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On Score Sequences of *k*-Hypertournaments†

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Given two nonnegative integers n and k with $n \geq k > 1$, a *k-hypertournament* on n vertices is a pair (V, A), where V is a set of vertices with $|V| = n$ and A is a set of k-tuples of vertices, called arcs, such that for any k-subset S of V, A contains exactly one of the $k!$ k-tuples whose entries belong to S. We show that a nondecreasing sequence $(r_1, r_2, \ldots, r_n)$ of nonnegative integers is a losing score sequence of a k-hypertournament if and only if for each j $(1 \leq j \leq n)$,

$$\sum_{i=1}^{j} r_i \geq \binom{j}{k},$$

with equality holding when $j = n$. We also show that a nondecreasing sequence $(s_1, s_2, \ldots, s_n)$ of nonnegative integers is a score sequence of some k-hypertournament if and only if for each j $(1 \leq j \leq n)$,

$$\sum_{i=1}^{j} s_i \geq j\binom{n-1}{k-1} + \binom{n-j}{k} - \binom{n}{k},$$

with equality holding when $j = n$.

Furthermore, we obtain a necessary and sufficient condition for a score sequence of a strong k-hypertournament. The above results generalize the corresponding theorems on tournaments.

1. INTRODUCTION

Hypertournaments have been studied by a number of authors (cf. Assous [1], Barbut and Bialostocki [2], Frankl [3], Gutin and Yeo [4]). These authors raise the problem of extending the most important results on tournaments to hypertournaments.

Given two nonnegative integers n and k with $n \geq k > 1$, a *k-hypertournament* on n vertices is a pair (V, A), where V is a set of vertices with $|V| = n$ and A is a set of k-tuples of vertices, called arcs, such that for any k-subset S of V, A contains exactly one of the $k!$ k-tuples whose entries belong to S. Note that if $n < k$, then $A = \emptyset$; we call this kind of hypertournament a *null-hypertournament* and the values of its scores are all equal to 0. Clearly a 2-hypertournament is merely a tournament.

Let $R = (r_1, r_2, \ldots, r_n)$ be an integer sequence. For $1 \leq i < j \leq n$, we denote $R(r_i^+, r_j^-) = (r_1, r_2, \ldots, r_i+1, \ldots, r_j-1, r_n)$; $R^+(r_i^+, r_j^-) = (r_1', r_2', \ldots, r_n')$ will denote a permutation of $R(r_i^+, r_j^-)$ such that $r_1' \leq r_2' \leq \cdots \leq r_n'$.

Let $H = (V, A)$ denote a k-hypertournament on n vertices. The vertices and arcs of H will be denoted by $V(H)$ and $A(H)$, respectively. An (x, y)-path in H is a sequence $(x =)v_1e_1v_2e_2v_3 \cdots v_{t-1}e_{t-1}v_t(= y)$ of distinct vertices $v_1, v_2, \ldots, v_t$, $t \geq 1$, and distinct arcs $e_1, \ldots, e_{t-1}$ such that v_{i+1} lies on the last entry in e_i, $1 \leq i \leq t-1$. Let $e = (v_1, v_2, \ldots, v_k)$ be an arc in H and $i < j \leq k$, we denote $e(v_i, v_j) = (v_1, \ldots, v_{i-1}, v_j, v_{i+1}, \ldots, v_{j-1}, v_i, v_j, \ldots, v_k)$, that is, the new arc obtained from e by exchanging v_i and v_j in e. Let S be a subset of V, we denote $H\langle S\rangle$ to be the subhypertournament induced by S, that is, an arc is kept in $H\langle S\rangle$ if and only if all the vertices belonging to this arc belong to S. A k-hypertournament H is *strong* if for any two vertices $x \in V$ and $y \in V$, H contains both an (x, y)-path and a (y, x)-path. A *strong component* of a k-hypertournament H is a maximal strong subhypertournament of H. For a pair of distinct vertices x and y in H, $A(x, y)$ denotes

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the set of all arcs of H in which x precedes y. For a given vertex $v \in V$, the score $d_H^+(v)$ (or simply $d^+(v)$) of v is denoted by $d_H^+(v) = \left|\bigcup_{u \in V} A(v,u)\right||$, that is, the number of arcs containing v and in which v is not the last element. Similarly, we define the losing score $d_H^-(v)$ (or simply $d^-(v)$) as the number of arcs containing v and in which v is the last element. The score sequence of a k-hypertournament is a nondecreasing sequence of nonnegative integers $(s_1, s_2, \ldots, s_n)$, where s_i is a score of some vertex in H. Let p, q be two integers, we denote $\binom{p}{q} = \frac{p!}{q!(p-q)!}$, with $\binom{p}{q} = 0$ if $p < q$.

2. Main Results

The main results of this paper are the following theorems.

THEOREM 1. *Given two nonnegative integers n and k with $n \geq k > 1$, a nondecreasing sequence $R = (r_{1,} r_2, \ldots, r_n)$ of nonnegative integers is a losing score sequence of some k -hypertournament if and only if for each j $(k \leq j \leq n)$,*

$$\sum_{i=1}^{j} r_i \geq \binom{j}{k}, \tag{1}$$

with equality holding when $j = n$.

THEOREM 2. *Given two nonnegative integers n and k with $n \geq k > 1$, a nondecreasing sequence $S = (s_1, s_2, \ldots, s_n)$ of nonnegative integers is a score sequence of some k-hypertournament if and only if for each $j (k \leq j \leq n)$,*

$$\sum_{i=1}^{j} s_i \geq j\binom{n-1}{k-1} + \binom{n-j}{k} - \binom{n}{k}, \tag{2}$$

with equality holding when $j = n$.

In order to prove Theorems 1 and 2, we need the following lemmas. Note that there are $\binom{n}{k}$ arcs in a k-hypertournament H of order $n \geq k$, and in each arc of H, only one vertex can be on the last entry; so we have $\sum_{i=1}^{n} d_H^-(v_i) = \binom{n}{k}$.

LEMMA 1. *Let H be a k-hypertournament of order n with $(s_1, s_2, \ldots, s_n)$ as its score sequence. Then*

$$\sum_{i=1}^{n} s_i = (k-1)\binom{n}{k}.$$

PROOF. If $n < k$, then $s_1 = s_2 = \cdots = s_n = 0$, hence the lemma holds; so we assume that $n \geq k$. Let t_i denote the losing score of v_i. Then $\sum_{i=1}^{n} t_i = \binom{n}{k}$. On the other hand, there are $\binom{n-1}{k-1}$ arcs containing a given vertex. Hence we have

$$\sum_{i=1}^{n} s_i = \sum_{i=1}^{n}\left[\binom{n-1}{k-1} - t_i\right] = n\binom{n-1}{k-1} - \binom{n}{k} = (k-1)\binom{n}{k}.$$

□

LEMMA 2. *Let $R = (r_1, r_2, \dots, r_n)$ be a losing score sequence of a k-hypertournament H. If $r_i < r_j$, then $R^+(r_i^+, r_j^-)$ is a losing score sequence of a k-hypertournament H'.*

PROOF. Let $u \in V(H)$ and $v \in V(H)$, such that $d^-(u) = r_i$ and $d^-(v) = r_j$, respectively. If there is an arc e containing both u and v with v as the last element in e, then let $e' = e(u, v)$ and $H' = (H - e) \cup e'$. It is clear that $R'(r_i^+, r_j^-)$ is the losing score sequence of H'. Thus, in the following, we assume that for every arc e containing both u and v, v is not the last element in e.

Since $r_i < r_j$, there must exist two arcs e_1 and e_2 such that $e_1 = (w_1, w_2, \dots, w_{l-1}, u, w_l, \dots, w_{k-1})$ and $e_2 = (w_1', w_2', \dots, w_{k-1}', v)$, where $(w_1', w_2', \dots, w_{k-1}')$ is a permutation of $(w_1, w_2, \dots, w_{k-1})$, $u \notin \{w_1, w_2, \dots, w_{k-1}\}$ and $v \notin \{w_1, w_2, \cdots, w_{k-1}\}$. Let i_0 be the integer such that $w_{i_0}' = w_{k-1}$, and let $e_1' = e_1(u, w_{k-1})$, $e_2' = e_2(v, w_{i_0}')$. Now we can construct $H' = (H - (e_1 \cup e_2)) \cup (e_1' \cup e_2')$. It is easy to check that $R^+(r_i^+, r_j^-)$ is the losing score sequence of H'. □

LEMMA 3. *Let $R = (r_1, r_2, \dots, r_n)$ with $r_1 \le r_2 \le \cdots \le r_n$ be a nonnegative integer sequence which satisfies* (1). *If $r_n < \binom{n-1}{k-1}$, then there exists p $(1 \le p \le n-1)$ such that $R(r_n^+, r_p^-)$ is nondecreasing and satisfies* (1).

PROOF. Let p be the maximum integer such that $r_{p-1} < r_p = r_{p+1} = \cdots = r_{n-1}$ with $r_0 = 0$ if $p = 1$. We shall show that $R(r_n^+, r_p^-)$ satisfies (1). In fact, we only need to show that for each j $(p \le j \le n-1)$,

$$\sum_{i=1}^{j} r_i > \binom{j}{k}. \tag{3}$$

Since $r_n < \binom{n-1}{k-1}$,

$$\sum_{i=1}^{n-1} r_i = \binom{n}{k} - r_n > \binom{n}{k} - \binom{n-1}{k-1} = \binom{n-1}{k}.$$

Hence if $p = n-1$, (3) holds. In the following, we assume that $p \le n-2$; then (3) holds for $j = n-1$. If there exists j_0 $(p \le j_0 \le n-2)$ such that

$$\sum_{i=1}^{j_0} r_i = \binom{j_0}{k},$$

we choose j_0 as large as possible. Since

$$\sum_{i=1}^{j_0+1} r_i > \binom{j_0+1}{k},$$

$$r_{j_0} = r_{j_0+1} = \sum_{i=1}^{j_0+1} r_i - \sum_{i=1}^{j_0} r_i > \binom{j_0+1}{k} - \binom{j_0}{k} = \binom{j_0}{k-1}.$$

It follows that

$$\sum_{i=1}^{j_0-1} r_i = \sum_{i=1}^{j_0} r_i - r_{j_0} < \binom{j_0}{k} - \binom{j_0}{k-1}$$

$$\Longrightarrow \sum_{i=1}^{j_0-1} r_i < \binom{j_0-1}{k} + \binom{j_0-1}{k-1} - \binom{j_0}{k-1}$$
$$\Longrightarrow \sum_{i=1}^{j_0-1} r_i < \binom{j_0-1}{k} - \binom{j_0-1}{k-2}$$
$$\Longrightarrow \sum_{i=1}^{j_0-1} r_i < \binom{j_0-1}{k},$$

a contradiction with the hypothesis on R. Hence (3) holds.

This completes the proof of this Lemma. □

LEMMA 4. *Let $H = (V, A)$ be a k-hypertournament of order n. Let $V_1 = \{v_1, v_2, \ldots, v_j\} \subset V$ and $V_2 = V - V_1$.*

(a) *If for every arc e containing vertices from both V_1 and V_2 no vertex of V_2 in e is on the last entry, then*

$$\sum_{i=1}^{j} d_H^+(v_i) = (k-1)\binom{j}{k} + \sum_{i=1}^{k-1}(i-1)\binom{j}{i}\binom{n-j}{k-i};$$

(b) *If H is strong, then*

$$\sum_{i=1}^{j} d_H^+(v_i) > (k-1)\binom{j}{k} + \sum_{i=1}^{k-1}(i-1)\binom{j}{i}\binom{n-j}{k-i}.$$

PROOF. Let $A_1 = A(H\langle\{v_1, v_2, \ldots, v_j\}\rangle)$ and let A_2 be the set of arcs containing vertices from both V_1 and V_2. For each $v \in V$ and $e \in A$, we define a function ρ as follows:

$$\rho(v, e) = \begin{cases} 1, & \text{if } v \text{ is in e and } v \text{ is not the last element in } e, \\ 0, & \text{otherwise.} \end{cases}$$

Note that there are $\sum_{i=1}^{k-1} \binom{j}{i}\binom{n-j}{k-i}$ arcs in A_2, and for each arc e of A_2 containing exactly i vertices of V_1 and $k-i$ vertices of V_2, we have $\sum_{v\in e} \rho(v, e) = i - 1$, since no vertex of V_2 in e is on the last entry. Hence we have

$$\begin{aligned}\sum_{i=1}^{j} d_H^+(v_i) &= \sum_{i=1}^{j} \sum_{e\in A_1\cup A_2} \rho(v_i, e) \\ &= \sum_{e\in A_1\cup A_2} \sum_{i=1}^{j} \rho(v_i, e) \\ &= \sum_{e\in A_1} \sum_{i=1}^{j} \rho(v_i, e) + \sum_{e\in A_2} \sum_{i=1}^{j} \rho(v_i, e) \\ &= (k-1)\binom{j}{k} + \sum_{i=1}^{k-1}(i-1)\binom{j}{i}\binom{n-j}{k-i}.\end{aligned}$$

Thus (a) holds. To prove (b), an analogous approach to that of (a) will follow and it is sufficient to note that there must exist an arc e containing exactly i vertices of V_1 and $k-i$ vertices of V_2, such that $\sum_{v\in e} \rho(v, e) = i > i - 1$. This is obvious since H is strong. □

PROOF OF THEOREM 1. For any j with $k \le j \le n$, let $v_1, v_2, \ldots, v_j$ be the vertices such that $d^-(v_i) = r_i$ for each $1 \le i \le j$, and let $H_1 = H\langle\{v_1, v_2, \ldots, v_j\}\rangle$. Hence

$$\sum_{i=1}^{j} r_i \ge \sum_{i=1}^{j} d^-_{H_1}(v_i) = \binom{j}{k}.$$

To prove the converse, we use induction on n. For $n = k$, the statement of Theorem 1 is valid, since there is only one arc and thus all the losing scores are equal to 0 except one, which is equal to 1. Thus we assume that $n > k$. Since

$$r_n = \sum_{i=1}^{n} r_i - \sum_{i=1}^{n-1} r_i \le \binom{n}{k} - \binom{n-1}{k} = \binom{n-1}{k-1},$$

we consider the following two cases.

Case 1. $r_n = \binom{n-1}{k-1}$. Then

$$\sum_{i=1}^{n-1} r_i = \sum_{i=1}^{n} r_i - r_n = \binom{n}{k} - \binom{n-1}{k-1} = \binom{n-1}{k}.$$

By induction hypothesis, $(r_1, r_2, \ldots, r_{n-1})$ is a losing score sequence of a k-hypertournament H' of order $n-1$. Now we can construct a k-hypertournament H of order n as follows. Let $V(H') = \{v_1, v_2, \ldots, v_{n-1}\}$. Adding a new vertex v_n, for each k-tuple containing v_n, we arrange v_n on the last entry. Denote E_1 to be the set of all these $\binom{n-1}{k-1}$ k-tuples. Let $E(H) = E(H') \cup E_1$. We can easily check that $(r_1, r_2, \ldots, r_n)$ is the losing score sequence of H.

Case 2. $r_n < \binom{n-1}{k-1}$. We apply Lemma 3 repeatedly until we obtain a new nondecreasing sequence $R' = (r'_1, r'_2, \ldots, r'_n)$ such that $r'_n = \binom{n-1}{k-1}$. By Case 1 we know that R' is a losing score sequence of a k-hypertournament. Now we apply Lemma 2 on R' repeatedly until we obtain the initial nondecreasing sequence $R = (r_1, r_2, \ldots, r_n)$. By Lemma 2, R is a losing score sequence of a hypertournament.

This completes the proof of Theorem 1. □

PROOF OF THEOREM 2. Let $(s_1, s_2, \ldots, s_n)$ be the score sequence of a k-hypertournament. Then there exists a k-hypertournament H with $V = \{v_1, v_2, \ldots, v_n\}$ such that $d^+_H(v_i) = s_i$ for $i = 1, 2, \ldots, n$. Note that $d^+(v_i) + d^-(v_i) = \binom{n-1}{k-1}$. Let $r_{n+1-i} = d^-(v_i)$; then $(r_1, r_2, \ldots, r_n)$ is the losing score sequence of H. Conversely, if $(r_1, r_2, \ldots, r_n)$ is the losing score sequence of a k-hypertournament H, $(s_1, s_2, \ldots, s_n)$ is the score sequence of H. Hence it is sufficient to show that conditions (1) and (2) are equivalent provided $s_i + r_{n+1-i} = \binom{n-1}{k-1}$.

First, suppose that (2) holds. Then

$$\begin{aligned}\sum_{i=1}^{j} r_i &= \sum_{i=1}^{j} \left\{\binom{n-1}{k-1} - s_{n+1-i}\right\}\\ &= j\binom{n-1}{k-1} - \left(\sum_{i=1}^{n} s_i - \sum_{i=1}^{n-j} s_i\right)\\ &= j\binom{n-1}{k-1} - (k-1)\binom{n}{k} + \sum_{i=1}^{n-j} s_i\\ &\ge j\binom{n-1}{k-1} - (k-1)\binom{n}{k} + (n-j)\binom{n-1}{k-1} + \binom{j}{k} - \binom{n}{k}\end{aligned}$$

$$\Longrightarrow \sum_{i=1}^{j} r_i \geq \binom{j}{k},$$

with the equality if $j = n$. Hence (1) holds.

Now suppose that (1) holds; using a similar argument as above, we can prove that (2) holds. This completes the proof of the theorem. □

Note that the statement of Theorem 1 is obvious for $j < k$, we have

COROLLARY 1 (LAUDAU [6]). *A nondecreasing sequence $(s_1, s_2, \ldots, s_n)$ of nonnegative integers is a score sequence of a tournament if and only if for each $1 \leq r \leq n-1$,*

$$\sum_{i=1}^{r} s_i \geq \binom{r}{2},$$

with equality holding when $j = n$.

With a slight alteration in the hypothesis of the previous theorem, we obtain a necessary and sufficient condition for a score sequence of a strong k-hypertournament. This result generalizes a theorem of Harary and Moser [5] about strong tournaments.

THEOREM 3. *A nondecreasing sequence $(s_1, s_2, \ldots, s_n)$ of nonnegative integers is a score sequence of a strong k-hypertournament with $n > k$ if and only if*

$$\sum_{i=1}^{j} s_i > j\binom{n-1}{k-1} + \binom{n-j}{k} - \binom{n}{k},$$

for $k \leq j \leq n-1$ and

$$\sum_{i=1}^{n} s_i = (k-1)\binom{n}{k}.$$

PROOF. Let H be a strong k-hypertournament and suppose that $(s_1, s_2, \ldots, s_n)$ is a score sequence of H with $s_1 \leq s_2 \leq \cdots \leq s_n$. By Lemma 1, we have

$$\sum_{i=1}^{n} s_i = (k-1)\binom{n}{k}.$$

For each $1 \leq i \leq n$, let v_i be the vertex of H with $d_H^+(v_i) = s_i$. Let $k \leq j \leq n-1$ and define $H_1 = H\langle\{v_1, v_2, \ldots, v_j\}\rangle$. Since H_1 is a k-hypertournament of order j,

$$\sum_{i=1}^{j} d_{H_1}^+(v_i) = (k-1)\binom{j}{k}.$$

Since H is a strong k-hypertournament, then, by Lemma 4(b), we have

$$\sum_{i=1}^{j} d_H^+(v_i) > (k-1)\binom{j}{k} + \sum_{i=1}^{k-1}(i-1)\binom{j}{i}\binom{n-j}{k-i},$$

it follows that

$$\sum_{i=1}^{j} s_i = \sum_{i=1}^{j} d_H^+(v_i) > (k-1)\binom{j}{k} + \sum_{i=1}^{k-1} i\binom{j}{i}\binom{n-j}{k-i} - \sum_{i=1}^{k-1}\binom{j}{i}\binom{n-j}{k-i}$$

$$\Longrightarrow \sum_{i=1}^{j} s_i > (k-1)\binom{j}{k} + j\sum_{i=1}^{k-1}\binom{j-1}{i-1}\binom{n-j}{k-i} - \sum_{i=1}^{k-1}\binom{j}{i}\binom{n-j}{k-i}$$

$$\Longrightarrow \sum_{i=1}^{j} s_i > (k-1)\binom{j}{k} + j\left[\binom{n-1}{k-1} - \binom{j-1}{k-1}\right] - \left[\binom{n}{k} - \binom{n-j}{k} - \binom{j}{k}\right]$$

$$\Longrightarrow \sum_{i=1}^{j} s_i > j\binom{n-1}{k-1} + \binom{n-j}{k} - \binom{n}{k}.$$

For the converse, we assume that $(s_1, s_2, \ldots, s_n)$ is a nondecreasing sequence of nonnegative integers such that

$$\sum_{i=1}^{j} s_i > j\binom{n-1}{k-1} + \binom{n-j}{k} - \binom{n}{k},$$

for $k \le j \le n-1$, and

$$\sum_{i+1}^{n} s_i = n\binom{n-1}{k-1} - \binom{n}{k} = (k-1)\binom{n}{k}.$$

By Theorem 2, $(s_1, s_2, \ldots, s_n)$ is the score sequence of a k-hypertournament. Let H be such a k-hypertournament, and we show that H is strong.

If H is not strong, then we can easily verify that $V(H)$ can be partitioned as $U \cup W$ such that for any arc e containing vertices from both U and W, no vertex of U is on the last entry of e. Thus, let $H_1 = H\langle W\rangle$ and $j = |V(H_1)|$, then, by Lemma 4(a), we have

$$\sum_{w\in W} d_H^+(w) = \sum_{w\in W} d_{H_1}^+(w) + \sum_{i=1}^{k-1}(i-1)\binom{j}{i}\binom{n-j}{k-i},$$

it follows that

$$\sum_{w\in W} d_H^+(w) = \sum_{w\in W} d_{H_1}^+(w) + j\binom{n-1}{k-1} + \binom{n-j}{k} - (k-1)\binom{j}{k} - \binom{n}{k}$$
$$= j\binom{n-1}{k-1} + \binom{n-j}{k} - \binom{n}{k}.$$

Since $s_1 \le s_2 \le \cdots \le s_n$,

$$\sum_{i=1}^{j} s_i \le \sum_{w\in W} d_H^+(w) = j\binom{n-1}{k-1} + \binom{n-j}{k} - \binom{n}{k},$$

which contradicts the initial hypothesis.

This completes the proof of the theorem. □

Using a similar argument as above, we have the following theorem.

THEOREM 4. *A nondecreasing sequence* $(r_1, r_2, \ldots, r_n)$ *of nonnegative integers with* $n > k$ *is a losing score sequence of a strong k-hypertournament if and only if, for* $k \le j \le n-1$,

$$\sum_{i=1}^{j} r_i > \binom{j}{k},$$

and

$$\sum_{i=1}^{n} r_i = \binom{n}{k}.$$

COROLLARY 2 (HARARY AND MOSER [5]). *A nondecreasing sequence* $(s_1, s_2, \ldots, s_n)$ *of nonnegative integers is a score sequence of a strong tournament if and only if for each* $1 \le j \le n-1$,

$$\sum_{i=1}^{j} s_i > \binom{j}{2},$$

and

$$\sum_{i=1}^{n} s_i = \binom{n}{2}.$$

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