The spectrum of $\alpha$-resolvable block designs with block size 3

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Abstract


A balanced incomplete block design $D(v, k, \lambda)$ is $\alpha$-resolvable if its blocks can be partitioned into classes such that each point of the design lies in exactly $\alpha$ blocks in each class. Necessary conditions on the parameters of such designs are those required for balanced incomplete block designs (that is, $vb = rk$, $\lambda(v-1) = r(k-1)$) together with the conditions $k \mid av$, and $\alpha \mid r$. It is shown here that if $k = 3$ then these conditions are also sufficient for the existence provided that $v \neq 6, \alpha \neq 1$.

1. Introduction

A balanced incomplete block design (BIBD) or order $v$, block size $k$, and index $\lambda$, where $v$, $k$, and $\lambda$ are integers satisfying the relations $v > k \geq 2$ and $\lambda > 0$, is a pair $(V, \mathcal{F})$ where $V$ is a $v$-set and $\mathcal{F}$ is a family of $k$-subsets of $V$, called blocks, which have the property that every pair of distinct points of $V$ occurs in precisely $\lambda$ blocks. It is trivial to show that in such a design, every point occurs in $r$ blocks, where $r$ is independent of the point chosen, and if $b$ is the number of blocks in $\mathcal{F}$, then the relations

\[ bk = rv, \]

\[ \lambda(v-1) = r(k-1) \]  \hspace{1cm} (2)

and therefore

\[ \lambda(v-1) \equiv 0 \pmod{k-1}, \]  \hspace{1cm} (3)

\[ \lambda v(v-1) \equiv 0 \pmod{k(k-1)} \]  \hspace{1cm} (4)

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must hold. A BIBD is said to be $\alpha$-resolvable if its blocks are partitioned into classes (called $\alpha$-resolution classes) such that each point of the design occurs in precisely $\alpha$ blocks in each class. Clearly, in such a design, the additional conditions

$$k \mid \alpha v \quad (5)$$

and

$$\alpha \mid r \quad (6)$$

must hold.

It is our purpose here to show that if $k = 3$, and $v \neq 6$, then conditions (3)–(6) are sufficient for the existence of an $\alpha$-resolvable BIBD.

2. A few special cases

A BIBD $(v, 3, \lambda)$ is called a $\lambda$-fold triple system and is denoted $TS(v, \lambda)$. For such systems, condition (5) becomes $3 \mid \alpha v$. For $v \equiv 0 \pmod{3}$, this places no condition on $\alpha$, while for $v \not\equiv 0 \pmod{3}$, this yields the condition $3 \mid \alpha$. Let us first consider the case in which $v \equiv 0 \pmod{3}$.

It is well known, see [2], that for positive $v \equiv 0 \pmod{3}$, $v \neq 6$, the conditions (1) and (2) imply the existence of a 1-resolvable (or simply, resolvable) triple system. If there are $r$ resolution classes, and if $r = \alpha t$, then one can collect $t$ larger classes each consisting of $\alpha$ different 1-resolution classes to obtain an $\alpha$-resolvable design. The case of six points presents special problems. Fortunately this case can be handled using the results of Hartman [7–8].

Lemma 2.1. The necessary conditions (3)–(6) are sufficient for the existence of an $\alpha$-resolvable $(6, 3, \lambda)$ design if $\alpha > 1$. A necessary and sufficient condition for the existence of a 1-resolvable $(6, 3, \lambda)$ design is $\lambda \equiv 0 \pmod{4}$.

Proof. Hartman has shown (see [7]) that a necessary and sufficient condition for the existence of a 1-resolvable $(6, 3, \lambda)$ is $\lambda \equiv 0 \pmod{4}$. So in this case, unions of $\alpha$ parallel classes will give an $\alpha$-resolvable $(6, 3, \lambda)$ design for only $\alpha \geq 1$ which satisfies (6). When $\lambda \equiv 2 \pmod{4}$ we have, by (6), that $\alpha \mid (5\lambda/2)$, hence $\alpha$ is odd, and consequently $\alpha \geq 3$. The unique $(6, 3, 2)$ design can be partitioned, as follows, into a 2-resolution class $P_2 = \{012, 034, 145, 235\}$ and a 3-resolution class $P_3 = \{015, 024, 123, 035, 134, 245\}$. The remaining blocks of a $(6, 3, 4n + 2)$ design may be constructed by taking $n$ copies of the set of all 3-subsets. The $\alpha$-resolution classes are constructed by adding $\alpha - 2$ parallel classes to $P_2$, $\alpha - 3$ parallel classes to $P_3$, and unions of $\alpha$ parallel classes for the rest. (The only case not covered by this construction is the trivial $\alpha = 5\lambda/2$ resolution with only one $\alpha$-resolution class.)
3. Triple systems with a point-regular group

A triple system is said to have a point regular group $G$ if (i) $G$ is a subgroup of the full automorphism group and (ii) for distinct points $x$ and $y$ there is precisely one element of $G$ mapping $a$ to $b$.

The following theorem is important in the case in which the order of a triple system is not divisible by three.

**Theorem 3.1.** A triple system $TS(v, \lambda)$ with a point-regular group $G$ exists if and only if one of the following holds:

1. $v = 1$ or $3 \pmod{6}$ for $\lambda = 1, 5, 7$ or $11 \pmod{12}$;
2. $v = 0, 1, 3, 4, 7$ or $9 \pmod{12}$ for $\lambda = 2$ or $10 \pmod{12}$;
3. $v = 1 \pmod{2}$ for $\lambda = 3$ or $9 \pmod{12}$;
4. $v = 0$ or $1 \pmod{3}$ for $\lambda = 4$ or $8 \pmod{12}$;
5. $v = 0$ or $3 \pmod{4}$ for $\lambda = 6 \pmod{12}$;
6. $v \geq 3$ for $\lambda = 0 \pmod{12}$.

**Proof.** By the results of Colbourn and Colbourn [4], the above conditions are sufficient for the existence of a cyclic triple system, with the two exceptions $v = 9$ and $\lambda = 1$ or $2$. Clearly these cases can be realised by the elementary group $G = EA(9)$. It remains to show that (1)–(6) are necessary. Note that these conditions are equivalent to the necessary existence conditions for $TS(v, \lambda)$ together with the additional condition

$$v = 2 \pmod{4} \implies \lambda = 0 \pmod{4}.$$ \hfill (*)

It remains to derive the necessity of (*). Assume $v = 2 \pmod{4}$. Then $G$ may be written as the semidirect product of $\mathbb{Z}_2 = \{0, 1\}$ with a group $H$ of (odd) order $v/2$. Assume first that $v$ is not a multiple of 3 (so that $D$ may be represented by a $(v, 3, \lambda)$-difference family in $G$). Then every base block yields either 0 or 4 differences with first coordinate 1. Writing $v = 4m + 2$ we need $\lambda(2m + 1)$ such differences and therefore 4 divides $\lambda$. Finally assume $v = 0 \pmod{3}$, say $v = 12m + 6$. Then any subgroup $U$ of $G$ of order 3 can only contain elements with first coordinate 0 (since the intersection of $U$ with $H$ cannot have size 1), and we get $\lambda = 0 \pmod{4}$ as before. \hfill \square

**Corollary 3.1.1.** A $(v, 3, \lambda)$-difference family exists if and only if the following conditions hold:

1. $\lambda(v - 1) = 0 \pmod{6}$;
2. $v = 2 \pmod{4}$ implies $\lambda = 0 \pmod{4}$. 

\vspace{1cm}
4. \( \alpha \)-Resolvable triple systems with \( (v, 3) = 1 \)

It remains to discuss the case of \( \alpha \)-resolvable triple systems in the case where 3 and \( v \) are relatively prime. In these cases, the condition \( k \mid \alpha v \) yields \( 3 \mid \alpha \), and a solution for the case \( \alpha = 3 \) will also provide solutions for all \( \alpha \) such that \( 3 \mid \alpha \) and \( \alpha \mid r \). In the event that there exists a triple system which admits a point regular group, then this system is clearly 3-resolvable. In view of the foregoing, we consider the following cases to show that the necessary conditions for an \( \alpha \)-resolvable design in the case \( k = 3 \) are also sufficient, provided that \( v \neq 6 \).

**Case 1:** \( \lambda = 1 \) or \( 5 \) (mod 6).

Here \( v = 3 \) (mod 6) and the assertion follows by the result of Ray-Chaudhuri and Wilson [9] on Kirkman triple systems; or \( v = 1 \) (mod 6), and we may use a \( (v, 3, \lambda) \)-difference family which exists by Theorem 3.1.

**Case 2:** \( \lambda = 2 \) or \( 4 \) (mod 6).

Then either \( v = 0 \) (mod 3) and the assertion follows from the results of Hanani [5] who provided the case \( \alpha = 1 \). Or \( v = 1 \) (mod 3), and the assertion follows from Theorem 3.1 unless \( v = 10 \) (mod 12) and \( \lambda = 2 \) or 10 (mod 12).

**Case 3:** \( \lambda = 3 \) (mod 6).

Then \( v \) is necessarily odd, and the assertion follows from Theorem 3.1.

**Case 4:** \( \lambda = 0 \) (mod 6) (and \( v \geq 3 \)).

The assertion follows from Theorem 3.1, unless \( v = 2 \) (mod 4) and \( \lambda = 6 \) (mod 12). Note that \( v = 6 \) (mod 12) is covered by Case 2.

The remaining cases can be disposed of by proving the assertion for the cases
(1) \( v = 10 \) (mod 12) and \( \lambda = 2 \); and
(2) \( v = 2 \) (mod 12) and \( \lambda = 6 \).

5. \( \alpha \)-Resolvable frames

The notion of a resolvable frame was introduced by Hanani [4] to investigate resolvable subsystems of resolvable triple systems of index \( \lambda = 1 \). The following version of this notion is useful here. Let \( V \) be a finite set of cardinality \( v \). Let \( k \) be any integer satisfying \( 2 < k < v \). Then a block is any subset of \( V \) which has cardinality \( k \). An \( \alpha \)-resolution class is a collection of blocks such that each point of \( V \) occurs in precisely \( \alpha \) blocks. A partial \( \alpha \)-resolution class is a collection of blocks in which each point of \( V \) occurs either \( \alpha \) or zero blocks. The set of points not occurring in a partial parallel class is called the complement of the class. Let \( G \) be a partition of \( V \) (the members of \( G \) are called groups). Then a \( (\lambda, \alpha) \) frame is the triple \( (V, G, P) \) where \( P \) is a collection of \( \alpha \)-partial resolution classes of the non-empty set \( V \), whose blocks are of cardinality 3, which satisfies the following conditions:

(1) The complement of each partial \( \alpha \)-resolution class \( P \) of \( P \) is a group \( G \) of \( G \);
(2) Each unordered pair of \( V \) which does not lie in some group of \( \mathcal{G} \) lies in precisely \( \lambda \) blocks of \( \mathcal{G} \), no unordered pair of elements which lie in some group of \( \mathcal{G} \) also lies in a block of \( \mathcal{P} \).

The type of the \((\lambda, \alpha)\) frame is the multiset \( \{|G|: G \in \mathcal{G}\} \). If the multiset contains \( u_1 \) 1's, \( u_2 \) 2's, etc., then the notation \( 1^{u_1}2^{u_2}\cdots \) is used to describe it.

For the definition of pairwise balanced design (PBD), group divisible design (GDD), and transversal design (TD), see [10].

The following construction for \((\lambda, \alpha)\) frames is that of [10, Construction 3.1].

**Lemma 5.1.** Let \((V, \mathcal{G}, \mathcal{A})\) be a GDD, and let \( w: X \rightarrow \mathbb{Z}^+ \cup \{0\} \). For each \( A \) in \( \mathcal{A} \), suppose that there is a \((\lambda, \alpha)\) frame of type \( \{w(x): x \in A\} \). Then there is a \((\lambda, \alpha)\) frame of type \( \{C_{xtc}w(x): G \in \mathcal{G}\} \).

6. Frames and \( \alpha \)-resolvable designs

A \((\lambda, \alpha)\) frame of type \( t^\alpha \) is said to be uniform. Let \( T = \{u: \) there exists a \((\lambda, \alpha)\) frame of type \( t^\alpha\}\}. Then it is easily shown that Lemma 5.1 implies that \( T \) is \( \text{PBD-closed} \).

**Lemma 6.1.** Let \( S = \{u: \) there exists a \((2, 3)\) frame of type \( 3^u\}\}. Then \( S = \{u: u \in \mathbb{Z}, u \geq 1, u \neq 2, 3\} \).

**Proof.** Assaf and Hartman [1] have given necessary and sufficient conditions for the existence of uniform \((\lambda, 1)\) frames with block size 3. This result implies the existence of \((2, 1)\) frames of type \( 3^u \) for all positive \( u \neq 2, 3 \). By amalgamating parallel classes, the required \((2, 3)\) frames are constructed. \( \square \)

**Corollary 6.1.** Let \( v = 3u + 1 \), where \( u \) is a positive integer, and let \( b, r, \lambda \) be positive integers such that conditions (3)-(6) of Section 1 hold. Then there exists an \( \alpha \)-resolvable \( \lambda \)-fold triple system of order \( v \).

**Proof.** If \( u \) is even, the result follows from Corollary 3.1.1. Suppose that \( u \) is odd. Then \( \lambda \) must be even, and the condition \( 3 \mid \alpha \) must hold. Therefore for specified \( v \) and \( \lambda \), such an \( \alpha \)-resolvable triple system exists provided that there exists a 3-resolvable, 2-fold triple system of order \( v \). If there exists a \((2, 3)\) frame of type \( 3^u \), then the required 2-fold triple system can be constructed as follows.

Trivially, there exists a 2-fold 3-resolvable triple system \((4, 3, 2)\) consisting of all 3-subsets of a 4-set. Let \( F \) be the \((2, 3)\) frame, and let \( \infty \) be a point not belonging to the point-set of the frame. To each group \( G \) of the frame, adjoin \( \infty \), then adjoin the blocks of a \((4, 3, 2)\) design defined on \( G \cup \{\infty\} \) to the blocks of the frame in such a way that these blocks are adjoined to the partial 3-resolution class whose complement is \( G \). The result is a 3-resolvable 2-fold triple system.
By the above lemma, this covers all cases except for \( v = 10 \) (recall that \( v = 7 \) is covered by Corollary 3.1.1). A 3-resolvable two-fold triple system of order 10, found by Royle, is exhibited in Table 1. (Royle also showed that each of the 960 two-fold triple systems of order 10 is 3-resolvable.) □

It remains to dispose of the case where \( v \) is congruent to 2 (mod 12). An orthogonal array \( OA(k, n) \) is an \( n^2 \times k \) array of the symbols \( N = \{1, 2, \ldots, n\} \) such that each of the ordered pairs \( \{i, j\}, i, j \in N \), occurs precisely once in any two specified columns of the array. Such an array is idempotent if the sub-arrays \( (1, 1, \ldots, 1), (2, 2, \ldots, 2), \ldots, (n, n, \ldots, n) \) appear as rows of the array. It is well known that a set of \( k - 2 \) mutually orthogonally latin squares of order \( n \) which contain a common transversal gives rise to an idempotent \( OA(k, n) \). In particular, since there exists a self-orthogonal latin square (i.e., a latin square orthogonal to its transpose) of order \( n \) for every positive integer \( n \) except for \( n = 2, 3 \) or 6, then there is an idempotent orthogonal array \( OA(4, n) \) for all such \( n \) except for \( n = 2, 3 \) or 6.

**Lemma 6.2.** Let \( n \) be a positive integer other than 2 or 3. Then there exists a (6, 3) frame of type \( 1^n \).

**Proof.** Let \( n \) be a positive integer other than 2, 3, or 6. Then there exists an idempotent orthogonal array \( OA(4, n) = A \). For each symbol \( k, k = 1, 2, \ldots, n \), let \( T_k \) denote the set of rows of \( A \) whose fourth entry is \( k \). Let \( U_k = T_k \setminus \{(k, k, k, k)\} \). Let \( S_k = \{(a, b, c) : (a, b, c, k) \in T_k\} \). Then it is easily shown that the collection of subsets \( \bigcup_{k=1}^n S_k \) form the blocks of a (6, 3) frame of type \( 1^n \) in which the set \( S_k \) is the complement of the group \( \{k\}, k = 1, 2, \ldots, n \).

For \( n = 6 \) there is a (6, 3) frame of type \( 1^6 \) whose underlying design is three copies of the design \( D(6, 3, 2) \). The 3-resolution of this design is unique and is given by \( \{(0, 2, 3) (mod\ 5)\} \) and \( \{(\infty, 1, 2), (\infty, 2, 3)(\infty, 3, 4)(1, 2, 4)(1, 3, 4)\} \) (mod 5). □

**Lemma 6.3.** Suppose that there exists a (6, 3) frame \( F \) of type \( \Pi_{i=1}^s g^n_i \). Suppose further that for each \( g_i, i = 1, 2, \ldots, s \), there exists a 3-resolvable 6-fold triple
system of order $g_i + 1$. Then there exists a 3-resolvable 6-fold triple system of order $v = \sum_{i=1}^{g_i} + 1$.

**Proof.** Counting arguments again show that in a $(6, 3)$ frame, the set of partial 3-resolution classes whose complement is the group $G$ contains precisely $|G|$ classes. Let $\infty$ be a point not occurring in the frame $F$. For each group $G$ of $F$, adjoin the blocks of a 3-resolvable 6-fold triple system on the set $G \cup \{\infty\}$ to the partial resolution classes whose complement is $G$. Since the number of 3-resolution classes in such a triple system is $|G|$, it easily follows that the resulting set of blocks yields the desired triple system of order $v$. \qed

**Lemma 6.4.** There exists a 3-resolvable 6-fold triple system of order 14 and 26.

**Proof.** A 3-resolvable 6-fold triple system of order 14 with point set $Z_{13} \cup \{\infty\}$ is generated by developing the following 3-resolution class modulo 13:

$$(2^i, 2^4+i, 2^5+i) \quad i = 0, 1, 2, 3 \text{ twice},$$
$$(0, 1, 4), (0, 2, 8), (0, 3, 5), (\infty, 6, 12), (\infty, 7, 11), (\infty, 9, 10).$$

A 3-resolvable 6-fold triple system of order 26 with point set $GF(25) \cup \{\infty\}$ is generated by adding each member of $GF(25)$ to the following 3-resolution class:

We denote by $x$ the root of the primitive polynomial $x^2 = x + 3$.

$$(x^i, x^8+i, x^{16+i}) \quad i = 0, 1, 2, 3 \text{ twice},$$
$$(x^4+i, x^{12+i}, x^{20+i}) \quad i = 0, 1, 2, 3 \text{ three times},$$
$$(0, x^8i+1, x^8i+3) \quad i = 0, 1, 2, (\infty, x^8i, x^8i+18) \quad i = 0, 1, 2. \quad \square$$

Let $U = \{u: \text{there exists a 3-resolvable } (u, 3, 6) \text{ design}\}$. By the foregoing results $U$ contains the set $W = \{u: u \geq 3, u \not\equiv 2 \pmod{12}\} \cup \{14, 26\}$. Let $X = \{u: u \geq 38, u \equiv 2 \pmod{12}\}$. It only remains to show that $X$ is contained in $U$, and that is the purpose of the next lemma.

**Lemma 6.5.** Let $u = 12s + 1$, where $s \geq 3$. Then there exists a (6, 3) frame $(U, \mathcal{G}, P)$ where $|U| = u$, and $|G| + 1 \in W$ for each $G \in \mathcal{G}$.

**Proof.** First consider the case of $s \geq 6$. Then $u$ can be written in the form $u = 4(3(s - 1)) + 13$, where $s - 1 \geq 5$. Let $T$ be a TD$(5, 3(s - 1))$. (Such exist, since there exist 3 mutually orthogonal latin squares of order $w$ for all $w \geq 11$.) By deleting all but 13 points from one of the groups of $T$, one obtains a GDD of group type $[3(s - 1)]^{13}1$ and block sizes belonging to $\{4, 5\}$. By Lemma 6.2, there exists a $(6, 3)$ frame of type $1^n$ for $n \in \{4, 5\}$. Therefore by Lemma 5.1, there is a $(6, 3)$ frame of type $[3(s - 1)]^{13}1$. 

For \( s = 3, 4, \) and 5, consider the following group divisible designs.

**Case:** \( s = 3 \) (\( u = 37 \)); **Group type** \( 8^45^1 \), block sizes \( \{4, 5\} \).

Delete 3 points from a \( \text{TD}(5, 8) \).

**Case:** \( s = 4 \) (\( u = 49 \)); **Group type** \( 7^7 \), block sizes \( \{7\} \).

Use a \( \text{TD}(7, 7) \).

**Case:** \( s = 5 \) (\( u = 61 \)); **Group type** \( 8^75^1 \), block sizes \( \{7, 8\} \).

Delete 3 points from a \( \text{TD}(8, 8) \).

In each case, an application of Lemmas 5.1 and 6.2 yields a \((6, 3)\) frame of the same type.

**Corollary 6.5.2.** Let \( X \) and \( U \) be defined as above. Then \( X \subseteq U \).

**Proof.** An application of Lemmas 6.3 and 6.4 yields the result.

The foregoing can be summarized in the following theorem.

**Theorem 6.6.** Let \( v \) and \( \alpha \) be integers such that \( v \geq 3 \) and \( \alpha \geq 1 \). If \( (v, \alpha) \neq (6, 1) \), then there exists an \( \alpha \)-resolvable \( \lambda \)-fold triple system if and only if the conditions \( \lambda(v - 1) \equiv 0 \pmod{2} \), \( \lambda v(v - 1) \equiv 0 \pmod{6} \), \( k \mid \alpha v \), and \( \alpha \mid r \) (where \( r = \lambda(v - 1)/2 \)) hold. Further, if \( (v, \alpha) = (6, 1) \) then there is a 1-resolvable design \((6, 3, \lambda)\) if and only if \( \lambda \equiv 0 \pmod{4} \).

**Proof.** As before, the triple systems above can be used to construct such systems for all admissible \( \alpha \) and \( \lambda \) in the case \( v = 2 \pmod{3} \).

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**References**


