtained as follows:
2.2. **Lemma.** Let \( \{f_j\} \) be an orthonormal basis for \( L^2_0(\Omega) \) and \( \phi \in H \) be of norm one. Then the family \( \{f_j \phi\} \) forms a modular basis for \( X \).

**Proof.** Observe that

\[
[f_j \phi, f_k \phi] = \int_\Omega f_j(\omega) \phi \otimes f_k(\omega) \phi \, \mu(d\omega)
\]

\[
= \int_\Omega f_j(\omega) f_k(\omega) (\phi \otimes \bar{\phi}) \, \mu(d\omega) = \langle f_j, f_k \rangle \phi \otimes \bar{\phi}.
\]

Hence we have that \( [f_j \phi, f_k \phi] = 0 \) for \( j \neq k \), \( [f_j \phi, f_j \phi] = [f_j \phi, f_j \phi] = 1 \), and \( \| f_j \phi \|_2 = \| f_j \| \cdot \| \phi \| = 1 \). Consequently \( \{f_j \phi\} \) is modular orthonormal. To see that this is a modular basis we show that the condition (\( \beta \)) above holds. Take any \( x \in X \) and let \( \{\phi_{\lambda}\} \) be an orthonormal basis for \( H \). Since \( \{f_j \phi_{\lambda}\}_{j, \lambda} \) forms an orthonormal basis for \( X \), we have that

\[
x = \sum_{j, \lambda} \langle x, f_j \phi_{\lambda} \rangle f_j \phi_{\lambda},
\]

where the series converges in the norm \( \| \cdot \|_2 \). Since the nonzero terms in the above sum are at most countable, we can choose an at most countable subset \( \{\phi_{\lambda}^N\}_{N=1}^\infty \subset \{\phi_{\lambda}\} \) \((1 \leq N < \infty)\) such that

\[
x = \sum_{n=1}^N \sum_j \langle x, f_j \phi_{\lambda}^n \rangle f_j \phi_{\lambda} = \sum_j \sum_{n=1}^N \langle x, f_j \phi_{\lambda}^n \rangle f_j \phi_{\lambda}.
\]

Writing \( \phi_n = \phi_{\lambda}^n \) and putting \( g_n = \sum_j \langle x, f_j \phi_n \rangle f_j \in L^2_0(\Omega) \), we see that \( x = \sum_{n=1}^N g_n \phi_n \). For each \( j \) it holds that

\[
[x, f_j \phi] f_j \phi = \left[ \sum_{n=1}^N g_n \phi_n, f_j \phi \right] f_j \phi = \sum_{n=1}^N \langle g_n, f_j \phi \rangle \phi_n \otimes \bar{\phi} = \sum_{n=1}^N \langle g_n, f_j \phi \rangle f_j \phi_n.
\]

Then we have that

\[
x = \sum_j \sum_{n=1}^N \langle x, f_j \phi_n \rangle f_j \phi_n = \sum_j \sum_{n=1}^N \left( \sum_{m=1}^N \langle g_m \phi_m, f_j \phi_n \rangle \right) f_j \phi_n
\]

\[
= \sum_j \sum_{m,n} \langle g_m, f_j \phi_n \rangle \phi_m \otimes \bar{\phi}_n = \sum_j \sum_{n=1}^N \langle g_n, f_j \phi_n \rangle f_j \phi_n
\]

\[
= \sum [x, f_j \phi] f_j \phi,
\]
from which the desired equality ($\beta$) is obtained. Therefore $\{f_j \phi\}$ is a modular basis for $X$.

A subset $Y$ of $X$ is called a submodule if it is a left $B(H)$-module and is closed w.r.t. $\| \cdot \|_2$. That is, $Y$ is a closed subspace of $X$ and $ax \in Y$ for every $a \in B(H)$ and $x \in Y$. In this case $Y$ is itself a normal Hilbert $B(H)$-module. Denote by $\mathfrak{S}(Y)$ the submodule generated by a subset $Y$ of $X$. $\mathfrak{S}(Y)$ is the closure of $\{ax + by; a, b \in B(H), x, y \in Y\}$.

2.3. Remark. (1) In view of [8, Theorem 4.2] for every submodule $Y$ of $X$ there is a unique closed subspace $K$ of $L_0^2(\Omega)$ such that $Y = K \otimes H$.

(2) Let $Y$ be a closed subspace of $X$ and $\{f_j \phi_j; (j, \lambda) \in \mathfrak{S}\}$ be an orthonormal basis of it where $\{f_j\}$ and $\{\phi_j\}$ are orthonormal families in $L_0^2(\Omega)$ and $H$, respectively. Then the submodule $\mathfrak{S}(Y)$ generated by $Y$ is obtained as follows: Put $J = \{j; (j, \lambda) \in \mathfrak{S}\}$ and let $K$ be the closed subspace of $L_0^2(\Omega)$ spanned by $\{f_j\}_{j \in J}$. Then we have that $\mathfrak{S}(Y) = K \otimes H$.

Let $\mathfrak{R}$ be the real line and $\mathfrak{B}$ be its Borel $\sigma$-algebra. $\mathfrak{B} \times \mathfrak{B}$ denotes the algebra generated by the family $R(\mathfrak{B} \times \mathfrak{B}) = \{A \times B; A, B \in \mathfrak{B}\}$ of all rectangles. We consider $X$-valued measures on $\mathfrak{B}$ and $T(H)$-valued bimeasures on $\mathfrak{B} \times \mathfrak{B}$. $ca(\mathfrak{B}; X)$ denotes the set of all $X$-valued bounded and countably additive (CA) measures on $\mathfrak{B}$. The variation of $\xi \in ca(\mathfrak{B}; X)$ is the function $|\xi|(\cdot)$ whose value on a set $A \in \mathfrak{B}$ is given by

$$|\xi|(A) = \sup \sum_{k=1}^{n} \|\xi(A_k)\|_2,$$

where the supremum is taken for all finite partitions $\{A_1, ..., A_n\} \subset \mathfrak{B}$ of $A$. The operator semivariation of $\xi$ is the function $\|\xi\|_0(\cdot)$ whose value on a set $A \in \mathfrak{B}$ is given by

$$\|\xi\|_0(A) = \sup \left\| \sum_{k=1}^{n} a_k \xi(A_k) \right\|_2,$$

(2.1)

where the supremum is taken for all finite partitions $\{A_1, ..., A_n\} \subset \mathfrak{B}$ of $A$ and for all finite subsets $\{a_1, ..., a_n\} \subset B(H)$ with $\|a_k\| \leq 1$, $1 \leq k \leq n$. The semivariation $\|\xi\|(A)$ ($A \in \mathfrak{B}$) is defined in (2.1) by replacing $a_k$ with $\lambda_k \in \mathfrak{C}$ (the complex number field) such that $|\lambda_k| \leq 1$, $1 \leq k \leq n$. Clearly we have for each $A \in \mathfrak{B}$

$$\|\xi(A)\|_2 \leq \|\xi\|(A) \leq \|\xi\|_0(A) \leq |\xi|(A).$$

A measure $\xi \in ca(\mathfrak{B}; X)$ is said to be of bounded operator semivariation (of
Denote by $bca(R; X)$ the set of all elements in $ca(R; X)$ of $BOS$. For $\zeta \in ca(R; X)$ and $x \in X$ define $\zeta \circ x$ by

$$(\zeta \circ x)(A) = [\zeta(A), x], \quad A \in \mathcal{B}. \quad (2.2)$$

Then we see that $\zeta \circ x$ is a $T(H)$-valued bounded and CA (in the trace norm) measure on $\mathcal{B}$, in symbols $\zeta \circ x \in ca(R; T(H))$. Moreover the following equality holds (cf. [1, Proposition 5.2]): for $A \in \mathcal{B}$

$$\|\zeta\|_{0}(A) = \sup \{ |(\zeta \circ x)(A); x \in X, \|x\|_{2} \leq 1 \}, \quad (2.3)$$

where $|(\zeta \circ x)(\cdot)|$ is the variation of the $T(H)$-valued measure $\zeta \circ x$.

$M = M(R \times R; T(H))$ denotes the set of all $T(H)$-valued bimeasures on $\mathcal{B} \times \mathcal{B}$ satisfying the following conditions:

1. $M$ is finitely additive on $\mathcal{B} \times \mathcal{B}$;
2. $M$ is a $T(H)$-valued positive definite kernel on $R(\mathcal{B} \times \mathcal{B})$ in the sense that $\sum_{j,k} a_{j}M(A_{j}, A_{k})a_{k}^{*} \geq 0$ for all finite subsets $\{A_{1}, \ldots, A_{n}\} \subseteq \mathcal{B}$ and $\{a_{1}, \ldots, a_{n}\} \subseteq B(H)$, where we denote the value of $M$ at $A \times B \in R(\mathcal{B} \times \mathcal{B})$ by $M(A, B)$ rather than $M(A \times B)$;
3. $M(A, \cdot), M(\cdot, A) \in ca(R; T(H))$ for each $A \in \mathcal{B}$.

The variation of $M \in M$ is the function $|M|(\cdot, \cdot)$ whose value on a set $A \times B \in \mathcal{B} \times \mathcal{B}$ is given by

$$|M|(A, B) = \sup \sum_{j=1}^{m} \sum_{k=1}^{n} \|M(A_{j}, B_{k})\|_{\tau},$$

where the supremum is taken for all finite measurable partitions $\{A_{1}, \ldots, A_{m}\}$ of $A$ and $\{B_{1}, \ldots, B_{n}\}$ of $B$. The operator semivariation of $M$ is the function $\|M\|_{0}(\cdot, \cdot)$ whose value on a set $A \times B \in R(\mathcal{B} \times \mathcal{B})$ is given by

$$\|M\|_{0}(A, B) = \sup \left\| \sum_{j=1}^{m} \sum_{k=1}^{n} a_{j}M(A_{j}, B_{k})b_{k}^{*} \right\|_{\tau},$$

where the supremum is taken for all finite measurable partitions $\{A_{1}, \ldots, A_{m}\}$ of $A$ and $\{B_{1}, \ldots, B_{n}\}$ of $B$, and for all finite subsets $\{a_{1}, \ldots, a_{m}\}, \{b_{1}, \ldots, b_{n}\} \subseteq B(H)$ with $\|a_{j}\|, \|b_{k}\| \leq 1, 1 \leq j \leq m, 1 \leq k \leq n$. $M_{e}$ and $M_{b}$ denote the sets of all elements in $M$ of bounded variation ($|M|(R, R) < \infty$) and of $BOS$ ($\|M\|_{0}(R, R) < \infty$), respectively. Clearly we have $M_{e} \subseteq M_{b}$.

For $\zeta \in ca(R; X)$ define $M_{\zeta}$ by

$$M_{\zeta}(A, B) = [\zeta(A), \zeta(B)], \quad A, B \in \mathcal{B}.$$
Then we see that $M_\xi \in M$. Moreover, the following holds (cf. [1, Lemma 3.6]): for $A, B \in \mathfrak{B}$

$$\|M_\xi\|_0(A, B) \leq \|\xi\|_0(A) \cdot \|\xi\|_0(B);$$

$$\|M_\xi\|_0(A, A) = \|\xi\|_0(A)^2;$$

$$|M_\xi|(A, B) \leq |\xi|(A) \cdot |\xi|(B).$$

(2.4)

Next we consider integration of $B(H)$-valued functions w.r.t. $X$-valued measures and $T(H)$-valued bimeasures. When $\xi \in ca(R; X)$ is orthogonally scattered, i.e., $[\xi(A), \xi(B)] = 0$ for every disjoint pair $A, B \in \mathfrak{B}$, a beautiful theory was obtained by Mandrekar and Salehi [6]. When $\xi$ is of BOS, the following consideration was made in [1]. A $B(H)$-valued simple function on $R$ is a function of the form

$$\sum_{j=1}^{n} 1_{A_j} a_j, \quad a_j \in B(H), \ A_j \in \mathfrak{B}, \ 1 \leq j \leq n,$$

where $1_A$ denotes the characteristic function of $A \in \mathfrak{B}$. Denote by $L^0(R; B(H))$ the set of all $B(H)$-valued simple functions on $R$. The integral of $\phi = \sum 1_{A_j} a_j \in L^0(R; B(H))$ w.r.t. $\xi$ over $A \in \mathfrak{B}$ is defined by

$$\int_A \phi d\xi = \sum_{j} a_j \xi(A \cap A_j).$$

For another $\Psi = \sum 1_{B_k} b_k \in L^0(R; B(H))$ the integral of $(\phi, \Psi)$ w.r.t. $M_\xi \in M_b$ (by (2.4)) over $A \times B \in \mathfrak{B} \times \mathfrak{B}$ is defined by

$$\int_{A \times B} \phi dM_\xi \Psi^* = \sum_{j,k} a_j M_\xi(A \cap A_j, B \cap B_k) b_k^*.$$

A set $A \in \mathfrak{B}$ is said to be $\xi$-null if $\|\xi\|_0(A) = 0$. The term $\xi$-almost everywhere (\(\xi\text{-a.e.}\)) refers to the complement of a $\xi$-null set. For $\phi \in L^0(R; B(H))$ define the $\xi$-essential sup norm by

$$\|\phi\|_{\infty} = \inf \{ \alpha > 0; \{s \in R; \|\phi(s)\| > \alpha \} \text{ is } \xi\text{-null}\}.$$

(2.6)

Then we see that for $\phi, \Psi \in L^0(R; B(H))$ and $A, B \in \mathfrak{B}$

$$\left\|\int_A \phi d\xi\right\|_2 \leq \|\phi\|_{\infty} \cdot \|\xi\|_0(A);$$

$$\left\|\int_{A \times B} \phi dM_\xi \Psi^*\right\|_1 \leq \|\phi\|_{\infty} \cdot \|\Psi\|_{\infty} \cdot \|M_\xi\|_0(A, B).$$

(2.7)

(2.8)
Denote by \( L^\infty(\mathbb{R}, \xi; B(H)) \) the set of all \( B(H) \)-valued functions on \( \mathbb{R} \) which are the \( \xi \)-a.e. uniform limits of sequences in \( L^0(\mathbb{R}; B(H)) \). That is, \( \Phi \in L^\infty(\mathbb{R}, \xi; B(H)) \) if and only if there is a sequence \( \{ \Phi_n \} \subseteq L^0(\mathbb{R}; B(H)) \) such that \( \| \Phi_n - \Phi \|_{\infty} \to 0 \) \((n \to \infty)\). With the norm \( \| \cdot \|_{\infty} \) defined in (2.6) \( L^\infty(\mathbb{R}, \xi; B(H)) \) becomes a Banach space. For \( \Phi, \Psi \in L^\infty(\mathbb{R}, \xi; B(H)) \) choose sequences \( \{ \Phi_n \}, \{ \Psi_n \} \subseteq L^0(\mathbb{R}; B(H)) \) such that \( \| \Phi_n - \Phi \|_{\infty}, \| \Psi_n - \Psi \|_{\infty} \to 0 \) \((n \to \infty)\). Then we define the integrals of \( \Phi \) w.r.t. \( \xi \) and of \( (\Phi, \Psi) \) w.r.t. \( M_\xi \), respectively, by

\[
\int \Phi d\xi = \lim_{n \to \infty} \int \Phi_n d\xi;
\]

\[
\iint_{A \times B} \Phi dM_\xi \Psi^* = \lim_{n,m \to \infty} \iint_{A \times B} \Phi_n dM_\xi \Psi_m^*
\]

for \( A, B \in \mathfrak{B} \). These integrals are well-defined because of the inequalities (2.7) and (2.8). For more information we refer to [1, 3].

3. Results

Let \( X = L^2_0(\Omega; H) \), \( \mathbb{R} \), and \( \mathfrak{B} \) be as in Section 2. We consider \( X \)-valued processes over \( \mathbb{R} \).

3.1. Definition. (1) A mapping \( t \to x(t) \) from \( \mathbb{R} \) into \( X \) is called an \( X \)-valued process over \( \mathbb{R} \) or a Hilbert space valued second order stochastic process over \( \mathbb{R} \). We denote it by \( \{x(t)\} \) or \( \tilde{x} \).

(2) The covariance function \( \Gamma \) of an \( X \)-valued process \( \{x(t)\} \) is defined by \( \Gamma(s, t) = [x(s), x(t)] \), \( s, t \in \mathbb{R} \).

(3) An \( X \)-valued process \( \{x(t)\} \) is said to be continuous if the mapping \( t \to x(t) \) is continuous in the norm \( \| \cdot \|_2 \).

(4) An \( X \)-valued process is said to be weakly harmonizable if its covariance function \( \Gamma \) is of the form

\[
\Gamma(s, t) = \iint_{\mathbb{R}^2} e^{i(su - tv)} M(du, dv), \quad s, t \in \mathbb{R}
\]

for some bimeasure \( M \in \mathcal{M}_b \) of \( \mathcal{B} \).

(5) An \( X \)-valued process is said to be strongly harmonizable if its covariance function \( \Gamma \) is of the form (3.1) for some bimeasure \( M \in \mathcal{M}_e \) of bounded variation.

(6) The time domain \( \mathfrak{S}(\tilde{x}) \) of an \( X \)-valued process \( \tilde{x} = \{x(t)\} \) is defined as a submodule \( \mathfrak{S}(\tilde{x}) = \mathfrak{S} \{x(t); t \in \mathbb{R}\} \).
3.2. Remark [2]. An $X$-valued process \{\{x(t)\}\} is weakly harmonizable if and only if there is some measure $\xi \in \text{bca}(\mathbb{R}; X)$ such that

$$x(t) = \int_{\mathbb{R}} e^{iut}\xi(du), \quad t \in \mathbb{R}$$

since every measure in $\text{ca}(\mathbb{R}; X)$ is necessarily regular. In this case the covariance function $\Gamma$ is given by (3.1) with $M = M_\xi$. Every $X$-valued weakly harmonizable process is (uniformly) continuous.

We now give a Gramian orthogonal series representation for an $X$-valued continuous process over $\mathbb{R}$.

3.3 Proposition. Let $\hat{x} = \{x(t)\}$ be an $X$-valued continuous process over $\mathbb{R}$. Then there exist sequences \{\{x_n\}\}_{n=1}^\infty \subset X$ and \{\{a_n(t)\}\}_{n=1}^\infty$ of $T(H)$-valued continuous (w.r.t. the trace norm) functions on $\mathbb{R}$ such that

$$x(t) = \sum_{n=1}^\infty a_n(t)x_n, \quad t \in \mathbb{R},$$

where the series converges in the norm $\| \cdot \|_2$ for each $t \in \mathbb{R}$, $[x_n, x_m] = 0$ for $n \neq m$ and $\|x_n\|_2 = 1$ for $n \geq 1$.

Proof. Let $Y$ be the closed subspace of $X$ spanned by $\{x(t); t \in \mathbb{R}\}$. Clearly we have that the time domain $\mathcal{S}(\hat{x})$ equals to the submodule $\mathcal{E}(Y)$. Since $\mathbb{R}$ is separable and $\hat{x}$ is continuous, $Y$ is separable. It follows from Remark 2.3(2) that $\mathcal{S}(\hat{x}) = K \otimes H$ for some closed subspace $K$ of $L_2(\Omega)$ and that $K$ is separable. Hence, by Lemma 2.2, there is a countable set $\{x_n\}_{n=1}^\infty \subset \mathcal{S}(\hat{x})$ forming a modular basis for $\mathcal{S}(\hat{x})$. Consequently we have that

$$x(t) = \sum_{n=1}^\infty \langle x(t), x_n \rangle x_n, \quad t \in \mathbb{R}.$$  

Putting $a_n(t) = \langle x(t), x_n \rangle$, $n \geq 1$, the desired results follow.

Our main result is the following.

3.4. Theorem. Let $\{x(t)\}$ be an $X$-valued weakly harmonizable process over $\mathbb{R}$. Then there exists a sequence $\{x_n(t)\}$, $n \geq 1$, of $X$-valued strongly harmonizable processes over $\mathbb{R}$ such that $x_n(t) \to x(t)$ ($n \to \infty$) for all $t \in \mathbb{R}$ in the norm $\| \cdot \|_2$. The convergence is uniform on each compact subset of $\mathbb{R}$.

Proof. Since $\hat{x} = \{x(t)\}$ is continuous (cf. Remark 3.2), it follows from
Proposition 3.3 that there exists a modular basis \( \{ x_n \}_{n=1}^{\infty} \) for the domain \( \mathcal{S}(\mathcal{x}) \) such that

\[
x(t) = \sum_{n=1}^{\infty} \left[ x(t), x_n \right] x_n, \quad t \in \mathbb{R}.
\]

On the other hand, again by Remark 3.2, there is some measure \( \xi \in bca(\mathbb{R}; X) \) such that

\[
x(t) = \int_{\mathbb{R}} e^{iu \xi}(du), \quad t \in \mathbb{R}.
\]

Define for each \( n \geq 1 \)

\[
x_n(t) = \sum_{k=1}^{n} \left[ x(t), x_k \right] x_k, \quad t \in \mathbb{R};
\]

\[
\xi_n(A) = \sum_{k=1}^{n} \left[ \xi(A), x_k \right] x_k, \quad A \in \mathcal{B}.
\]

Then we see that \( \xi_n \in ca(\mathbb{R}; X) \) and that for \( t \in \mathbb{R} \)

\[
x_n(t) = \sum_{k=1}^{n} \left[ \int_{\mathbb{R}} e^{iu \xi}(du), x_k \right] x_k
\]

\[
= \int_{\mathbb{R}} e^{iu \xi} \sum_{k=1}^{n} \left[ \xi(du), x_k \right] x_k = \int_{\mathbb{R}} e^{iu \xi_n(du)}.
\]

To show that, for each \( n \geq 1 \), \( \{ x_n(t) \} \) is strongly harmonizable it is sufficient to prove that \( \xi_n \) is of bounded variation because of the inequality (2.5). Let \( \{ A_1, \ldots, A_m \} \subset \mathcal{B} \) be a finite partition of \( \mathbb{R} \). Then we have

\[
\sum_{j=1}^{m} \left\| \xi_n(A_j) \right\|_2 = \sum_{j=1}^{m} \left\| \sum_{k=1}^{n} \left[ \xi(A_j), x_k \right] x_k \right\|_2
\]

\[
\leq \sum_{j=1}^{m} \sum_{k=1}^{n} \left\| \left[ \xi(A_j), x_k \right] x_k \right\|_2
\]

\[
\leq \sum_{k=1}^{n} \sum_{j=1}^{m} \| \xi \circ x_k(A_j) \|_\infty \quad \text{(see (2.2))}
\]

\[
\leq \sum_{k=1}^{n} | \xi \circ x_k |(\mathbb{R})
\]

\[
\leq n \cdot \| \xi \|_\alpha(\mathbb{R}) \quad \text{(by (2.3))}.
\]
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Hence $|\xi_n|_p(R) \leq n \cdot \|\xi\|_P(R) < \infty$ for each $n \geq 1$. It is clear that $x_n(t) \to x(t)$ ($n \to \infty$) in the norm $\|\cdot\|_2$ for each $t \in \mathbb{R}$. Uniform convergence on each compact subset of $\mathbb{R}$ follows from the (metric) approximation property of the Hilbert space $\mathcal{H}(\mathbb{R})$.

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