Quasi-symmetric functions and up–down compositions

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ABSTRACT

Carlitz (1973) [5] and Rawlings (2000) [13] studied two different analogues of up–down permutations for compositions with parts in \{1, \ldots, n\}. Cristea and Prodinger (2008/2009) [7] studied additional analogues for compositions with unbounded parts. We show that the results of Carlitz, Rawlings, and Cristea and Prodinger on up–down compositions are special cases of four different analogues of generalized Euler numbers for compositions. That is, for any \(s \geq 2\), we consider classes of compositions that can be divided into an initial set of blocks of size \(s\) followed by a block of size \(j\) where \(0 \leq j \leq s - 1\). We then consider the classes of such compositions where all the blocks are strictly increasing (weakly increasing) and there are strict (weak) decreases between blocks. We show that the weight generating functions of such compositions \(w = w_1 \cdots w_m\), where the weight of \(w\) is \(\prod_{i=1}^{m} z_{w_i}\), are always the quotients of sums of quasi-symmetric functions. Moreover, we give a direct combinatorial proof of our results via simple involutions.

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1. Introduction

Let \(\mathbb{P} = \{1, 2, 3, \ldots\}\) denote the set of positive integers, \(\mathbb{E} = \{2, 4, 6, \ldots\}\) denote the set of even integers in \(\mathbb{P}\), and \(\mathbb{O} = \{1, 3, 5, \ldots\}\) denote the set of odd integers in \(\mathbb{P}\). Let \(\mathbb{P}_n = \{1, \ldots, n\}\), \(\mathbb{E}_n = \mathbb{E} \cap \mathbb{P}_n\), and \(\mathbb{O}_n = \mathbb{O} \cap \mathbb{P}_n\). Let \(S_n\) denote the set of all permutations of \(\mathbb{P}_n\). Then if \(\sigma = \sigma_1 \sigma_2 \cdots \sigma_n \in S_n\), we define \(\text{Des}(\sigma) = \{i : \sigma_i > \sigma_{i+1}\}\) and \(\text{Ris}(\sigma) = \{i : \sigma_i < \sigma_{i+1}\}\). We say that \(\sigma\) is an up–down permutation if

\(\sigma_1 < \sigma_2 > \sigma_3 < \sigma_4 > \cdots\)

or, equivalently, if \(\text{Des}(\sigma) = \mathbb{E}_{n-1}\) and \(\text{Ris}(\sigma) = \mathbb{O}_{n-1}\). Similarly, we say that \(\sigma\) is a down–up permutation if

\(\sigma_1 > \sigma_2 < \sigma_3 > \sigma_4 < \cdots\)

or, equivalently, if \(\text{Ris}(\sigma) = \mathbb{E}_{n-1}\) and \(\text{Des}(\sigma) = \mathbb{O}_{n-1}\). Clearly, if \(\sigma = \sigma_1 \sigma_2 \cdots \sigma_n \in S_n\) is an up–down permutation, then the complement of \(\sigma\),

\(\sigma^c = (n+1-\sigma_1)(n+1-\sigma_2)\cdots(n+1-\sigma_n)\)

is a down–up permutation. Thus, the number of up–down permutations in \(S_n\) is equal to the number of down–up permutations in \(S_n\). Let \(\text{UD}_n\) denote the number of up–down permutations in \(S_n\). Then André [1,2] proved the following.

\[
\sec(t) = 1 + \sum_{n \in \mathbb{E}} \text{UD}_n \frac{t^n}{n!} \quad \text{and} \quad \tan(t) = \sum_{n \in \mathbb{O}} \text{UD}_n \frac{t^n}{n!}. \tag{1}
\]

\[
\sec(t) = 1 + \sum_{n \in \mathbb{E}} \text{UD}_n \frac{t^n}{n!} \quad \text{and} \quad \tan(t) = \sum_{n \in \mathbb{O}} \text{UD}_n \frac{t^n}{n!}. \tag{2}
\]

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If $s \geq 2$ and $1 \leq j \leq s - 1$, let $sP = \{s, 2s, 3s, \ldots\}$ and $j + sP = \{j, s + j, 2s + j, \ldots\}$. For any $n > 0$, let $(sP)_n = sP \cap \mathbb{P}_n$ and $(j + sP)_n = (j + sP) \cap \mathbb{P}_n$. Let $E_{n,i}$ denote the number of permutations $\sigma \in S_n$ such that $\text{Des}(\sigma) = (sP)_{n-1}$. The $E_{n,i}$’s are called generalized Euler numbers \cite{11}. There are well-known generating functions for $q$-analogues of the generalized Euler numbers; see Stanley’s book \cite{15}, page 148. Various divisibility properties of the $q$-Euler numbers have been studied in \cite{3,4,8}, and of the generalized $q$-Euler numbers in \cite{9,14}. More general generating functions for statistics on permutations $\sigma \in S_n$ such that $\text{Des}(\sigma) = (j + sP)_{n-1}$ were given by Mendes et al. \cite{12}.

Carlitz \cite{5} and Rawlings \cite{13} proved two different analogues of André’s results for compositions. To state their results, we first need to introduce some more notation. A composition $w$ is a sequence of positive integers $w = (w_1, \ldots, w_m)$. We call the $w_i$ parts of $w$ and let $\ell(w) = m$ denote the length of $w$ (i.e. the number of parts). We also let $|w| = \sum_{i=1}^{m} w_i$ (saying that $w$ is a composition of $|w|$), and $z(w) = \prod_{i=1}^{m} z^{w_i}$. For example, if $w = 1 \ 2 \ 1 \ 3 \ 2 \ 4 \ 5 \ 4$, then $\ell(w) = 8$, $|w| = 22$, and $z(w) = z^2 z^1 z^2 z^1 z^4 z^2$. Let $\mathbb{P}^+$ denote the set of compositions with parts from $\mathbb{P}$, $\mathbb{P}^+$ denote the set of all non-empty compositions in $\mathbb{P}^+$, and $\mathbb{P}^m$ denote the set of all composition of length $m$. Define $\mathbb{P}^s$, $\mathbb{P}^w$, and $\mathbb{P}^n$ similarly with parts from $\mathbb{P}_n$. For convenience, we let $\epsilon$ denote the empty composition. Given $w = w_1 w_2 \cdots w_m \in \mathbb{P}^m$, we define the descent set $\text{Des}(w)$, the weak descent set $\text{WDes}(w)$, the rise set $\text{Ris}(w)$, and the weak rise set $\text{WRis}(w)$ as follows:

\[
\begin{align*}
\text{Des}(w) &= \{i : w_i > w_{i+1}\}, \\
\text{WDes}(w) &= \{i : w_i \geq w_{i+1}\}, \\
\text{Ris}(w) &= \{i : w_i < w_{i+1}\}, \quad \text{and} \\
\text{WRis}(w) &= \{i : w_i \leq w_{i+1}\}.
\end{align*}
\]

\textbf{Definition 1.} Let $w = w_1 w_2 \cdots w_m \in \mathbb{P}^m$.

1. We say that $w$ is a strict up–down composition if $w_1 < w_2 > w_3 < w_4 > w_5 \cdots$, or, equivalently if $\text{Ris}(w) = \emptyset_{m-1}$ and $\text{Des}(w) = \emptyset_{m-1}$.

2. We say that $w$ is a strict down–up composition if $w_1 > w_2 < w_3 > w_4 < w_5 \cdots$, or, equivalently if $\text{Des}(w) = \emptyset_{m-1}$ and $\text{Ris}(w) = \emptyset_{m-1}$.

3. We say that $w$ is a weak up–down composition if $w_1 \leq w_2 \geq w_3 \leq w_4 \geq w_5 \cdots$, or, equivalently if $\text{WRis}(w) = \emptyset_{m-1}$ and $\text{WDes}(w) = \emptyset_{m-1}$.

4. We say that $w$ is a weak down–up composition if $w_1 \geq w_2 \leq w_3 \geq w_4 \leq w_5 \cdots$, or, equivalently if $\text{WDes}(w) = \emptyset_{m-1}$ and $\text{WRis}(w) = \emptyset_{m-1}$.

By convention, the empty composition $\epsilon$ and one part composition $w_1$ are considered to be simultaneously strict up–down compositions, strict down–up compositions, weak up–down compositions, and weak down–up compositions. We let $\text{SUD}_{n}$, $\text{SDU}_{n}$, $\text{WUD}_{n}$, and $\text{WDU}_{n}$ denote the sets of all compositions in $\mathbb{P}^n$ which are strict up–down, strict down–up, weak up–down, and weak down–up, respectively. Clearly, if $w = w_1 w_2 \cdots w_m \in \mathbb{P}_{n}^m$, then $w \in \text{SUD}_{n}$ (or $\text{WUD}_{n}$) if and only if the complement of $w$ relative to $n$, $w^{\epsilon,n} = (n + 1 - w_1)(n + 1 - w_2) \cdots (n + 1 - w_m) \in \text{SDU}_{n}$ (or $\text{WDU}_{n}$).

We let $\text{SUD}_{n,m}$, $\text{SDU}_{n,m}$, $\text{WUD}_{n,m}$, and $\text{WDU}_{n,m}$ denote the sets of all compositions in $\mathbb{P}_{n}^m$ which are strict up–down, strict down–up, weak up–down, and weak down–up, respectively.

Carlitz \cite{5,6} proved analogues of André’s formulas for strict up–down compositions. In particular, Carlitz \cite{5} considered the following generating functions.

\[
\begin{align*}
F_n(z_1, \ldots, z_s) &= \sum_{m \in \mathbb{N}} \sum_{w \in \text{SUD}_{n,m}} z(w), \\
G_n(z_1, \ldots, z_s) &= 1 + \sum_{m \in \mathbb{N}} \sum_{w \in \text{SUD}_{n,m}} z(w), \\
F_n(z) &= \sum_{m \in \mathbb{N}} |\text{SUD}_{n,m}| z^m, \quad \text{and} \\
G_n(z) &= 1 + \sum_{m \in \mathbb{N}} |\text{SUD}_{n,m}| z^m.
\end{align*}
\]

For example, if $n = 2$, then clearly $\text{SUD}_{2,1} = \{1, 2\}$ and $\text{SUD}_{2,2m} = \{(1 2)^m\}$ and $\text{SUD}_{2,2m+1} = \{(1 2)^m 1\}$ for $m \geq 1$. Thus

\[
\begin{align*}
G_2(z_1, z_2) &= \frac{1}{1 - z_1 z_2}, \\
G_2(z) &= \frac{1}{1 - z^2}, \\
F_2(z_1, z_2) &= z_2 + \frac{z_1}{1 - z_1 z_2} = \frac{z_1 + z_2 - z_1 z_2^2}{1 - z_1 z_2}, \quad \text{and} \\
F_2(z) &= \frac{2z - z^3}{1 - z^2}.
\end{align*}
\]
In general, Carlitz [5] proved that
\[
G_n(z_1, \ldots, z_n) = \frac{1}{Q_n(z_1, \ldots, z_n)} \quad \text{and} \quad F_n(z_1, \ldots, z_n) = \frac{P_n(z_1, \ldots, z_n)}{Q_n(z_1, \ldots, z_n)},
\]
where
\[
P_{n+1}(z_1, \ldots, z_{n+1}) = (1 - z_{n+1}^2)P_n(z_1, \ldots, z_n) + z_{n+1}Q_n(z_1, \ldots, z_n)
\]
and
\[
Q_{n+1}(z_1, \ldots, z_{n+1}) = -z_{n+1}P_n(z_1, \ldots, z_n) + Q_n(z_1, \ldots, z_n).
\]
In particular, he used these recursions to prove the following formulas:
\[
G_n(z) = \frac{1}{Q_n(z)},
\]
and
\[
F_n(z) = \frac{P_n(z)}{Q_n(z)}.
\]
where
\[
P_n(z) = \sum_{k=0}^{n} (-1)^k \left( \frac{n+k}{2k+1} \right) z^{2k+1} \quad \text{and} \quad Q_n(z) = \sum_{k=0}^{n} (-1)^k \left( \frac{n+k-1}{2k} \right) z^{2k}.
\]

Rawlings proved \( q \)-analogues of (5) and (6) for weak down–up compositions. That is, let \([n] = 1 + q + \cdots + q^{n-1} = \frac{1 - q^n}{1 - q}\), \([n]_q! = [n]_q[n - 1]_q \cdots [1]_q\), and \( \left[ \frac{n!}{k!q!} \right]_q = [n]_q [n-1]_q \cdots [k+1]_q \). Let
\[
B_n(q, z) = \sum_{k \geq 0} (-1)^k q^{k(k+1)} \left[ \frac{n+k}{2k+1} \right] q^z \quad \text{and} \quad A_n(q, z) = \sum_{k \geq 0} (-1)^k q^{k^2+3k+1} \left[ \frac{n+k}{2k+1} \right] q^z.
\]
Then Rawlings [13] proved that
\[
1 + \sum_{m \geq 0} \sum_{w \in WDU_{n,m}} q^{[w]} \cdot x^{f(w)} = \frac{1}{B_n(q, z)}
\]
and
\[
\sum_{m \geq 0} \sum_{w \in WDU_{n,m}} q^{[w]} \cdot x^{f(w)} = \frac{A_n(q, z)}{B_n(q, z)}.
\]
Note that the coefficient of \( qz^m \) in (7) or (8) can be interpreted as the number of compositions of \( i \) with \( m \) parts, where each part is less than or equal to \( n \).

Cristea and Prodinger [7] obtained similar results for different classes of up–down compositions with a probability distribution. Specifically, they consider compositions with unbounded parts, where each part \( j \) occurs with probability \( pq_j \) for some \( 0 < q < 1 \) and \( p = 1-q \). They also count rises (strict rises) on the lower level, meaning up–down subcompositions \( abc \) with \( a \leq c \) (\( a < c \)). They found generating functions for the probability that an up–down composition of some length has a particular number of rises on the lower level. For example, let \( p(n, r) \) be the probability that an up–down composition of length \( 2n+1 \) with the pattern \( < \cdots < \geq \cdots \geq \) has \( r \) strict rises on the lower level. Then they proved that
\[
f(z, x) = \sum_{n \geq 0, r \geq 1} p(n, r)z^{2n+1}x^r = \frac{\sum_{k \geq 0} \frac{p^{k+1}q^k(1+q(k+1))}{(q)_{2k+1}} \prod_{j=0}^{k-1} T(x, q^{2j})}{\sum_{k \geq 0} \frac{p^{k+1}q^k}{(q)_{2k}} \prod_{j=0}^{k-1} T(x, q^{2j-1})},
\]
where \( T(x, u) = x - 1 - uq(x - q) \). Cristea and Prodinger obtained similar results for three other up–down patterns and compositions of even length. By taking \( \lim_{q \to 1} f(z, 1) \), they recovered the classical formulas (1) and (2).
This paper was motivated by our attempt to give direct proofs via involutions of the formulas of Carlitz and Rawlings as well as certain specializations of the formulas of Cristea and Prodinger described above. That is, Carlitz [5] proved (5) and (6) by recursions. Rawlings [13] developed much more general recursions for generating functions of compositions and proved (7) and (8) as special cases of these recursions. Cristea and Prodinger [7] used the method of “adding a new slice”, or looking at how the probability of a composition changes when two additional parts are placed at the end of it (maintaining the same up-down pattern). Recursive machinery, though very powerful, requires an iterated expression to be simplifiable. For example, Cristea and Prodinger obtained (9) by iterating to obtain:

\[ F(z, x, u) = \frac{\operatorname{puz}}{(uz)_1} + \frac{\operatorname{puz}^2}{(uz)_1} F(z, x, q) + \frac{\operatorname{puT}_x}{(uz)_3} + \frac{\operatorname{puq}^2 z^2}{(uz)_1} F(z, x, q) + \frac{\operatorname{puq^2 z^2}}{(uz)_1} \]

simplifying \( F(z, x, u) \), and setting \( u = 1 \). Although they were able to obtain relatively clean formulas for their generating functions, the same machinery would be much more cumbersome to use for variations on the problem, such as considering patterns like \( <<> <<> \cdots <<> \) or including additional variables to track which parts are used in the composition. In contrast, the method of this paper reduces all such problems to finding generating functions for compositions whose parts are weakly increasing, and these generating functions are usually easy to express directly. We mention an example of further generalizations this method can afford in Section 4.

The main goal of this paper is to show that all of the formulas of Carlitz, Rawlings, and Cristea and Prodinger described above can be proved directly by simple involutions. In fact, we shall give direct combinatorial proofs of generalizations of these formulas. That is, we shall prove formulas for the analogues of generalized Euler numbers for compositions. To this end, we define the following classes of compositions (these generalize the four patterns considered by Cristea and Prodinger in [7]).

**Definition 2.** Let \( s \geq 2 \).

1. \( \text{SU}^{s-1} \text{SD}_{n,m} \) is the set of all compositions \( w \in \mathbb{P}^n \) such that Des\((w) = (sP)_{m-1} \) and Ris\((w) = P_{m-1} - (sP)_{m-1} \).
2. \( \text{WU}^{s-1} \text{SD}_{n,m} \) is the set of all compositions \( w \in \mathbb{P}^n \) such that Des\((w) = (sP)_{m-1} \) and WRis\((w) = P_{m-1} - (sP)_{m-1} \).
3. \( \text{SU}^{s-1} \text{WD}_{n,m} \) is the set of all compositions \( w \in \mathbb{P}^n \) such that WDes\((w) = (sP)_{m-1} \) and Ris\((w) = P_{m-1} - (sP)_{m-1} \).
4. \( \text{WU}^{s-1} \text{WD}_{n,m} \) is the set of all compositions \( w \in \mathbb{P}^n \) such that WDes\((w) = (sP)_{m-1} \) and WRis\((w) = P_{m-1} - (sP)_{m-1} \).

For example, \( \text{SU}^{s-1} \text{SD}_{n,m} \) consists of all compositions of length \( m \) with parts from \( \mathbb{P}^n \) that start out with \( s-1 \) strict increases followed by a strict decrease, then another sequence of \( s-1 \) strict increases followed by a strict decrease, etc. For example, we can describe \( \text{SU}^2 \text{SD}_{n,m} \) as the set of all compositions \( w = w_1 \cdots w_m \) where each \( w_i \leq n \), such that \( w_1 > w_{i+1} \) if \( i \equiv 0 \) mod 3 and \( w_1 < w_{i+1} \) if \( i \not\equiv 0 \) mod 3. Alternatively, \( \text{SU}^2 \text{SD}_{n,m} \) consists of all compositions in \( w = w_1 \cdots w_m \in \mathbb{P}^n \) such that:

\[ w_1 < w_2 < w_3 > w_4 < w_5 < w_6 > w_7 < w_8 < w_9 > w_{10} \cdots \]

Similarly, \( \text{WU}^2 \text{SD}_{n,m} \) consists of all compositions \( w = w_1 \cdots w_m \in \mathbb{P}^n \) such that \( w_1 > w_{i+1} \) if \( i \equiv 0 \) mod 3 and \( w_1 \leq w_{i+1} \) if \( i \not\equiv 0 \) mod 3. That is, \( \text{WU}^2 \text{SD}_{n,m} \) denotes the set of all compositions \( w = w_1 \cdots w_m \in \mathbb{P}^n \) such that:

\[ w_1 \leq w_2 \leq w_3 > w_4 \leq w_5 \leq w_6 > w_7 \leq w_8 \leq w_9 > w_{10} \cdots \]

It will be useful for later developments to have a pictorial representation of these classes of compositions. The idea is that we are interested in compositions \( w \) that we can partition into an initial sequence of blocks of size \( s \) and ending in a block of size \( j \) where \( 0 \leq j \leq s-1 \). The parts in any given block are either strictly increasing if we pick \( \text{SU}^{s-1} \) or weakly increasing if we pick \( \text{WU}^{s-1} \). Then, either we have strict decreases between blocks (as pictured in the top of Fig. 1) if we are considering either \( \text{SU}^{s-1} \text{SD} \) or \( \text{WU}^{s-1} \text{SD} \) or we have weak decreases between blocks (as pictured at the bottom of Fig. 1) if we are considering either \( \text{SU}^{s-1} \text{WD} \) or \( \text{WU}^{s-1} \text{WD} \).

It is then easy to see that the collection of compositions studied by Carlitz [5] is \( \text{SU}^1 \text{SD}_{n,m} \) and the collection of compositions studied by Rawlings [13] is \( \text{WU}^1 \text{WD}_{n,m} \). Cristea and Prodinger [7] also consider \( \text{WU}^1 \text{SD} \) and \( \text{SU}^1 \text{WD} \) with parts from \( \mathbb{P} \) having geometric probabilities. This given, we define the following generating functions for
any \( s \geq 2 \):
\[
H_{n,s,0}^{\text{SU}^{-1}\text{SD}}(z_1, \ldots, z_n) = 1 + \sum_{w \in \text{MEF}} \sum_{w \in \text{SU}^{-1}\text{SD}_{j,m}} z(w) \quad \text{and} \\
H_{n,s,j}^{\text{SU}^{-1}\text{SD}}(z_1, \ldots, z_n) = \sum_{w \in \text{MEF}} \sum_{w \in \text{SU}^{-1}\text{SD}_{j,m}} z(w) \quad \text{for } j = 1, \ldots, s - 1.
\]

We define \( H_{n,s,j}^{\text{WD}^{-1}\text{SD}}(z_1, \ldots, z_n) \), \( H_{n,s,j}^{\text{SU}^{-1}\text{WD}}(z_1, \ldots, z_n) \), and \( H_{n,s,j}^{\text{WD}^{-1}\text{WD}}(z_1, \ldots, z_n) \) for \( j = 0, \ldots, s - 1 \) similarly. We give an explicit expression for each of these generating functions in terms of Gessel quasi-symmetric functions [10]. Our expressions can then be specializes to explicit formulas like (5)–(8).

The outline of this paper is as follows. In Section 2, we shall define the Gessel quasi-symmetric functions and some additional classes of compositions that can be defined in terms of quasi-symmetric functions that we will need for our proofs. In Section 3, we state and prove our generating functions for \( H_{n,s,j}^{\text{SU}^{-1}\text{SD}}(z_1, \ldots, z_n) \), \( H_{n,s,j}^{\text{SU}^{-1}\text{WD}}(z_1, \ldots, z_n) \), \( H_{n,s,j}^{\text{WD}^{-1}\text{SD}}(z_1, \ldots, z_n) \), and \( H_{n,s,j}^{\text{WD}^{-1}\text{WD}}(z_1, \ldots, z_n) \) and give some specializations. Finally, in Section 4, we shall end with a brief discussion about some extensions of our work.

2. Quasi-symmetric functions

Let \( \gamma = (\gamma_1, \ldots, \gamma_t) \) be a composition. We let
\[
\text{Set}(\gamma) = \{\gamma_1, \gamma_1 + \gamma_2, \ldots, \gamma_1 + \gamma_2 + \cdots + \gamma_{t-1}\}.
\]
For example, if \( \gamma = (2, 3, 1, 1, 2) \), \( \text{Set}(\gamma) = \{2, 5, 6, 7\} \). Then Gessel [10] defined the quasi-symmetric function
\[
Q_\gamma(z_1, \ldots, z_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_{|\gamma|} \leq n} z_{i_1}z_{i_2} \cdots z_{i_{|\gamma|}}.
\]
(10)
Thus, for example, if \( \gamma = (2, 3, 1, 1, 2) \), then
\[
Q_\gamma(z_1, \ldots, z_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_9 < j} z_j.
\]
We shall also need explicit expression for the specializations
\[
Q_\gamma(z_1, \ldots, z_n)|_{z_i \rightarrow q} \quad \text{and} \quad Q_\gamma(z_1, \ldots, z_n)|_{z_i \rightarrow q^2}.
\]

Lemma 3.
\[
Q_\gamma(z_1, \ldots, z_n)|_{z_i \rightarrow q} = \left( \frac{n + |\gamma| - \ell(\gamma)}{|\gamma|} \right) q^{\sum_{i \in \gamma} \ell(i)} z^{|\gamma|}.
\]
(11)
and
\[
Q_\gamma(z_1, \ldots, z_n)|_{z_i \rightarrow q^2} = q^{\sum_{i \in \gamma} \ell(i)} \left( \frac{n + |\gamma| - \ell(\gamma)}{|\gamma|} \right) z^{|\gamma|}.
\]
(12)

Proof. For the specialization \( Q_\gamma(z_1, \ldots, z_n)|_{z_i \rightarrow q} \), we must count the number of sequences \( 1 \leq i_1 \leq i_2 \leq \cdots \leq i_{|\gamma|} \leq n \) such that \( i_j \leq i_{j+1} \) if \( j \in \text{Set}(\gamma) \). Let \( s(\gamma) = a_1 \cdots a_{|\gamma|} \) where \( a_1 = 1 \) and \( a_{i+1} = a_i \) if \( i \notin \text{Set}(\gamma) \) and \( a_{i+1} = a_i + 1 \) if \( i \in \text{Set}(\gamma) \). Thus \( s(\gamma) \) is the minimal sequence of this type. For example, if \( \gamma = (2, 3, 1, 1, 2) \), then \( s(\gamma) = 12223 \). Now if \( 1 \leq i_1 \leq \cdots \leq i_{|\gamma|} \leq n \) is a sequence such that \( i_j < i_{j+1} \) if \( j \in \text{Set}(\gamma) \), then it easy to see that we have designed \( s(\gamma) = a_1 \cdots a_{|\gamma|} \) so that if \( b_j = i_j - a_j \), for \( j = 1, \ldots, |\gamma| \), then \( 0 \leq b_1 \leq b_2 \leq \cdots \leq b_{|\gamma|} \leq n - 1 - |\text{Set}(\gamma)| \). Note that \( |\text{Set}(\gamma)| = \ell(\gamma) - 1 \). Thus, the number of such sequences \( b_1 \cdots b_{|\gamma|} \) is the number of partitions contained in the \( |\gamma| \times (n - \ell(\gamma)) \) rectangle, which is well known to be \( \binom{n + |\gamma| - \ell(\gamma)}{|\gamma|} \). Thus, the number of sequences \( 1 \leq i_1 \leq i_2 \leq \cdots \leq i_{|\gamma|} \leq n \) such that \( i_j < i_{j+1} \) if \( j \in \text{Set}(\gamma) \) equals \( \binom{n + |\gamma| - \ell(\gamma)}{|\gamma|} \), which yields (11).

For the specialization \( Q_\gamma(z_1, \ldots, z_n)|_{z_i \rightarrow q^2} \), note that
\[
\sum_{0 \leq b_1 \leq b_2 \leq \cdots \leq b_{|\gamma|} \leq n - \ell(\gamma)} q^{b_1 + \cdots + b_{|\gamma|}} = \left[ \frac{n + |\gamma| - \ell(\gamma)}{|\gamma|} \right]_q.
\]
Wedefine increasing, but this timewewanteither weak increasesor strict increases between the blocks. In pictures, we want to

$$P - n \leq WU,$$ $SU,$ and $WU$.

Thus

$$\sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq \ell(y) \leq \ell(y)} q^{\gamma(i \cdots j(y))} \left[ \frac{n + |\gamma| - \ell(y)}{|\gamma|} \right]_q = q^{\gamma(n)} \left[ \frac{n + |\gamma| - \ell(y)}{|\gamma|} \right]_q = \sum_{\gamma} q^{\gamma(n)} \left[ \frac{n + |\gamma| - \ell(y)}{|\gamma|} \right]_q. \quad \Box$$

Next, we define several more classes of compositions. In particular, we are interested in compositions $w$ that we can partition into blocks of size $s$ and ending in a block of size $j$ where $0 \leq j \leq s - 1$ like those considered for the classes in $SU^{j=1}WU, WU^{j=1}WU, SU^{j=1}SU$, and $WU^{j=1}SU$. That is, parts in a given block are either strictly increasing or weakly increasing, but this time we want either weak increases or strict increases between the blocks. In pictures, we want to consider compositions as depicted in Fig. 2.

Formally, we consider the following set of compositions.

**Definition 4.** Let $s \geq 2$.

1. $SU^{j=1}WU_{n,m}$ is the set of all compositions $w = w_1 \cdots w_m \in \mathbb{P}_n$ such that $w_i \leq w_{i+1}$ if $i \in sP$ and $w_i < w_{i+1}$ if $i \not\in sP$.
2. $WU^{j=1}WU_{n,m}$ is the set of all compositions $w = w_1 \cdots w_m \in \mathbb{P}_n$ such that $w_i \leq w_{i+1}$ if $i \in sP$ and $w_i \leq w_{i+1}$ if $i \not\in sP$.

Thus $WU^{j=1}WU_{n,m}$ is just the set of all weakly increasing compositions in $\mathbb{P}_n$.

3. $SU^{j=1}SU_{n,m}$ is the set of all compositions $w = w_1 \cdots w_m \in \mathbb{P}_n$ such that $w_i < w_{i+1}$ if $i \in sP$ and $w_i < w_{i+1}$ if $i \not\in sP$. Thus $SU^{j=1}SU_{n,m}$ is just the set of all strictly increasing compositions in $\mathbb{P}_n$.

4. $WU^{j=1}SU_{n,m}$ is the set of all compositions $w = w_1 \cdots w_m \in \mathbb{P}_n$ such that $w_i < w_{i+1}$ if $i \in sP$ and $w_i \leq w_{i+1}$ if $i \not\in sP$.

We then define the following generating functions for any $s \geq 2$:

$$p_{n,s,0}^{SU^{j=1}WU}(z_1, \ldots, z_n) = 1 + \sum_{k \geq 1} (-1)^k \sum_{w \in SU^{j=1}WU_{n,k}} z(w) \quad \text{and}$$

$$p_{n,s,j}^{SU^{j=1}WU}(z_1, \ldots, z_n) = \sum_{k \geq 0} (-1)^k \sum_{w \in SU^{j=1}WU_{n,k+1}} z(w) \quad \text{for} \ j = 1, \ldots, s - 1.$$

We define $p_{n,s,j}^{WU^{j=1}WU}(z_1, \ldots, z_n), p_{n,s,j}^{SU^{j=1}SU}(z_1, \ldots, z_n),$ and $p_{n,s,j}^{WU^{j=1}SU}(z_1, \ldots, z_n)$ for $j = 0, \ldots, s - 1$ similarly. We can express each of these generating functions in terms of quasi-symmetric functions. That is, for any $s \geq 2$,

$$p_{n,s,j}^{SU^{j=1}WU}(z_1, \ldots, z_n) = 1 + \sum_{k \geq 1} (-1)^k Q_{(1^{s-2}2^{j-1})}^{(1^{s-2}2^{j-1})} (z_1, \ldots, z_n) \quad \text{and}$$

$$p_{n,s,j}^{SU^{j=1}WU}(z_1, \ldots, z_n) = \sum_{k \geq 0} (-1)^k Q_{(1^{s-2}2^{j-1})}^{(1^{s-2}2^{j-1})} (z_1, \ldots, z_n) \quad \text{for} \ j = 1, \ldots, s - 1.$$

It then follows from Lemma 3 that for $s \geq 2$ and $j = 1, \ldots, s - 1$,

$$p_{n,s,0}^{SU^{j=1}WU}(z_1, \ldots, z_n) |_{z_i = q^{z_i}} = 1 + \sum_{k \geq 1} (-1)^k q^{(k-1)2 + (s-1)k} \left[ \frac{n + k - 1}{k} \right]_q z^k,$$ and

$$p_{n,s,j}^{SU^{j=1}WU}(z_1, \ldots, z_n) |_{z_i = q^{z_i}} = \sum_{k \geq 0} (-1)^k q^{(k-1)2 + (s-1)k} \left[ \frac{n + k}{k} \right]_q z^{k+j}.$$ 

Note that both of these specializations are finite sums as $\left[ \frac{n + k}{k} \right]_q = 0$ for $k > \frac{n}{s} + 1$.

Similarly, for $s \geq 2$ and $j = 1, \ldots, s - 1$,

$$p_{n,s,0}^{SU^{j=1}WU}(z_1, \ldots, z_n) = 1 + \sum_{k \geq 1} (-1)^k Q_{(s,k)}(z_1, \ldots, z_n) \quad \text{and}$$

$$p_{n,s,j}^{SU^{j=1}WU}(z_1, \ldots, z_n) = \sum_{k \geq 0} (-1)^k Q_{(ks+j)}(z_1, \ldots, z_n).$$
and with the specializations
\[
P_{n,s,0}^{\text{WU}^{-1}\text{SU}}(z_1, \ldots, z_n) = 1 + \sum_{k \geq 1} (-1)^k q^{k s} \left[ \frac{n + ks - 1}{ks} \right]_q z^{ks}, \quad \text{and}
\]
\[
P_{n,s,j}^{\text{WU}^{-1}\text{SU}}(z_1, \ldots, z_n) = \sum_{k \geq 0} (-1)^k q^{ks+j} \left[ \frac{n + ks + j - 1}{ks + j} \right]_q z^{ks+j}.
\]
Note that, in this case, the specializations are infinite sums.

We also have, for any \( s \geq 2 \) and \( 1 \leq j \leq s - 1 \),
\[
P_{n,s,0}^{\text{SU}^{-1}\text{WU}}(z_1, \ldots, z_n) = 1 + \sum_{k \geq 1} (-1)^k Q_{1|ks}(z_1, \ldots, z_n) \quad \text{and}
\]
\[
P_{n,s,j}^{\text{SU}^{-1}\text{WU}}(z_1, \ldots, z_n) = \sum_{k \geq 0} (-1)^k Q_{1|ks+j}(z_1, \ldots, z_n)
\]
with the specializations
\[
P_{n,s,0}^{\text{SU}^{-1}\text{SU}}(z_1, \ldots, z_n) = 1 + \sum_{k \geq 1} (-1)^k q^{(\frac{k+1}{2})} \left[ \frac{2(n + ks - 1)}{ks} \right]_q z^{ks}, \quad \text{and}
\]
\[
P_{n,s,j}^{\text{SU}^{-1}\text{SU}}(z_1, \ldots, z_n) = \sum_{k \geq 0} (-1)^k q^{(\frac{k+1}{2})} \left[ \frac{n + ks + j + 1}{ks + j} \right]_q z^{ks+j}.
\]
In this case, the specializations are finite sums.

Finally, for any \( s \geq 2 \) and \( j = 1, \ldots, s - 1 \),
\[
P_{n,s,0}^{\text{WU}^{-1}\text{SU}}(z_1, \ldots, z_n) = 1 + \sum_{k \geq 1} (-1)^k Q_{2|ks}(z_1, \ldots, z_n) \quad \text{and}
\]
\[
P_{n,s,j}^{\text{WU}^{-1}\text{SU}}(z_1, \ldots, z_n) = \sum_{k \geq 0} (-1)^k Q_{2|ks+j}(z_1, \ldots, z_n)
\]
with the specializations
\[
P_{n,s,0}^{\text{SU}^{-1}\text{SU}}(z_1, \ldots, z_n) = 1 + \sum_{k \geq 1} (-1)^k q^{(\frac{k+1}{2})} \left[ \frac{n + k(s - 1)}{ks} \right]_q z^{ks}, \quad \text{and}
\]
\[
P_{n,s,j}^{\text{SU}^{-1}\text{SU}}(z_1, \ldots, z_n) = \sum_{k \geq 0} (-1)^k q^{(\frac{k+1}{2})}\left[ \frac{1 + k(s - 1) + j - 1}{ks + j} \right]_q z^{ks+j}.
\]
In this case, the specializations are finite sums.

3. Main results

In this section, we shall prove our desired formulas. Our first theorem is the following.

**Theorem 5.** Let \( s \geq 2 \). Then
\[
H_{n,s,0}^{\text{SU}^{-1}\text{SD}}(z_1, \ldots, z_n) = \frac{1}{\sum_{m \in \text{SU}} \sum_{w \in \text{SU}^{-1}\text{SD}_{n,m}} z(w)} = \frac{1}{p_{n,s,0}^{\text{SU}^{-1}\text{SU}}(z_1, \ldots, z_n)} = 1 + \sum_{k \geq 1} (-1)^k Q_{1|1|2|k-1|1|1}(z_1, \ldots, z_n),
\]
\[
H_{n,s,0}^{\text{WU}^{-1}\text{SD}}(z_1, \ldots, z_n) = \frac{1}{\sum_{m \in \text{SU}} \sum_{w \in \text{SU}^{-1}\text{SD}_{n,m}} z(w)} = \frac{1}{p_{n,s,0}^{\text{WU}^{-1}\text{SU}}(z_1, \ldots, z_n)} = 1 + \sum_{k \geq 1} (-1)^k Q_{2|k}(z_1, \ldots, z_n).
\]
\[
H_{n,s,0}^{\text{SD}}(z_1, \ldots, z_n) = 1 + \sum_{m \in \mathbb{S}^p} \sum_{w \in \mathbb{S}^{n-1}^{\text{SD}}_{n,m}} z(w) = \frac{1}{p_{n,s,0}^{\text{SD}}(z_1, \ldots, z_n)} = 1 + \sum_{k \geq 1} (-1)^k Q_{1,k}(z_1, \ldots, z_n),
\]

and
\[
H_{n,s,0}^{\text{WD}}(z_1, \ldots, z_n) = 1 + \sum_{m \in \mathbb{S}^p} \sum_{w \in \mathbb{W}^{n-1}^{\text{WD}}_{n,m}} z(w) = \frac{1}{p_{n,s,0}^{\text{WD}}(z_1, \ldots, z_n)} = 1 + \sum_{k \geq 1} (-1)^k Q_{0,k}(z_1, \ldots, z_n).
\]

**Proof.** We start by proving (13). We must show that
\[
H_{n,s,0}^{\text{SD}}(z_1, \ldots, z_n) \cdot p_{n,s,0}^{\text{SD}}(z_1, \ldots, z_n) = 1.
\]

Now we can interpret the LHS of (17) as
\[
\sum_{(a,b) \in T} z(a)z(b)(-1)^{\ell(b)/s}
\]
where \(T\) is the set of all pairs of compositions \((a, b)\) such that \(a \in \{\epsilon\} \cup \bigcup_{m \in \mathbb{S}^p} \mathbb{S}^{r-1} \mathbb{D}_{n,m} \) and \(b \in \{\epsilon\} \cup \bigcup_{m \in \mathbb{S}^p} \mathbb{S}^{n-1} \mathbb{U}_{n,m} \).

The empty composition \(\epsilon\) accounts for the leading 1 in the series of \(H_{n,s,0}^{\text{SD}}(z_1, \ldots, z_n)\) and \(p_{n,s,0}^{\text{WD}}(z_1, \ldots, z_n)\). Thus, in general, \(a\) consists of a number of strictly increasing blocks of size \(s\) where there are strict decreases between blocks and \(b\) consists of a number of strictly increasing blocks of size \(s\) where there are weak increases between blocks. We will define a sign-reversing, weight-preserving involution \(I_1\) on the collection of all such pairs of compositions \((a, b)\). The definition of \(I_1\) proceeds in 4 cases.

**Case 1.** The last block of \(a\) is \(a_{k+1} < \cdots < a_{k+s}\) and the first block of \(b\) is \(b_1 < \cdots < b_l\).

If \(a_{k+s} > b_1\), then \(I_1(a, b) = (\bar{a}, \bar{b})\), where \(\bar{a}\) is the result of inserting the first block of \(b\) at the end of \(a\) and \(\bar{b}\) is the result of removing the first block of \(b\) from \(a\). Clearly \((\bar{a}, \bar{b})\) is again a pair in \(T\). However if \(a_{k+s} \leq b_1\), then we let \(I_1(a, b) = (\bar{a}, \bar{b})\) where \(\bar{a}\) is the result of removing the last block of \(a\) from \(a\) and \(\bar{b}\) is the result of inserting the last block of \(a\) at the start of \(b\). Clearly \((\bar{a}, \bar{b})\) is again a pair in \(T\).

**Case 2.** The first block of \(b\) is \(b_1 < \cdots < b_l\) and \(a = \epsilon\).

Then \(I_1(a, b) = (\bar{a}, \bar{b})\), where \(\bar{a} = b_1 \cdots b_l\) and \(\bar{b}\) is the result of removing the first block of \(b\) from \(a\). Clearly \((\bar{a}, \bar{b})\) is again a pair in \(T\).

**Case 3.** The last block of \(a\) is \(a_{k+1} < \cdots < a_{k+s}\) and \(b = \epsilon\).

Then \(I_1(a, b) = (\bar{a}, \bar{b})\), where \(\bar{a}\) is the result of removing the last block of \(a\) from \(a\) and \(\bar{b} = a_{k+1} \cdots a_{k+s}\).

**Case 4.** \(a = b = \epsilon\).

Then \(I_1(a, b) = (a, b)\).

It is easy to see that \(I_1\) is a sign-reversing, weight-preserving involution with trivial fixed point \((\epsilon, \epsilon)\), so that \(I_1\) proves (17).

The exact same involution will prove that
\[
H_{n,s,0}^{\text{WD}}(z_1, \ldots, z_n) \cdot p_{n,s,0}^{\text{WD}}(z_1, \ldots, z_n) = 1,
\]

since the only difference in this case is that the blocks are weakly increasing.

The same proof, with minor modifications, will also prove
\[
H_{n,s,0}^{\text{SD}}(z_1, \ldots, z_n) \cdot p_{n,s,0}^{\text{SD}}(z_1, \ldots, z_n) = 1
\]

and
\[
H_{n,s,0}^{\text{WD}}(z_1, \ldots, z_n) \cdot p_{n,s,0}^{\text{WD}}(z_1, \ldots, z_n) = 1.
\]

Using Lemma 3, we immediately have the following corollaries.
Corollary 6. Let \( s \geq 2 \). Then

\[
1 + \sum_{m \in \mathbb{Z}^d} \sum_{w \in SU^{d-1}SD_n, m} q^{w} z^f(w) = \frac{1}{1 + \sum_{k \geq 1} (-1)^k q^{(k-1)\binom{k+1}{2} + (s-1)\binom{k+1}{2}} \left[ \frac{n + k}{ks} \right] z^k},
\]

(22)

\[
1 + \sum_{m \in \mathbb{Z}^d} \sum_{w \in WU^{d-1}SD_n, m} q^{w} z^f(w) = \frac{1}{1 + \sum_{k \geq 1} (-1)^k q^{(k-1)\binom{k+1}{2}} \left[ \frac{n + ks - 1}{ks} \right] z^k},
\]

(23)

\[
1 + \sum_{m \in \mathbb{Z}^d} \sum_{w \in SU^{d-1}WD_n, m} q^{w} z^f(w) = \frac{1}{1 + \sum_{k \geq 1} (-1)^k q^{(k-1)\binom{k+1}{2}} \left[ \frac{n + k}{ks} \right] z^k},
\]

(24)

and

\[
1 + \sum_{m \in \mathbb{Z}^d} \sum_{w \in WU^{d-1}WD_n, m} q^{w} z^f(w) = \frac{1}{1 + \sum_{k \geq 1} (-1)^k q^{(k+1)\binom{k}{2}} \left[ \frac{n + k(s - 1)}{ks} \right] z^k}.
\]

(25)

Our next theorem will give the other generating functions mentioned in the introduction.

Theorem 7. Let \( s \geq 2 \) and \( 1 \leq j \leq s - 1 \). Then

\[
H_{n,s,j}^{SU^{d-1}SD}(z_1, \ldots, z_n) = \sum_{m \in \mathbb{Z}^d} \sum_{w \in SU^{d-1}SD_n, m} z(w)
\]

\[
p_{SU^{d-1}WD}(z_1, \ldots, z_n)
\]

\[
p_{SU^{d-1}WD}(z_1, \ldots, z_n)
\]

\[
\sum_{k \geq 0} (-1)^k Q_{(1^{(s-2)})(s-1)}(z_1, \ldots, z_n)
\]

\[
1 + \sum_{k \geq 1} (-1)^k Q_{(k)}(z_1, \ldots, z_n)
\]

(26)

\[
H_{n,s,j}^{WU^{d-1}SD}(z_1, \ldots, z_n) = \sum_{m \in \mathbb{Z}^d} \sum_{w \in WU^{d-1}SD_n, m} z(w)
\]

\[
p_{WU^{d-1}WD}(z_1, \ldots, z_n)
\]

\[
p_{WU^{d-1}WD}(z_1, \ldots, z_n)
\]

\[
\sum_{k \geq 0} (-1)^k Q_{(k+j)}(z_1, \ldots, z_n)
\]

\[
1 + \sum_{k \geq 1} (-1)^k Q_{(k)}(z_1, \ldots, z_n)
\]

(27)

\[
H_{n,s,j}^{SU^{d-1}WD}(z_1, \ldots, z_n) = \sum_{m \in \mathbb{Z}^d} \sum_{w \in SU^{d-1}WD_n, m} z(w)
\]

\[
p_{SU^{d-1}WD}(z_1, \ldots, z_n)
\]

\[
p_{SU^{d-1}WD}(z_1, \ldots, z_n)
\]

\[
\sum_{k \geq 0} (-1)^k Q_{(1^{(s-2)})(s-1)}(z_1, \ldots, z_n)
\]

\[
1 + \sum_{k \geq 1} (-1)^k Q_{(k)}(z_1, \ldots, z_n)
\]

(28)

\[
H_{n,s,j}^{WU^{d-1}WD}(z_1, \ldots, z_n) = \sum_{m \in \mathbb{Z}^d} \sum_{w \in WU^{d-1}WD_n, m} z(w)
\]

\[
p_{WU^{d-1}WD}(z_1, \ldots, z_n)
\]

\[
p_{WU^{d-1}WD}(z_1, \ldots, z_n)
\]

\[
\sum_{k \geq 0} (-1)^k Q_{(k+j)}(z_1, \ldots, z_n)
\]

\[
1 + \sum_{k \geq 1} (-1)^k Q_{(k)}(z_1, \ldots, z_n)
\]

(29)
**Proof.** We start by proving (26). Since we know that

$$H_{n,s,0}^{SU^{-1}SD}(z_1, \ldots, z_n) = \frac{1}{p_{n,s,0}^{SU^{-1}WU}(z_1, \ldots, z_n)},$$

we must show that

$$H_{n,s,0}^{SU^{-1}SD}(z_1, \ldots, z_n) \cdot p_{n,s,0}^{SU^{-1}WU}(z_1, \ldots, z_n) = H_{n,s,j}^{SU^{-1}SD}(z_1, \ldots, z_n).$$ (30)

Now we can interpret the LHS of (30) as

$$\sum_{(a,b) \in V} z(a)z(b)(-1)^{|(b)-j)/s}$$ (31)

where $V$ is the set of all pairs of compositions $(a, b)$ such that $a \in \{\epsilon\} \cup \bigcup_{m \in \mathbb{Z}^+} SU^{j-1}SD_{n,m}$ and $b \in \bigcup_{m \in \mathbb{Z}^+} SU^{j-1}WU_{n,m}$. Thus, in general, $a$ consists of a number of strictly increasing blocks of size $s$, where there are strict decreases between blocks, and $b$ consists of a number of strictly increasing blocks of size $s$ followed by a strictly increasing block of size $j$, where there are weak increases between blocks. We will define a sign-reversing, weight-preserving involution $I_3$ on the collection of all such pairs of compositions $(a, b)$. The definition of $I_3$ proceeds in 4 cases.

**Case 1.** The last block of $a$ is $a_{k_3+1} < \cdots < a_{k_3+3}$ and the first block of $b$ is $b_1 < \cdots < b_i$.

If $a_{k_3+3} > b_i$, then $I_3(a, b) = (\tilde{a}, \tilde{b})$ where $\tilde{a}$ is the result of inserting the first block of $b$ at the end of $a$ and $\tilde{b}$ is the result of removing the first block of $b$ from $b$. Clearly $(\tilde{a}, \tilde{b})$ is again a pair in $V$. However if $a_{k_3+3} \leq b_i$, then we let $I_3(a, b) = (\tilde{a}, \tilde{b})$ where $\tilde{a}$ is the result of removing the last block of $a$ from $a$ and $\tilde{b}$ is the result of inserting the last block of $a$ at the start of $b$. Clearly $(\tilde{a}, \tilde{b})$ is again a pair in $V$.

**Case 2.** The first block of $b$ is $b_1 < \cdots < b_i$ and $a = \epsilon$.

Then $I_3(a, b) = (\tilde{a}, \tilde{b})$ where $\tilde{a} = b_1 \cdots b_i$ and $\tilde{b}$ is the result of removing the first block of $b$ from $b$. Clearly $(\tilde{a}, \tilde{b})$ is again a pair in $V$.

**Case 3.** The last block of $a$ is $a_{k_3+1} < \cdots < a_{k_3+3}$ and $b = b_1 < \cdots < b_j$ where $a_{k_3+3} \leq b_1$.

Then $I_3(a, b) = (\tilde{a}, \tilde{b})$ where $\tilde{a}$ is the result of removing the last block of $a$ from $a$ and $\tilde{b}$ is the result of inserting the last block of $a$ at the start of $b$.

**Case 4.** The last block of $a$ is $a_{k_3+1} < \cdots < a_{k_3+3}$ and $b = b_1 < \cdots < b_j$ where $a_{k_3+3} > b_1$.

Then $I_3(a, b) = (a, b)$.

It is easy to see that $I_3$ is a sign-reversing, weight-preserving involution, so that $I_3$ proves that the LHS of (30) reduces to summing the weights of the pairs of compositions $(a, b)$ in Case 4. To do this, first observe that the signs of all the pairs of compositions in Case 4 are positive. Moreover, it is easy to see that if we insert $b$ at the end of $a$, we will create a composition in $\bigcup_{m \in \mathbb{Z}^+} SU^{j-1}SD_{n,m}$ and that all compositions in $\bigcup_{m \in \mathbb{Z}^+} SU^{j-1}SD_{n,m}$ arise from the pairs of compositions in Case 4 in this way. Thus, the sum of the weights in Case 4 is equal to $H_{n,s,j}^{SU^{j-1}SD}(z_1, \ldots, z_n)$ as desired.

The exact same involution will prove that

$$H_{n,s,0}^{WD^{-1}SD}(z_1, \ldots, z_n) \cdot p_{n,s,0}^{WD^{-1}WU}(z_1, \ldots, z_n) = H_{n,s,j}^{WD^{-1}SD}(z_1, \ldots, z_n)$$ (32)

since the only difference in this case is that the blocks are weakly increasing.

The same proof, with minor modifications, will also prove

$$H_{n,s,0}^{JW^{-1}WD}(z_1, \ldots, z_n) \cdot p_{n,s,0}^{JW^{-1}SU}(z_1, \ldots, z_n) = H_{n,s,j}^{JW^{-1}WD}(z_1, \ldots, z_n)$$ (33)

and

$$H_{n,s,0}^{WJ^{-1}WD}(z_1, \ldots, z_n) \cdot p_{n,s,j}^{WJ^{-1}SU}(z_1, \ldots, z_n) = H_{n,s,j}^{WJ^{-1}WD}(z_1, \ldots, z_n). \qed$$ (34)

Using Lemma 3, we immediately have the following corollaries.
Corollary 8. Let $s \geq 2$ and $1 \leq j \leq s - 1$. Then

$$
\sum_{m \in j + \mathbb{P}} \sum_{w \in \mathcal{P}^{s-1}SD_{n,m}} q^{w|z^{(w)}} = \frac{\sum (-1)^k q^{\binom{k(s-1)+j+1}{2} \binom{k+1}{2} + \binom{k+1}{2}}}{1 + \sum_{k \geq 1} (-1)^k q^{\binom{k(s-1)+j+1}{2} \binom{k+1}{2} + \binom{k+1}{2}}}
$$

(35)

$$
\sum_{m \in j + \mathbb{P}} \sum_{w \in \mathcal{W}^{s-1}SD_{n,m}} q^{w|z^{(w)}} = \frac{\sum (-1)^k q^{kjs} z^{ks+j}}{1 + \sum_{k \geq 1} (-1)^k q^{kjs} z^{ks}}
$$

(36)

$$
\sum_{m \in j + \mathbb{P}} \sum_{w \in \mathcal{W}^{s-1}SD_{n,m}} q^{w|z^{(w)}} = \frac{\sum (-1)^k q^{\binom{k+1}{2}} z^{ks+j}}{1 + \sum_{k \geq 1} (-1)^k q^{\binom{k+1}{2}} z^{ks}} \quad \text{and}
$$

(37)

$$
\sum_{m \in j + \mathbb{P}} \sum_{w \in \mathcal{W}^{s-1}SD_{n,m}} q^{w|z^{(w)}} = \frac{\sum (-1)^k q^{\binom{k+1}{2} + j(k+1)} z^{ks+j}}{1 + \sum_{k \geq 1} (-1)^k q^{\binom{k+1}{2} + j(k+1)} z^{ks}}
$$

(38)

4. Extensions

One major advantage of the involution method in this paper is that it readily admits extensions.

It should be clear from our definitions of the involutions in Section 3 that they did not depend on the nature of what was in the blocks. We only needed that the blocks in the pairs of compositions $(\alpha, \beta)$ are of the same type. Thus, the same type of theorems will hold for any type of block conditions. There are many different conditions one could examine, but for the sake of brevity we will end with one simple example. Suppose that we consider a block condition on $a_1 \cdots a_t$, where we require that $a_1 \cdots a_i \geq r$ for $i = 1, \ldots, s - 1$. That is, fix $s \geq 2$ and $r \geq 1$. We then define the following classes of compositions.

1. $SU^{s-1}SD_{n,m}$ is the set of all compositions $w = w_1 \cdots w_m \in \mathbb{P}^n$ such that $w_i > w_{i+1}$ if $i \in \mathbb{P}$ and $r + w_i \leq w_{i+1}$ if $i \notin \mathbb{P}$.
2. $SU^{s-1}SU^{s-1}WU_{n,m}$ is the set of all compositions $w = w_1 \cdots w_m \in \mathbb{P}^n$ such that $w_i \leq w_{i+1}$ if $i \in \mathbb{P}$ and $r + w_i \leq w_{i+1}$ if $i \notin \mathbb{P}$.

We also define the following set of generating functions.

$$
H_{n,s,0}^{SU^{s-1}SD}(z_1, \ldots, z_n) = 1 + \sum_{k \geq 1} \sum_{w \in SU^{s-1}SD_{n,k}} z(w),
$$

$$
H_{n,s,r}^{SU^{s-1}SD}(z_1, \ldots, z_n) = \sum_{k \geq 0} \sum_{w \in SU^{s-1}SD_{n,k+1}} z(w) \quad \text{for} \quad j = 1, \ldots, s - 1,
$$

$$
p_{n,s,0}^{SU^{s-1}SU^{s-1}WU}(z_1, \ldots, z_n) = 1 + \sum_{k \geq 1} (-1)^k \sum_{w \in SU^{s-1}SU^{s-1}WU_{n,k}} z(w)
$$

and

$$
p_{n,s,r}^{SU^{s-1}SU^{s-1}WU}(z_1, \ldots, z_n) = \sum_{k \geq 0} (-1)^k \sum_{w \in SU^{s-1}SU^{s-1}WU_{n,k+1}} z(w) \quad \text{for} \quad j = 1, \ldots, s - 1.
$$

Then we can use the same proofs as in Theorems 5 and 7 to prove that

$$
H_{n,s,0}^{SU^{s-1}SD}(z_1, \ldots, z_n) = \frac{1}{p_{n,s,0}^{SU^{s-1}SU^{s-1}WU}(z_1, \ldots, z_n)},
$$

(39)

and

$$
H_{n,s,r}^{SU^{s-1}SD}(z_1, \ldots, z_n) = \frac{p_{n,s,r}^{SU^{s-1}SU^{s-1}WU}(z_1, \ldots, z_n)}{p_{n,s,0}^{SU^{s-1}SU^{s-1}WU}(z_1, \ldots, z_n)}.
$$

(40)

In this case, we cannot express the $p_{n,s,r}^{SU^{s-1}SU^{s-1}WU}(z_1, \ldots, z_n)$ as a sum of quasi-symmetric functions, but we can still give explicit expressions for the specializations where we replace $z_i$ by $q^i z$ for $i = 1, \ldots, n$. That is, suppose that $m = ks + j$ where
0 \leq j \leq s - 1, and we are given a composition \( a_1 \cdots a_m \in \mathcal{S} U^{s-1} WU_n, m \). Then, let \( b = b_1 \cdots b_{k=j} \) be such that \( b_1 = 1 \) and \( b_{i+1} - b_i = r \) if \( i \not\in \mathcal{P} \) and \( b_{i+1} = b_i \) if \( i \in \mathcal{P} \). For example if \( s = 3, r = 2, \) and \( m = 10, \) then \( b_1 \cdots b_{10} = 1355799111313 \). Note that the largest part in \( b \) is \( b_{k=j} = 1 + r(k(s-1) + [j-1]^+), \) where \([j-1]^+ = \max(j-1, 0), \) and that

\[
|b| = \sum_{i=0}^{k(s-1)+j-1} (1 + ir) + \sum_{i=1}^{k} i(s-1)r + 1
\]

\[
= ks + j + r \left( \sum_{i=0}^{k(s-1)+j-1} i \right) + r(s-1) \sum_{i=1}^{k} i
\]

\[
= ks + j + r \left( \frac{k(s-1) + j}{2} \right) + r(s-1) \left( \frac{k + 1}{2} \right).
\]

It is then easy to see that we have designed \( b \) so that if \( c_i = a_i - b_i, \) then

\[
0 \leq c_1 \leq c_2 \leq \cdots \leq c_{k+j} \leq n - (1 + r(k(s-1) + [j-1]^+)).
\]

Thus, the sequences \( c = c_1 \cdots c_{k+j} \) that arise in this way are just the partitions that lie in the \((k + j) \times (n - (1 + r(k(s-1) + [j-1]^+))) \) rectangle. Since

\[
\sum_{0 \leq i_1 \leq i_2 \leq \cdots \leq i_{k+j} \leq n - (1 + r(k(s-1) + [j-1]^+))} q^{i_1 + i_2 + \cdots + i_{k+j}} = \left[ n + ks + j - (1 + r(k(s-1) + [j-1]^+)) \right] \frac{ks+j}{q},
\]

it follows that

\[
\sum_{a_1 \cdots a_{k+j} \in \mathcal{S} U^{s-1} WU_n, k+j} q^{a_1 + a_2 + \cdots + a_{k+j}} = \left[ n + ks + j - (1 + r(k(s-1) + [j-1]^+)) \right] \frac{ks+j}{q}.
\]

Thus, for \( s \geq 2, r \geq 1, \) and \( j = 1, \ldots, s - 1, \)

\[
p_{n,s,r,0}^{S' U^{s-1} WU}(z_1, \ldots, z_n) |_{z_1 \rightarrow qz_1} = 1 + \sum_{k \geq 1} (-1)^k q^{ks+r \left( \frac{k(s-1)}{2} \right) + r(s-1) \left( \frac{k+1}{2} \right)} \left[ n + kr - (r-1)ks - 1 \right] \frac{ks}{q} z^{ks}
\]

and

\[
p_{n,s,r}^{S' U^{s-1} WU}(z_1, \ldots, z_n) = \sum_{k \geq 0} (-1)^k q^{ks+j+r \left( \frac{k(s-1)+j}{2} \right) + r(s-1) \left( \frac{k+1}{2} \right)} \left[ n + kr - (r-1)(ks + j - 1) \right] \frac{ks+j}{q} z^{ks+j}.
\]

Thus we have the following theorem.

**Theorem 9.** For \( s \geq 2, r \geq 1, \) and \( j = 1, \ldots, s - 1, \)

\[
1 + \sum_{m \in \mathcal{S}} \sum_{w \in \mathcal{S} U^{s-1} \mathcal{D}_n, m} q^{[w]} z^{f(w)} = \frac{1}{1 + \sum_{k \geq 1} (-1)^k q^{ks+r \left( \frac{k(s-1)}{2} \right) + r(s-1) \left( \frac{k+1}{2} \right)} \left[ n + kr - (r-1)ks - 1 \right] \frac{ks}{q} z^{ks}}.
\]

and

\[
\sum_{m \in \mathcal{S}^{s+k+n+j} \in \mathcal{S} U^{s-1} \mathcal{D}_n, m} q^{[w]} z^{f(w)} = \frac{\sum_{k \geq 1} (-1)^k q^{ks+j+r \left( \frac{k(s-1)+j}{2} \right) + r(s-1) \left( \frac{k+1}{2} \right)} \left[ n + kr - (r-1)(ks + j - 1) \right] \frac{ks+j}{q} z^{ks+j}}{1 + \sum_{k \geq 1} (-1)^k q^{ks+r \left( \frac{k(s-1)}{2} \right) + r(s-1) \left( \frac{k+1}{2} \right)} \left[ n + kr - (r-1)ks - 1 \right] \frac{ks}{q} z^{ks}}.
\]

Finally, we end this section with a description of related work of Mendes et al. [12] on permutations with regular descent patterns and how we can use the results of this paper to prove analogous results for compositions. Mendes et al. defined \( C_{i,j,k}^{s+k+n+j} \) to be the set of \( \sigma \in \mathcal{S}_{i+k+n+j} \) with \( \text{Des}(\sigma) \subseteq \{ i, i + k, \ldots, i + nk \} \) and let \( c_{i,j,k}^{s+k+n+j} = |C_{i,j,k}^{s+k+n+j}|. \) Similarly, they defined \( E_{i+k+n+j} \) to be the set of \( \sigma \in \mathcal{S}_{i+k+n+j} \) with \( \text{Des}(\sigma) = \{ i, i+k, \ldots, i+ nk \} \) and let \( E_{i,j,k}^{s+k+n+j} = |E_{i,j,k}^{s+k+n+j}|. \) For \( \sigma \in \mathcal{S}_{i+k+n+j}, \) they defined \( R_{i,j,k}^{s+k+n+j} = \{ \sigma: 0 \leq s \leq n \) and \( s_i < s_{i+k} \) and \( r_i < s_{i+k+1} \) and \( s_{i+k} + s_{i+k+1} \} \) and let \( R_{i,j,k}^{s+k+n+j} = |R_{i,j,k}^{s+k+n+j}|. \) Thus \( E_{i,j,k}^{s+k+n+j} \) is the number of \( \sigma \in C_{i,j,k}^{s+k+n+j} \) such that \( R_{i,j,k}^{s+k+n+j} = \emptyset. \) They then considered generating functions of the form

\[
\sum_{n \geq 0} \frac{1}{(i+kn+j)!} \sum_{\sigma \in \mathcal{S}_{i+k+n+j}} z^{R_{i,j,k}^{s+k+n+j}}.
\]
Note that this generating function is similar to the generating function for $SU^k SD_{n+1,m}$ considered in this paper. There are two differences however. That is, instead of starting with $n$ increasing blocks of size $k$ followed by an increasing block of size $j$, in (43), permutations are allowed to start with an increasing block of size $i$, followed by $n$ increasing blocks of size $k$, followed by an increasing block of size $j$. The second difference is that Mendes et al. did not require a strict descents between any two consecutive blocks but, instead kept track of the number of times there were strict increases between consecutive blocks.

In fact, Mendes et al. [12] considered generalizations of (43) for $L$-tuples of permutations. To state their results we need some notation. Given a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, let

\[
\begin{align*}
\text{des}(\sigma) &= |\text{Des}(\sigma)|, \\
\text{rise}(\sigma) &= |\text{Rise}(\sigma)|, \\
\text{inv}(\sigma)(\sigma) &= \sum_{i<j} \chi(\sigma_i > \sigma_j), \\
\text{coinv}(\sigma) &= \sum_{i<j} \chi(\sigma_i < \sigma_j),
\end{align*}
\]

where for any statement $A$, $\chi(A) = 1$ if $A$ is true and $\chi(A) = 0$ if $A$ is false. Note these definitions make sense for any sequence of numbers, not just permutations. Let

\[
[n]_{p,q} = \frac{p^n - q^n}{p - q},
\]

\[
[n]_{p,q}! = [n]_{p,q}[n - 1]_{p,q} \cdots [1]_{p,q},
\]

\[
\begin{align*}
\begin{bmatrix} n \end{bmatrix}_{p,q} &= \begin{bmatrix} [n]_{p,q}! \end{bmatrix}_{p,q}, \\
\begin{bmatrix} n \lambda_1, \ldots, \lambda_k \end{bmatrix}_{p,q} &= \begin{bmatrix} [n]_{p,q}! \end{bmatrix}_{p,q},
\end{align*}
\]

be the standard $p,q$-analogues of $n, n!,$ and $(\lambda_1, \ldots, \lambda_k)$, respectively.

Now suppose $\Sigma = (\sigma^{(1)}, \ldots, \sigma^{(L)})$ is a sequence of permutations with $\sigma^{(t)} \in C_{i+kn+j}^{i,j,k}$. Define

\[
\text{Comris}_{i,k}(\Sigma) = \{ s : 0 \leq s \leq n \text{ and for all } 1 \leq t \leq L, \sigma^{(t)}_{i+sk} < \sigma^{(t)}_{i+sk+1} \}
\]

and let $\text{comris}_{i,k}(\Sigma) = |\text{Comris}_{i,k}(\Sigma)|$. Given two sequences of indeterminates, $Q = (q_1, \ldots, q_L)$ and $P = (p_1, \ldots, p_L)$, let

\[
Q^m = q_1^m \cdots q_L^m, \quad P^m = p_1^m \cdots p_L^m,
\]

\[
[n]_{P,Q} = \prod_{i=1}^l [n]_{p_i,q_i}, \quad [n]_{P,Q}! = \prod_{i=1}^l [n]_{p_i,q_i}!,
\]

\[
\begin{align*}
\text{Q}^{\text{inv}}(\Sigma) &= \prod_{i=1}^l q_i^{\text{inv}(\sigma^{(i)})}, \\
\text{P}^{\text{conv}}(\Sigma) &= \prod_{i=1}^l p_i^{\text{conv}(\sigma^{(i)})}, \quad \text{and}
\end{align*}
\]

\[
\begin{bmatrix} n \lambda_1, \ldots, \lambda_k \end{bmatrix}_{P,Q} = \prod_{i=1}^l \begin{bmatrix} n \lambda_1, \ldots, \lambda_k \end{bmatrix}_{p_i,q_i}.
\]

In addition, we set

\[
e_{P,Q,i}(x) = \sum_{n \geq 1} \frac{t_{kn} p_{\binom{k}{2} n}}{[k]_{P,Q}!}\] and $e_{P,Q,j}(x) = \sum_{n \geq 1} \frac{t_{kn} p_{\binom{k}{2} n}}{[k(n - 1) + j]_{P,Q}!}.$

Then in the case where $i = 0$ and $j = k$, Mendes et al. [12] proved that

\[
1 + \sum_{n \geq 1} \frac{t_{kn}}{[k]_{P,Q}!} \sum_{\Sigma \in \text{Comris}_{0,k}(\Sigma)} x^{\text{comris}_{0,k}(\Sigma)} \text{Q}^{\text{inv}}(\Sigma) \text{P}^{\text{conv}}(\Sigma) = \frac{1 - q}{q - x + e_{P,Q,k}(x - 1)^{1/k}}.
\]

In the case, where $i = 0$ and $1 \leq j \leq k - 1$, they proved that

\[
\sum_{n \geq 1} \frac{t_{kn}}{[k(n - 1) + j]_{P,Q}!} \sum_{\Sigma \in \text{Comris}_{0,k}(\Sigma)} x^{\text{comris}_{0,k}(\Sigma)} \text{Q}^{\text{inv}}(\Sigma) \text{P}^{\text{conv}}(\Sigma) = \frac{-e_{P,Q,j}(x - 1)^{1/k}}{q - x + e_{P,Q,k}(x - 1)^{1/k}}.
\]
Finally, in the case where $1 \leq i, j \leq k - 1$, they proved that

\[
\sum_{n \geq 2} \frac{t^{kn}}{[i + k(n - 2) + j]!P,Q!} \sum_{\sigma \in (e_i^{/k} e_j^{(k(n - 2) - 2)})} \chi_{\text{comris},k}^{\sigma} \left( \prod_{\text{mv}}(x) \right) P^{\text{coinv}}(x)
\]

\[
= \sum_{n \geq 2} \frac{x^{n-1}P\left(\frac{i+k(n-2)+j}{2}\right)t^{kn}}{[i + k(n - 2) + j]!P,Q!} - \sum_{n \geq 2} \frac{(n-1)x^{n-2}P\left(\frac{i+k(n-2)+j}{2}\right)t^{kn}}{[i + k(n - 2) + j]!P,Q!} + \frac{e_{P,Q,k}^\sigma(t(x - 1)^{1/k})e_{P,Q,k}^\sigma(t(x - 1)^{1/k})}{(1 - x) (n + 2)}.
\] (46)

Mendes et al. [12] proved these results by applying certain ring homomorphisms defined on the ring $\Lambda$ of symmetric functions in infinitely many variables to certain symmetric function identities. In the case of (46), the symmetric function identity involved a new class of symmetric functions $p_{n,a_1,\ldots,a_r}$. We can prove analogous results for $L$-tuples of compositions using the same methods. Such results will appear in a later paper. In fact, the formulas derived in this paper are needed as building blocks to derive such analogues. We should note that one could have used the methods of Mendes et al. [12] to derive all the formulas in this paper as well, but such a derivation would require the use of the same involutions defined in this paper combined with machinery of deriving generating function identities by applying carefully defined homomorphisms to symmetric function identities in $\Lambda$. One of the key points of this paper is that such machinery is not needed since we have shown that we can derive our formulas directly via simple involutions.

References