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Bijective preimages of ω_1

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Abstract

We study the structure of spaces admitting a continuous bijection to the space of all countable ordinals with its usual order topology. We relate regularity, zero-dimensionality and pseudonormality. We examine the effect of covering properties and ω_1-compactness and show that locally compact examples have a particularly nice structure assuming MA + $\neg$CH. We show that various conjectures concerning normality-type properties in products can be settled (modulo set-theory) amongst such spaces.

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1. Preamble

In [28], Reed defines the class $\mathcal{C}$ of spaces $(X,\mathcal{T})$, where X has size ω_1 and $\mathcal{T}$ is the join of two topologies $\mathcal{T}_{\mathbb{R}}$ (which makes X homeomorphic to a subset of $\mathbb{R}$) and $\mathcal{T}_{\omega_1}$ (which makes X homeomorphic to the ordinal space ω_1). Reed calls $\mathcal{C}$ the class of 'intesection' topologies since such spaces have a base of the form $\{B\cap G\colon B\in\mathcal{T}_{\mathbb{R}},\ G\in\mathcal{T}_{\omega_1}\}$. This construction was inspired by various specific constructions, for example, Pol's perfectly normal, locally metrizable, nonmetrizable space, Pol and Pol's hereditarily normal, strongly zero-dimensional space with a subspace of positive dimension (see [28]), and has also been studied by van Douwen [7], Jones [17] and Kunen [20]. Motivated by Reed's definition, we define $\mathcal{W}$ to be the class of all continuous bijective preimages of the space of countable ordinals and we analyse the structure of such spaces. In [12], we characterize bijective preimages of arbitrary ordinals.

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We begin with some remarks concerning regularity and first countability and then look at covering properties, ω_1-compactness, normality and countable paracompactness, and the effect of Martin's axiom together with local compactness on $\mathcal{W}$. Covering properties, as one might expect, have a significant effect on members of $\mathcal{W}$; for example, a regular X in $\mathcal{W}$ is paracompact if and only if it has a club set of isolated points. On the other hand, ω_1-compactness ensures that much of the structure of ω_1 remains, since only stationary sets can be both closed and uncountable. We end with a few examples, mostly concerning normality-type properties in products. It is not suprising that many of these examples are set-theoretic since, assuming MA + $\neg$CH, any locally compact X in $\mathcal{W}$ is either a normal nonmetrizable Moore space, a metrizable LOTS or contains a club set which has its usual order topology (Theorem 6.1), whilst there is a locally compact Dowker space in $\mathcal{W}$ assuming $\diamondsuit^*$ [13]. Fleissner was prompted to call de Caux's Dowker construction a litmus test for set-theoretic models. The same could be said of $\mathcal{W}$.

Obviously, every X in $\mathcal{C}$ is a member of $\mathcal{W}$ and some results about $\mathcal{W}$ generalize results about $\mathcal{C}$. However, there are differences and it is worth comparing the two classes. No member of $\mathcal{C}$ can be locally compact and the tension between $\mathbb{R}$ and ω_1 gives a global nature to constructions in $\mathcal{C}$, whereas in $\mathcal{W}$ it is natural to aim for locally compact examples, defined inductively. If X is in $\mathcal{W}$ and is ω_1-compact, then it is strongly collectionwise Hausdorff if it is regular, and collectionwise normal if and only if it is normal. In $\mathcal{W}$ countable paracompactness does not imply normality (Example 7.2, also [13] for an ω_1-compact, strongly collectionwise Hausdorff example) and, for locally compact spaces, the converse is consistent and independent (Theorem 6.1 and [13]). In $\mathcal{C}$ normality, countable paracompactness, strong collectionwise Hausdorffness, collectionwise normality and ω_1-compactness all coincide. Reed proves that under MA + $\neg$CH every X in $\mathcal{C}$ is perfect, and Kunen shows that no member of $\mathcal{C}$ is both normal and perfect. This situation generalizes to $\mathcal{W}$, since no X in $\mathcal{W}$ can be both ω_1-compact and perfect. Kunen also shows that there is a model of set theory in which $\mathcal{C}$ contains both normal and perfect elements, and that, assuming CH, every X in $\mathcal{C}$ contains a closed unbounded (club) set D which is a normal subspace. Since D is also a member of $\mathcal{C}$ and there is a nonnormal X in $\mathcal{C}$ (see Example 7.2), this is about as close as possible to reversing the situation under MA + $\neg$CH. One might compare this with our result under Martin's axiom: in $\mathcal{C}$, where no element can be locally compact, it is the Q-sets assured by MA + $\neg$CH that have the significant effect; in $\mathcal{W}$ it is the effect of local compactness together with MA + $\neg$CH that is important.

All spaces are Hausdorff and our notation is standard, as found in [10,19,21]. We use the fact that a nonstationary subset of ω_1 is σ-discrete and metrizable (see [8]) and that a stationary subset of ω_1 may be partitioned into ω_1 many disjoint stationary sets. We distinguish between σ-closed discrete and σ-discrete subsets. The limit type of a point in a scattered space is denoted $\mathrm{lt}(x)$. A space is κ-compact if every subset of size κ has a limit point, has the DFCC (or DCCC), if every discrete collection of open sets is finite (or countable). A space is pseudonormal if every pair of disjoint closed sets can be separated by disjoint open sets, provided at least one of them is countable.

Given an X in $\mathcal{W}$ there will be several possible maps from X to ω_1, however, we ignore this, fixing a map and regarding an element of $\mathcal{W}$ as a copy of ω_1 together with a topology which refines the usual order topology. We may refer to points of a given X in $\mathcal{W}$ by their corresponding names in ω_1 and we often talk about a subset of X as being nonstationary, stationary or club if it is in ω_1. A *basic* open set about a point x is always taken to be a subset of a basic open ω_1-interval, $(\gamma, x]$.

Some basic facts are summarized in the following lemma, the proof of which is trivial, bearing in mind the following: Examples 1.6.19 and 1.6.20 of [10] can easily be modified to show that members of $\mathcal{W}$ need not be either Fréchet or sequential. Since initial segments are compact, countably compact X in $\mathcal{W}$ are homeomorphic to ω_1. If X is not homeomorphic to ω_1, there must be an ω-sequence which does not have a limit. Hence the DFCC and regular, pseudocompact X in $\mathcal{W}$ are homeomorphic to ω_1. (The first countable, nonregular space described in Example 2.1 below is pseudocompact but not homeomorphic to ω_1.)

Lemma 1.1. *If X is a member of $\mathcal{W}$, then X is a locally countable, countably tight, Hausdorff scattered space of cardinality ω_1 with countable pseudocharacter and character $\leqslant \mathfrak{c}$, but need not be Fréchet or sequential.*

Further, X cannot be Lindelöf or have the CCC and, if it is countably compact, has the DFCC or is both regular and pseudocompact, then it is homeomorphic to ω_1.

Let D be nonstationary and $C = \{x_\alpha\colon\ \alpha \in \omega_1\}$ a disjoint club and let D_α be the set $\{y \in X\colon\ x_\alpha < y < x_{\alpha+1}\}$. Then $\{D_\alpha\colon\ \alpha \in \omega_1\}$ is a collection of open (in ω_1, as well as X) sets whose union is nonstationary and misses C and $\{C\} \cup \{D_\alpha\colon\ \alpha \in \omega_1\}$ partitions X. Thus we have

Lemma 1.2. *Let X be a member of $\mathcal{W}$. If D is a nonstationary subset of X, then D can be covered by a collection $\mathcal{U}$ of pairwise disjoint, countable sets which are open in ω_1 and whose union is nonstationary. If X is regular (and first countable), then the union is paracompact (metrizable). In fact X is first countable and regular if and only if nonstationary sets are metrizable.*

2. Local properties

Example 2.1. Let $X = \omega_1$ have the usual order topology. If, in addition, we declare sets of the form

$$\{\omega^2\} \cup \bigcup \{(\omega k, \omega(k+1))\colon\ n \leqslant k \leqslant \omega\}$$

to be open, then X is first countable but fails to be either regular or locally compact at the point ω^2. Since every sequence of successor ordinals below ω^2 has a limit, every continuous function from $(0, \omega^2]$ to $\mathbb{R}$ is bounded and X is pseudocompact. It is clear that this space does not have the DFCC and is not homeomorphic to ω_1. If instead we declare sets of the form

$$\{\omega^2\} \cup \bigcup \{ (\omega k + m_k, \omega(k+1)] \colon\ m_k \in \omega \}$$

to be open, for any sequence $\{m_k\}_{k\in\omega}$ from ω, then X is regular but fails to be either first countable or locally compact at the point ω^2. If we declare sets of the form

$$\{\omega^2\} \cup \bigcup \{ (\omega k + m_k, \omega(k+1)) \colon\ m_k \in \omega \}$$

to be open, then regularity, first countability and local compactness all fail at ω^2. Furthermore, if we isolate every point ωk below ω^2, the resulting space is regular and first countable but not locally compact.

Again, since a compact topology coincides with a coarser Hausdorff one, we have

Lemma 2.2. *Let X be a member of $\mathcal{W}$ and suppose that X is locally compact at some point x. If C is a compact neighbourhood of x, then the subspace topology on C is the same as the topology induced on C by the usual ω_1 topology. In particular, if X is locally compact, then it is regular and first countable.*

It is easy to see that first countable, collectionwise Hausdorff spaces are regular and, if the subspace $(\beta,\alpha]$ of some X in $\mathcal{W}$ is collectionwise Hausdorff and $\mathrm{lt}(\alpha)$ is a successor, then X is regular at α. However, Example 3 of [23] describes an hereditarily collectionwise Hausdorff refinememnt (at the point ω^ω) of the usual topology on the countable ordinal space $\omega^\omega + 1$ which fails to be regular at ω^ω. Hence collectionwise Hausdorffness does not imply regularity. On the other hand, if X is regular, then it is collectionwise Hausdorff with respect to nonstationary closed discrete sets by Lemma 1.2 and, as regularity is hereditary, regular X in $\mathcal{W}$ are collectionwise Hausdorff with respect to any discrete set that is not stationary.

Lemma 2.3. *If Y is a closed discrete subset of some X in $\mathcal{W}$ and Y is separated by open sets (i.e., there are disjoint open neighbourhoods about each point), then all but a nonstationary subset of Y consists of isolated points. If X is not collectionwise Hausdorff, then it has a closed discrete stationary set of nonisolated points.*

If X in $\mathcal{W}$ is regular and collectionwise Hausdorff, then it is collectionwise normal with respect to closed nonstationary sets and, if X in $\mathcal{W}$ is normal and collectionwise Hausdorff, then it is collectionwise normal with respect to collections containing countably many stationary sets.

Proof. The first paragraph is trivial by the pressing down lemma.

Let $\{D_\alpha\colon\ \alpha \in \omega_1\}$ be a discrete collection of closed, nonstationary subsets. By Lemma 1.2, each D_α can be partitioned into a discrete collection of countable clopen sets $\{D_{\alpha,\beta}\colon\ \beta \in \omega_1\}$. Let $\{C_\delta\colon\ \delta \in \omega_1\}$ list $\{D_{\alpha,\beta}\colon\ \alpha,\beta \in \omega_1\}$, let $\{c_{\delta,n}\}_{n\in\omega}$ list C_δ and let $B_n = \{c_{\delta,n}\colon\ \delta \in \omega_1\}$. It is sufficient to separate $\{C_\delta\}$, which is a discrete collection of closed sets. B_n is a closed discrete subset of X and, by the first part, all but a nonstationary subset N_n of B_n consists of isolated points. Let $N = \bigcup_n N_n$. N is nonstationary and X is regular, so N is contained in a nonstationary, open paracompact subset M. We can therefore separate $\{C_\delta \cap M\colon \delta \in \omega_1\}$ and are done. The last claim follows similarly. $\square$

As we point out later, normal X in $\mathcal{W}$ are collectionwise Hausdorff assuming $V = L$, whilst the ladder space built over a stationary set (Example 7.3) is always locally compact, regular, first countable (and normal assuming MA + $\neg$CH) but never collectionwise Hausdorff.

Given Lemma 1.2, it should be clear that X is regular and first countable if and only if it is locally metrizable if and only if nonstationary subsets are metrizable and can be covered by a metrizable set which is open in ω_1. Given that locally countable, Tychonoff spaces are zero-dimensional as well as Lemma 1.2, the proof of the following proposition should also be clear.

Proposition 2.4. *For any X in $\mathcal{W}$, the following are equivalent:*

(i) *X is regular;*

(ii) *X is Tychonoff;*

(iii) *X is (hereditarily) pseudonormal;*

(iv) *if C and D are any two disjoint closed subsets, at least one of which is countable, then there is a continuous map from X to $[0,1]$ taking C to $\{0\}$ and D to $\{1\}$;*

(v) *any two disjoint closed nonstationary subsets of X can be separated by disjoint open nonstationary sets;*

(vi) *X is zero-dimensional.*

For regular (i.e., zero-dimensional) X in W, 2^{ω_1} is a universal space (see [10]). For arbitrary X in $\mathcal{W}$, $2^{\mathcal{P}\omega_1}$ is universal: given $\mathcal{T}$ refining the usual topology on ω_1 define $f\colon (X,\mathcal{T}) \to 2^{\mathcal{P}\omega_1}$ by $f(x,U) = \chi_U(x)$ where $\chi_U(x)$ is 1 if and only if $x \in U \in \mathcal{T}$ and 0 otherwise (see [29, 2.4]).

Of course we cannot expect to deduce normality from regularity and, as the next example shows, we cannot even expect to be able to separate a nonstationary closed set from a disjoint stationary set.

Example 2.5. Let X be the set ω_1 and let

$$W = \{\alpha + \omega\colon \alpha \in \omega_1\} \quad \text{and} \quad R = \{\alpha\colon \operatorname{lt}(\alpha) \geqslant 2\}.$$

Partition R into ω stationary sets $\{S_n\colon n \in \omega\}$. Topologize X by giving each of the sets $X - R$ and $T_n = S_n \cup \{\alpha + n\colon \alpha \in \omega_1\}$, $n \in \omega$, the subspace topology inherited from the usual topology on ω_1 and declaring each T_n open. Since regularity is preserved in subspaces, and each of the sets $X - R$ and S_n, $n \in \omega$, are mutually disjoint, X is regular. W and R are disjoint closed subsets of X, W is nonstationary and R is stationary. However, it is easy to see using the pressing down lemma that they cannot be separated by disjoint open sets. See also Example 7.3, where a locally compact example is constructed assuming ♣. Theorem 6.1 suggests that some set-theoretic assumption is needed in the locally compact case.

3. Covering properties

Recall that a space is said to be *weakly θ-refinable* if every open cover has an open refinement $\mathcal{G} = \bigcup_{n\in\omega} \mathcal{G}_n$ such that, for each x in X, x meets only finitely many open sets from $\mathcal{G}_n$, for some n. If there exist such $\mathcal{G}_n$, each covering X, then X is said to be *θ-refinable* or *submetacompact*. X is *subparacompact* if every open cover has a σ-discrete closed refinement. A space is screenable if every open cover has a σ-disjoint open refinement and is strongly paracompact if every open cover has a star-finite open refinement.

It is clear that ω_1 and other stationary sets have an extreme dislike for uncountable, locally countable open covers. We would, therefore, expect elements of $\mathcal{W}$ which satisfy covering properties to look very different from ω_1. This is indeed the case, stronger covering properties having stronger effect on ω_1. For example, it is certainly impossible to tell which subsets are the preimages of stationary sets for any paracompact X in $\mathcal{W}$. This is not the case for θ-refinable X in $\mathcal{W}$; noncollectionwise Hausdorffness of the the ladder space is witnessed by a closed discrete stationary set, and assuming MA + ¬CH the space is θ-refinable hence σ-closed discrete.

Proposition 3.1. *Let X be a member of $\mathcal{W}$.*

(1) *X is σ-discrete if and only if it is weakly θ-refinable.*

(2) *X is σ-closed discrete if and only if it is θ-refinable if and only if it is weakly θ-refinable and perfect if and only if it is weakly θ-refinable and has a G_δ-diagonal if and only if it is subparacompact.*

(3) *X is developable (a Moore space) if and only if it is (regular), first countable and σ-closed discrete.*

(4) *X is screenable if and only if it is meta-Lindelöf if and only if it is σ-metacompact if and only if it is σ-para-Lindelöf if and only if it has a club set of isolated points.*

(5) *If X is metacompact then it is screenable. If X is regular then it is screenable if and only if it is (strongly) paracompact. Moreover, if X is also first countable, then it is screenable if and only if it is metrizable*

Proof. Most of the first three equivalences follow directly from [24], but note also that subparacompact spaces are θ-refinable and, if $X = \bigcup_n X_n$, where each X_n is closed discrete, and $\mathcal{U}$ is any open cover, then

$$\{\{U \cap X_n\colon\ u \in \mathcal{U}\}\colon\ n \in \omega\}$$

is a σ-discrete closed refinement.

If a space is screenable or (σ-)para-Lindelöf, then it is meta-Lindelöf, so let us suppose that X is meta-Lindelöf. Let $\mathcal{V}$ be any point countable open refinement of any open cover consisting of countable sets. Unless every stationary set contains an isolated point x, the pressing down lemma provides a contradiction to the point countability of $\mathcal{V}$. Hence there is a club set of isolated points.

Conversely, if $C = \{x_\lambda\}_{\lambda\in\omega_1}$ is a club set of isolated points (with $x_0 = 0$),

$$\{C\} \cup \bigl\{\{y\colon\ x_\lambda < y < x_{\lambda+1}\}\colon\ \lambda \in \omega_1\bigr\}$$

partitions X into a discrete collection of countable, clopen subsets. The rest follows easily, noting that paracompact, regular, first countable scattered spaces are metrizable [24]. □

In fact, by the above and [10, 6.3.2(f)], first countable, regular, paracompact X in $\mathcal{W}$ are LOTS. Given that monotonically normal X in $\mathcal{W}$ are either paracompact or contain a closed stationary subset with its usual topology [2], one might ask whether X in $\mathcal{W}$ is first countable and monotonically normal if and only if it is a LOTS.

Example 3.2. Let $X = \omega_1$. Let neighbourhoods about the ordinal ω^2 be as for the nonregular space described in Example 2.1 and isolate every other point. With this topology X is not regular and is not metacompact but does have a club set of isolated points.

It is clear that any paracompact X in $\mathcal{W}$ is σ-closed discrete. How far is being σ-closed discrete from having a club set of isolated points? By Lemma 2.3 and Proposition 3.1, the following is immediate.

Lemma 3.3. *Let X in $\mathcal{W}$ be σ-closed discrete. If X is collectionwise Hausdorff, then it has a club of isolated points. If X is, in addition, regular* (*and first countable*), *then X is collectionwise Hausdorff if and only if it is paracompact* (*metrizable*).

Assuming $V = L$ (in fact $\diamondsuit$ for stationary systems on ω_1) normal X in $\mathcal{W}$ are collectionwise Hausdorff (see [31], $\diamondsuit$ will not suffice for the same reasons given in [31]), hence collectionwise normal with respect to closed nonstationary subsets. The same is true of countably paracompact X in $\mathcal{W}$. Under MA + $\neg$CH [6] (also in a model in which GCH holds [31]) the ladder space of Example 7.3 is a σ-closed discrete, normal Moore space which is clearly not collectionwise Hausdorff. Hence it is consistent and independent that σ-closed discrete, (first countable) normal or countably paracompact X in $\mathcal{W}$ are collectionwise Hausdorff and hence paracompact (metrizable). Notice that in any case normal, σ-closed discrete X in $\mathcal{W}$ are countably paracompact (since they are Moore spaces). Are normality and countable paracompactness equivalent for σ-closed discrete X in $\mathcal{W}$? (Certainly they are if MA + $\neg$CH or $V = L$.)

4. ω_1-compactness

We would like some topological property that reflects stationarity in $\mathcal{W}$. One candidate might be the fact that nonstationary sets are σ-discrete and metrizable in ω_1. Another that every continuous function from a stationary set to $\mathbb{R}$ is eventually constant. The space described in Example 7.1 satisfies such a property and this is put to use in [14]. However, any X in $\mathcal{C}$ is a continuous preimage of $\mathbb{R}$, so in general this approach will not be effective. It turns out that ω_1-compactness is the correct condition.

Lemma 4.1. *Let X be a member of $\mathcal{W}$. X is ω_1-compact if and only if every nonstationary closed subset is countable.*

Proof. If X is not ω_1-compact, then it contains an uncountable closed discrete set K say which certainly has an uncountable closed nonstationary subset. Conversely suppose that X contains an uncountable closed set H that is not stationary. Let C be a club set disjoint from H and let K be a subset of H such that between any two elements of K there is an element of C and K is an uncountable closed discrete subset. □

Similar facts are true of $\mathcal{C}$ (see [20,28]).

The proof of the following proposition follows trivially from Proposition 3.1 and the fact that discrete collections are countable in the presence of ω_1-compactness

Proposition 4.2. *Let $X \in \mathcal{W}$ be ω_1-compact. Every discrete collection of subsets is countable and X*

(i) *has the DCCC;*
(ii) *is neither perfect nor subparacompact;*
(iii) *is collectionwise Hausdorff if it is regular;*
(iv) *is normal if and only if it is collectionwise normal.*

Of course, X can simultaneously fail to be ω_1-compact and perfect: let $X = \omega_1$ have the topology generated from the usual topology by declaring the set of successors closed (note also that no stationary subset is σ-discrete).

In his thesis and in [9], Tree has made an extensive study of generalizations of ω_1-compactness and the Lindelöf property. Certain of these properties are worth mentioning in the context of $\mathcal{W}$.

A space X is said to be n-star-Lindelöf if for every open cover $\mathcal{U}$ there is a countable subcollection $\mathcal{V}$ of $\mathcal{U}$ such that $\mathrm{st}^n(\bigcup \mathcal{V}, \mathcal{U}) = X$ and is said to be strongly n-star-Lindelöf if the subcollection $\mathcal{V}$ can be replaced by a countable set of points from X. X is said to be ω-star-Lindelöf if for every open cover $\mathcal{U}$ there exists an n and a countable subcollection $\mathcal{V}$ such that $\mathrm{st}^n(\bigcup \mathcal{V}, \mathcal{U}) = X$. (Recall that for a subset B and a countable collection of subsets $\mathcal{A}$, $\mathrm{st}(B, \mathcal{A})$ is the set $\bigcup\{A \in \mathcal{A}\colon\ A \cap B \neq \emptyset\}$ and that $\mathrm{st}^{n+1}(B, \mathcal{A})$ is defined inductively as $\mathrm{st}(\mathrm{st}^n(B, \mathcal{A}), \mathcal{A})$.)

We can summarize the relevant results of [9]: If X is Lindelöf, then it is ω_1-compact. If it is ω_1-compact, then it is strongly 1-star-Lindelöf. If X has the CCC, then it is 1-star-Lindelöf. If X is regular and ω-star-Lindelöf, then it has the DCCC and, if it has the DCCC, it is 2-star-Lindelöf. If X has the DCCC and is perfectly normal, then it has the CCC. If X is strongly n-star-Lindelöf, then it is n-star-Lindelöf, and, if it is

n-star-Lindelöf, then it is strongly $(n+1)$-star-Lindelöf. X is ω-star-Lindelöf if it is n-star-Lindelöf for any n. It is easy to show that locally countable space is n-star-Lindelöf space if and only if it is strongly n-star-Lindelöf.

Proposition 4.3. *Any* 1*-star-Lindelöf X in $\mathcal{W}$ is ω_1-compact.*

Any normal or strongly collectionwise Hausdorff X in $\mathcal{W}$ has the DCCC if and only if it is ω_1-compact.

Proof. Suppose that X is not ω_1-compact and let D be an uncountable closed subset of X, which is discrete in the usual order topology on ω_1. For each point x of D, choose a basic open set meeting D in just the point x. For every other point of X, pick a (countable) basic open neighbourhood which misses D. The open cover consisting of all these neighbourhoods has the property that the first star about any countable subset of X will miss U_x for some (in fact uncountably many) x in D. Therefore X is not strongly 1-star-Lindelöf, in which case it is not 1-star-Lindelöf.

We have already shown that ω_1-compact X in $\mathcal{W}$ have the DCCC, so suppose that X is not ω_1-compact. Let D be an uncountable closed discrete subset that is not stationary. Let $C = \{x_\lambda\}_{\lambda\in\omega_1}$ be a club set disjoint from D. For each λ, pick a point y_λ and an open subset U_λ of $\{y \in X\colon x_\lambda < y < x_{\lambda+1}\}$ such that $U_\lambda \cap D = \{y_\lambda\}$. By normality pick an open set V such that $\{y_\lambda\colon \lambda \in \omega_1\} \subseteq V \subseteq \overline{V} \subseteq \bigcup_{\lambda\in\omega_1} U_\lambda$. Then the collection of open sets $\{V \cap U_\lambda\colon \lambda \in \omega_1\}$ is discrete. Strong collectionwise Hausdorffness also gives such a collection. □

So, for regular X in $\mathcal{W}$, X is strongly 2-star-Lindelöf if and only if it has the DCCC if and only if it is ω-star-Lindelöf, and X is 1-star-Lindelöf if and only if it is ω_1-compact. For normal X in $\mathcal{W}$, all these properties coincide.

Clearly, ω_1 itself distinguishes ω_1-compactness from the Lindelöf property and the CCC. The following example is a modification of an example due to Reed [9]. It is essentially a subspace of the larger Reed machine ([26,27]) over ω_1. It is also an example of a DCCC Moore space that is not DFCC (see Lemma 1.2).

Example 4.4. There is a strongly 2-star-Lindelöf Moore space in $\mathcal{W}$ which is not 1-star-Lindelöf.

For each $\alpha \in \omega_1$, let $\{B_n(\alpha)\colon n \in \omega\}$ be a decreasing, countable neighbourhood base in ω_1 at the point α. Let Q be the set, including 0, of all finite rational sums of the form $\sum_{i=0}^{n} 1/2^{k_i}$ where $k_{i+1} > k_i$. Partition ω_1 in to countably many disjoint stationary sets, indexed by Q, and let $X = \bigcup_{q\in Q} S_q = \omega_1$. For convenience, we denote points of X as (α, q), where α is in S_q and q is in Q.

Suppose that $x = (\alpha, q)$ and that $q = \sum_{i=0}^{m} 1/2^{k_i}$. The nth neighbourhood about x is defined to be the set $N_n(x) = \{x\} \cup (X \cap \bigcup_{k\geqslant n}(B_k(\alpha) \times I_k))$, where I_k is the interval $[q + 1/2^{m+k+1}, q + 1/2^{m+k})$. Let X have the topology generated by these basic open sets. X is a Moore space just as for Reed's original example and, since the topology refines the usual topology on ω_1, X is in $\mathcal{W}$.

Since Q is countable the pressing down lemma yields: $(*)$ If U is any open set containing a stationary subset of S_q, then U contains $((\alpha, \omega_1) \times (q, p]) \cap X$ for some α in ω_1 and some $p > q$ in Q.

Clearly X is not ω_1-compact. Suppose that $\mathcal{U} = \{U_\alpha\colon \alpha \in \omega_1\}$ is an uncountable collection of open sets in X. Without loss of generality we can assume that, for some q in Q, each U_α is a basic open set about a point x_α in S_q. For each x in S_q let B_x be a basic open neighbourhood. By $(*)$ some B_x meets uncountably many U_α. Hence $\mathcal{U}$ is not a discrete collection and X must have the DCCC. By the above, X is strongly 2-star-Lindelöf but not 1-star-Lindelöf.

Again the ladder space provides a locally compact example assuming $\diamondsuit$ (see Example 7.3).

5. Covering properties and ω_1-compactness

As we can see, no X in $\mathcal{W}$ can be both ω_1-compact and paracompact. The following simple modification from [12] of Balogh and Rudin's difficult result [2] illustrates this well.

Theorem 5.1. *A monotonically normal space is paracompact if and only if it does not contain a closed subspace, which is homeomorphic to a stationary subset of some regular cardinal κ if and only if it does not contains a closed subset, which is homeomorphic to some κ-compact X in $\mathcal{W}_\kappa$ for some regular κ.*

Theorem 5.2. *Let $X \in \mathcal{W}$ be either ω_1-compact or σ-closed discrete, or a free topological sum of ω_1-compact and σ-closed discrete clopen subsets. $X \times \omega_1$ is normal if and only if X is normal, countably paracompact and collectionwise Hausdorff.*

Proof. By [16, Theorem 2.3], if $X \times \omega_1$ is normal then X is countably paracompact and (ω_1-)collectionwise normal. If X is ω_1-compact, then normality of the product follows by [16, 3.3]. (Notice that in this case X is collectionwise Hausdorff.) If X is regular, σ-closed discrete and collectionwise Hausdorff, then it has a clopen partition into countable regular pieces, by Lemma 1.2, Proposition 3.1 and Lemma 3.3, and, therefore, has normal product with ω_1. □

Assuming MA + ¬CH, the ladder space of Example 7.3 is a normal, σ-closed discrete, countably paracompact, locally compact Moore space which is not collectionwise Hausdorff and so does not have normal product with ω_1. What happens if σ-closed discrete is replaced by σ-discrete? (In [13], assuming $\diamondsuit^*$, the space $Z \in \mathcal{W}$ is σ-discrete, collectionwise normal, countably paracompact, locally compact and ω_1-compact so that $Z \times \omega_1$ is normal but $Z^2 \times \omega_1$ is not since Z^2 is a Dowker space.) That the theorem is about the best possible can be seen from the following modification of the space Δ constructed by Bešlagić and Rudin [3], also used in [16].

Example 5.3. Bešlagić and Rudin use the axiom $\diamondsuit^{++}$ to construct a collectionwise normal, countably paracompact space Δ, which is shown in [16] to have nonnormal product with ω_1. The point set for Δ is $\{(\gamma, \delta) \in \omega_1^2\colon\ \delta < \gamma\}$. The proofs of collectionwise normality, countable paracompactness and of the non-normality of $\Delta \times \omega_1$ follow essentially from of [3, Lemma 1.2]. We shall associate ω_1 with a subset E of Δ in such a way that the subspace topology on E inherited from Δ satisfies this lemma. It is then easy to verify that E is collectionwise normal and countably paracompact but has nonnormal product with ω_1. It is also easy to check that E is in $\mathcal{W}$. To get the lemma to hold for E, we need an apparent strengthening of $\diamondsuit^{++}$. We use the notation from [3] to state this strengthening and point out that, in fact, it follows immediately from Fleissner's discussion of how to partition the set D in the statement of the axiom into a stationary, co-stationary set [11, p. 72]. Fleissner gives two methods of partitioning D and from the first it is clear that we can state the following version of $\diamondsuit^{++}$:

There is a sequence $\{\mathcal{A}_\alpha\colon\ \alpha \in \omega_1\}$ such that for all $\alpha \in \omega_1$:

(1) (i) $\mathcal{A}_\alpha$ is a family of subsets of α; (ii) $|\mathcal{A}_\alpha| \leqslant \omega$; (iii) $(\alpha - \beta) \in \mathcal{A}_\alpha$ for all $\beta \in \alpha$; (iv) $\mathcal{A}_\alpha$ is closed under finite intersections.

(2) If X is a subset of ω_1, there is a club C_X such that (v) $(X \cap \gamma) \in \mathcal{A}_\gamma$ and $(C_X \cap \gamma) \in \mathcal{A}_\gamma$ for all $\gamma \in C_X$.

(3) Also there are disjoint stationary sets $\{D_\gamma\}_{\gamma \in \omega_1}$ such that, if

$$\mathcal{C}_\alpha = \{C \in \mathcal{A}_\alpha\colon\ C \text{ is club in } \alpha\}$$

and, for X a subset of ω_1,

$$S_X = \{\alpha\colon\ X \cap C \neq \emptyset \text{ for all } C \in \mathcal{C}_\alpha\},$$

then: (vi) $\mathcal{C}_\delta$ is closed under finite intersections for all δ in $\bigcup_\gamma D_\gamma$; (vii) if $\mathcal{S}$ is a countable collection of stationary sets then $\bigcup\{S_X\colon\ X \in \mathcal{S}\} \cap D_\gamma$ is stationary for all $\gamma \in \omega_1$.

Without loss of generality, we assume that D_γ is a subset of (γ, ω_1) and that $\{D_\gamma\}_\gamma$ partitions ω_1. Let $E = \bigcup_\gamma D_\gamma \times \{\gamma\}$ be associated with ω_1 by the projection map $(\gamma, \delta) \mapsto \gamma$.

6. Martin's axiom and local compactness

In this section we prove:

Theorem 6.1. (MA + $\neg$CH) *Suppose that X in $\mathcal{W}$ is locally compact. X is countably metacompact. Further, either*

(i) *X contains a closed subspace homeomorphic to ω_1, or*

(ii) *X is a σ-closed discrete, normal, nonmetrizable Moore space that is not collectionwise Hausdorff, or*

(iii) *X is a metrizable LOTS.*

(As a corollary it is consistent and independent that normality implies countable paracompactness in locally compact members of $\mathcal{W}$. It is also clear that ω_1-compact, locally compact X in $\mathcal{W}$ are, assuming MA+ $\neg$CH, homeomorphic to a copy of ω_1 together with a countable clopen set; assuming $\diamondsuit^*$ there is a Dowker space in $\mathcal{W}$ which is ω_1-compact and locally compact.)

Our proof is based on the following three results from [1,6,18]. To state them we recall some terminology: A family $\mathcal{C}$ separates disjoint members of $\mathcal{A}$ and $\mathcal{B}$ if, given disjoint A from $\mathcal{A}$ and B from $\mathcal{B}$, there are C and D in $\mathcal{C}$ such that $A \subseteq C - D$ and $B \subseteq D - C$. A ladder on a limit α in ω_1 is a strictly increasing sequence $\{\alpha_n\}_{n\in\omega}$ cofinal in α, a ladder system is a collection of ladders for each limit α. A colouring of a ladder system is a collection of functions $\{f_\alpha\colon f_\alpha(\alpha_n) \in 2 \text{ for all } n \in \omega\}$. A uniformization of a colouring of a ladder system is a function $f : \omega_1 \to 2$ with the property that for each limit α there is $n \in \omega$ such that $f(\alpha_k) = f_\alpha(\alpha_k)$, whenever $n \leqslant k$.

Theorem 6.2 (Balogh). (MA+ $\neg$CH) *If X is a locally countable, locally compact space of cardinality less than $\mathfrak{c}$, then X is either σ-closed discrete or contains a perfect preimage of ω_1.*

Theorem 6.3 (Juhász and Weiss). (MA+ $\neg$CH) *Let H_0 and H_1 be subsets of a space X such that $\overline{H_i} \cap H_j = \emptyset$, and $|H_i| = \kappa \leqslant \mathfrak{c}$, $i \neq j$. If, for $i \in 2$, there is a family of closed subsets $\mathcal{A}_i$, which is closed under finite intersections and contains a neighbourhood base for points of H_i, and a family $\mathcal{C}$, which is countable and separates disjoint members of $\mathcal{A}_0$ and $\mathcal{A}_1$, then H_0 and H_1 can be separated by disjoint open sets.*

Theorem 6.4 (Devlin and Shelah). (MA + $\neg$CH) *Every colouring of a ladder system has a uniformization.*

The proof of the following lemma is easy

Lemma 6.5. *Every perfect preimage of ω_1 is a countably compact, noncompact space and no space containing a perfect preimage of ω_1 is σ-discrete.*

Lemma 6.6. (MA + $\neg$CH) *If $X \in \mathcal{W}$ is locally compact and σ-discrete, then it is normal.*

Proof. If X is locally compact and σ-discrete, then it is a σ-closed discrete Moore space by Theorem 6.2 and Proposition 3.1 and can be written as a union of closed discrete sets D_n. By Theorem 6.3, it is enough to separate disjoint, closed (discrete) subsets of each D_k. Let H and K be two such subsets. Since X is a Moore space, D_k is a G_δ and is an intersection of open sets $\bigcap U_n$. For each α in $H \cup K$ choose a neighbourhood base of compact, clopen sets $\{B_\alpha(n)\}_{n\in\omega}$ such that $B_\alpha(n)$ is a subset of U_n. For each limit α, define a ladder $\{\alpha_n\}$, where $\alpha_n = \sup(B_\alpha(n) - U_{n+1})$. The colouring f_α of $B_\alpha(0)$ where f_α takes the value 0 if α is in H and 1 if α is in K induces a colouring of the ladder system. Uniformization of this colouring chooses disjoint neighbourhoods of H and K. $\square$

Proof of Theorem 6.1. By Theorem 6.2, either X contains a perfect preimage of ω_1, or it is σ-closed discrete. If the first holds, then, by Lemma 6.5, X contains a countably compact, noncompact subspace K. This subspace is closed, since X is first countable, and since it is uncountable, it is also an element of $\mathcal{W}$ in its own right. Lemma 1.1, then, implies that K is homeomorphic to ω_1. If X is σ-closed discrete, then, by Lemma 6.6, it is a normal Moore space. By Lemma 3.3, if X is collectionwise Hausdorff, then it is paracompact and, since it is first countable, it is a metrizable LOTS, as mentioned above.

Moore spaces are countably metacompact. Suppose that X contains a closed copy K of ω_1, and that $\{D_n\}_{n\in\omega}$ is a decreasing sequence of closed sets with empty intersection. If every D_n meets K, then there is an n such that D_m has countable intersection with K for all $n < m$. Otherwise, the D_n are nonstationary and, by Lemma 1.2, can be covered by an open, metrizable set. In either case it is easy to see that X is countably metacompact. □

7. Some examples

Dowker proved that a topological space is normal and countably paracompact if and only if its product with the closed unit interval is normal. There is a sequence of similar results. A common theme links these results—they all involve some notion related to being perfect: $X \times [0,1]$ is P if and only if X is Q for pairs of properties $(P;Q)$

(1) (monotonically normal; monotonically normal and (semi-)stratifiable)
(2) (hereditarily normal; perfectly normal)
(3) (normal; normal and countably paracompact)
(4) (δ-normal; countably paracompact)
(5) (perfect (and normal); perfect (and normal))
(6) (orthocompact; countably metacompact)

For references and definitions see [15,25,30,22]. As we have seen ω_1 is decidedly nonperfect and it turns out that for each pair $(P;Q)$ (excepting, of course, the fifth) there is a space in $\mathcal{W}$ satisfying P but not Q, at least modulo some set-theoretic assumption. For the first two ω_1 itself will do and, for the third, the $\diamondsuit^*$ Dowker space [13]. In Example 7.1 a simple modification of the space described in [14, 3.1], based partly on Davies' almost Dowker space [5], gives an example that will do for the fourth and sixth pairings. (A space is orthocompact if every open cover has a refinement every subset of which has open intersection. A set is a regular G_δ if it is a countable intersection of the closures of open sets each containing it. A space is δ-normal if every pair of disjoint closed sets, one of which is a regular G_δ can be separated by disjoint open sets.)

Example 7.1. There is a pseudonormal, δ-normal, orthocompact, almost-Dowker space in $\mathcal{W}$.

Let $X = \omega_1$ and partition X into stationary sets $\{S\} \cup \{S_\alpha\colon \alpha \in \omega_1\} \cup \{T_n\colon n \in \omega\}$. We identify X with a subset of $\omega_1^2 \cup (\omega_1 \times \omega)$: If α is in S then identify α with (α,α) in ω_1^2. If α is in S_β then identify α with (α,β) in ω_1^2. If α is in T_n then identify α with (α,n) in $\omega_1 \times \omega$. Let R be the set $\{(\alpha,\beta)\colon \alpha < \beta \in \omega_1\}$.

If α is not in S then α is isolated. If α is in S then choose a countable, decreasing clopen neighbourhood base $\{B_\alpha(n)\}_{n\in\omega}$ for α. Let the nth basic open neighbourhood of (α,α) in X be the set

$$\{(\alpha,\alpha)\} \cup \left(B^2_\alpha(n) \cap R\right) \cup \bigcup_{j>n} \left(B_\alpha(j) \times \{j\}\right) \cap T_j.$$

Bearing in mind the identification of X made above, with the topology generated by these sets it is clear that X is a first countable, zero-dimensional member of $\mathcal{W}$. Since a diagonal intersection of club sets is club, only a nonstationary subset of S is isolated. We give outline proofs only (see [5,14]).

X is not countably metacompact since the closed subspace $S \cup \bigcup_\alpha S_\alpha$ is not: let $\{D_j\}_{j\in\omega}$ be a decreasing sequence of stationary subsets of S, each D_j is closed but, by the pressing down lemma applied twice to each D_j, if $\{U_j\}$ is a sequence of open sets, U_j containing D_j, then $\bigcap U_j$ is nonempty. Since X is Tychonoff, it is an almost Dowker space.

X is orthocompact because every point of $X - S$ is isolated and S is a closed discrete subset, so that every open cover has a refinement, the intersection of any two elements of which consists entirely of isolated points.

X is δ-normal: Consider the (Moore) subspace $S \cup \bigcup_n T_n$. Let C be any closed set, D a disjoint regular G_δ and $E = D \cap S$. Using the pressing down lemma it is not hard to show that E is either a countable or co-countable subset of S. Since at most one of $C \cap S$ and E can be co-countable, X is pseudonormal and all points in $X - S$ are isolated, C and D can be separated by disjoint open sets.

It is also possible to modify the other construction that Davies describes in [5] to obtain a Tychonoff space in $\mathcal{W}$ that has a point countable base but is not perfect. Note also that the subspace $S \cup \bigcup_n T_n$ is a Moore space hence perfect and countably metacompact.

The Dowker space mentioned above shows that normality is not hereditary in $\mathcal{W}$ and the example mentioned in Example 5.3 satisfies the same properties as the space Δ [3]: it is a normal space with an open cover having no closed shrinking such that every increasing open cover has a clopen shrinking. Assuming $\diamondsuit^*$, there is a locally compact anti-Dowker (countably paracomapct but not normal) space in $\mathcal{W}$ [13] which is both strongly collectionwise Hausdorff and ω_1-compact, both of which (along with countable paracompactness) imply normality in $\mathcal{C}$. In the next example we construct an anti-Dowker space in ZFC. The space is based on an example due to Reed [28] which we outline first.

Example 7.2. There is an anti-Dowker space in $\mathcal{W}$.

First we describe Reed's example of a pseudonormal, collectionwise Hausdorff, non-normal space in $\mathcal{C}$:

Let $X = \omega_1$, and let $\mathcal{T}_{\omega_1}$ be the usual topology on ω_1. Let L be the set $\{\alpha+n\colon n \leqslant \omega\}$, L_2 the set $\{\alpha+\omega^2\colon \alpha \in \omega_1\}$ and let $R = \omega_1 - (L \cup L_2)$. For each $\alpha+\omega^2$ in L_2, $\{\alpha+\omega n\}_n$ is cofinal in $\alpha+\omega^2$ and $\{(\alpha+\omega n, \alpha+\omega(n+1)]\colon n \in \omega, \alpha \in \omega_1\}$ partitions L into disjoint ω_1-intervals. Let $\mathbb{C}$ be a Cantor subset of $\mathbb{R}$, let $\mathcal{B}$ denote a countable base for $\mathbb{R}$ and, for each x in $\mathbb{R}$, let $\{B(x,i)\colon i \in \omega\}$ be a decreasing $\mathbb{R}$-neighbourhood base at x.

We identify the points of X with points of the reals in the following way: Identify L_2 with a subset of $\mathbb{C}$. Associate R with a subset of $\mathbb{R} - \mathbb{C}$ so that $B \cap R$ is stationary for every $B \in \mathcal{B}$. For each $\alpha + \omega^2$ in L_2 and each $n \in \omega$, let $(\alpha + \omega n, \alpha + \omega(n+1)]$ be associated with a countable dense subset of $B(\alpha, n)$. With this identification, let $\mathcal{T}_{\mathbb{R}}$ be the topology that X inherits from $\mathbb{R}$.

Let X have the "intersection" topology $\mathcal{T}$ generated by $\mathcal{T}_{\omega_1} \cup \mathcal{T}_{\mathbb{R}}$. Clearly X is in $\mathcal{C}$, and is therefore first countable, pseudonormal and collectionwise Hausdorff. L_2 is a subset of a Cantor set and is $\mathcal{T}_{\mathbb{R}}$-closed and R is $\mathcal{T}_{\omega_1}$-closed so R and L_2 are disjoint closed subsets of X.

Suppose that U and V are disjoint open sets separating L_2 and R. For each α in L_2, there is an $i_\alpha \in \omega$ such that $(\alpha_{i_\alpha}, \alpha] \cap B(\alpha, i_\alpha)$ is a subset of U. There is an uncountable subset M of L_2, a B in $\mathcal{B}$ and an $i \in \omega$ such that $i_\alpha = i$ and $B(\alpha, i_\alpha) = B$, for all α in M. $R \cap B$ is stationary, so there is a λ in $R \cap B$ which is a $\mathcal{T}_{\omega_1}$-limit of M. But, if $(\beta, \lambda]$ is any $\mathcal{T}_{\omega_1}$-open set containing λ, then $(\alpha_i, \alpha_{i+1}] \subseteq (\beta, \lambda]$ for some α in M, where $(\alpha_i, \alpha_{i+1}]$ is $\mathcal{T}_{\mathbb{R}}$-dense in B. Hence λ is a $\mathcal{T}$-limit of U contained in R and U and V are not disjoint.

Now let $Y = \omega_1$. Let the sets L and L_2, and the topologies $\mathcal{T}_{\omega_1}$ and $\mathcal{T}$ be as defined above. Partition $\omega_1 - L$ into two disjoint stationary sets S_1 and S_2, with $L_2 \subseteq S_1$. We topologize Y so that it is an anti-Dowker space as follows:

The subspace topology on both S_1 and S_2 is precisely the subspace topology inherited from ω_1. If x is in S_2, then a basic open set about x is of the form

$$(B \cap S_2) \cup \bigcup_{y \in B \cap S_2} A_y,$$

where A_y is a basic $\mathcal{T}$-open set and B is a basic open interval from $\mathcal{T}_{\omega_1}$. Basic open sets about points in $L \cup L_2$ are inherited from $\mathcal{T}_{\omega_1}$.

Y with the topology $\mathcal{T}$ is just X and is pseudonormal. If x is in S_1, then the set $L_x = \{y \in L_2\colon\ y < x\}$ is countable and $\mathcal{T}$-closed. By $\mathcal{T}$-pseudonormality, therefore, we can pick an $\mathcal{T}$-open set U_x containing L_x, whose closure misses S_2. Let basic open sets about x in S_1 be those inherited from the usual topology $\mathcal{T}_{\omega_1}$ restricted to the set $S_1 \cup \overline{U_x}^{\mathcal{T}}$.

Clearly Y with this topology is a first countable member of $\mathcal{W}$. To see that it is regular, consider the three cases:

(a) If x is an element of either L or L_2, then it has a base of clopen sets inherited from ω_1.

(b) Let x be an element of S_2. Since X is regular, it is zero-dimensional and there is a $\mathcal{T}$-clopen A_x set containing x disjoint from L_2. If $B_x = \{y \in S_2\colon\ y_x \leqslant y \leqslant x\}$, where y_x is the least element of A_x, then $A_x \cup B_x$ is a clopen set containing x. By construction, x has a base of such clopen sets.

(c) Let x be an element of $S_1 - L_2$. Since the subspace $S_2 \cup L \cup L_2$ is regular, it is pseudonormal. Therefore, there is an open set U_x containing $\{y \in L_2\colon\ y < x\}$ whose closure misses S_2. If $B_x = \{y \in S_1\colon\ y \leqslant x\}$, then $\overline{U_x \cap B_x}$ is a closed neighbourhood of x which misses S_2.

With this information one can see that Y is regular.

The proof that X is not normal only requires that R is stationary. The same argument shows that the disjoint closed sets S_1 and S_2 of Y cannot be separated by disjoint open sets.

To see that Y is countably paracompact, let $\{D_n\}_{n\in\omega}$ be a decreasing sequence of closed subsets with empty intersection. We require a decreasing sequence $\{U_n\}$ of open sets, $D_n \subseteq U_n$, such that $\bigcap \overline{U}_n$ is empty. If some D_n is countable, then we are done. Suppose that each D_n is uncountable. The subspace topology on both S_1 and S_2 is precisely the subspace topology inherited from ω_1. Hence the intersection $\bigcup_m D_m \cap S_i$ is nonempty only if $D_n \cap (S_1 \cup S_2)$ is countable for some n. By construction, α in $S_1 \cup S_2$ is a limit of a cofinal sequence $\{\alpha_n + j_n\}_{n\in\omega}$ in L if and only if it is a limit of $\{\alpha_n + \omega\}$. Hence D_n lies in a clopen set U, which is contained in the paracompact subspace $(0, \beta] \cup L$.

Example 7.3. The ladder space.

Partition $X = \omega_1$ into two disjoint sets A and B. For each α in A, which is a limit of B, choose a sequence $\{\alpha_n\}_{n\in\omega}$ from B cofinal in α. Let neighbourhoods of any such α be α and all but finitely many points of $\{\alpha_n\}$ and isolate all the other points of X. With this topology X is a ladder space (see [31]). X is clearly a locally compact, first countable, regular member of $\mathcal{W}$ of scattered length 2. By the pressing down lemma, if A is stationary then X is not collectionwise Hausdorff. By Theorem 6.1, assuming $\mathrm{MA} + \neg\mathrm{CH}$ X is a σ-closed discrete, hereditarily normal Moore space, which is neither collectionwise Hausdorff or ω_1-compact.

Under $\diamondsuit$ for stationary systems on ω_1, normal X in $\mathcal{W}$ are collectionwise Hausdorff (see [31]). Hence no ladder space with A stationary is normal assuming $\diamondsuit$ for stationary systems.

If we assume $\clubsuit$, then we may take the ladder space to be a strongly 2-star-Lindelöf space, which is not 1-star-Lindelöf (4.4): Let $\{R_\alpha\colon\ \alpha \in \mathrm{LIM} \cap \omega_1\}$ be a $\clubsuit$-sequence. Let A be the set of all limit ordinals and B the set of all nonlimits If $R_\alpha \cap B$ is infinite, then let $\{\alpha_n\}$ be $R_\alpha \cap W$ indexed increasingly; otherwise let $\{\alpha_n\}$ be some arbitrary sequence from B which is cofinal in α. Let $\mathcal{U}$ be an uncountable collection of open sets and U be an uncountable subset of B meeting uncountably many members of $\mathcal{U}$. By $\clubsuit$, $\{\alpha_n\}$ is a subset of U for some α, so $\mathcal{U}$ is not a discrete collection of open sets and Y has the DCCC. Clearly X is not ω_1-compact so we are done by Proposition 4.3. Notice also that X is σ-discrete, regular and locally compact but is neither σ-closed discrete nor normal (in fact it is not possible to separate the nonstationary set $W = \{\alpha + \omega\colon\ \alpha \in \omega_1\}$ from the stationary $A - W$).

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