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Comparison Theorems for Differential Equations

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1. Introduction

This note is concerned with extensions of the following:

Basic Comparison Theorem. *Suppose $u(t)$ and $v(t)$ are continuous on the interval $[a, b]$ of the real line R, and differentiable on $(a, b]$, f is a continuous mapping from $R \times R$ to R and*

$$u(a) < v(a), \qquad \frac{du}{dt} - f(t, u) < \frac{dv}{dt} - f(t, v) \qquad \text{on } (a, b]. \tag{1.1}$$

Then $u < v$ on $[a, b]$.

Let us suppose $u \geqq v$ somewhere on $[a, b]$. Then, since $u - v$ is continuous on $[a, b]$, and $u(a) - v(a) < 0$, there is a point c in $(a, b]$ such that $u(c) = v(c)$ and $u < v$ on $[a, c)$. But then $(du/dt)(c) \geqq (dv/dt)(c)$ and $f(c, u(c)) = f(c, v(c))$.

Since this violates the inequality (1.1) at c, no such c exists in $[a, b]$. This result is our prototype weak comparison theorem.

If $f(t, u)$ satisfies a Lipschitz condition of the form

$$|f(t, u_1) - f(t, u_2)| \leqq K|u_1 - u_2|, \tag{1.2}$$

then a stronger result can be stated.

Strong Comparison Theorem. *Suppose $u(t)$, $v(t)$ are continuous on*

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$[a, b]$ and differentiable on $(a, b]$, f is a continuous mapping from $R \times R$ to R, satisfying the Lipschitz condition (1.2), *and*

$$u(a) < v(a), \qquad \frac{du}{dt} - f(t, u) \leqq \frac{dv}{dt} - f(t, v) \qquad \text{on } (a, b].$$

Then $u < v$ on $[a, b]$.

Proof. Suppose $v(a) - u(a) = A > 0$, and let

$$w = u + \frac{A}{2} e^{-2K(t-a)}.$$

Then

$$\begin{aligned}\frac{dw}{dt} - f(t, w) &= \frac{du}{dt} - AKe^{-2K(t-a)} - f(t, u) - [f(t, w) - f(t, u)] \\ &\leqq \left[\frac{du}{dt} - f(t, u)\right] + K|w - u| - AKe^{-2K(t-a)} \\ &\leqq \frac{dv}{dt} - f(t, v) + \frac{KA}{2} e^{-2K(t-a)} - KAe^{-2K(t-a)}, \qquad \text{on } (a, b] \\ &< \frac{dv}{dt} - f(t, v), \qquad \text{on } (a, b].\end{aligned}$$

Now $w(a) < v(a)$ and so by the *Weak Comparison Theorem*, $w < v$ on $[a, b]$ and $u < w < v$ on $[a, b]$.

An obvious implication of this result is that two solutions of the differential equation

$$\frac{du}{dt} = f(t, u), \qquad \text{on } (a, b) \tag{1.3}$$

can only intersect at singular points where $f(t, u)$ does not satisfy (1.2). These results also provide a basis for deriving upper and lower bounds for solutions of the differential equation (1.3) above. Extensions of these results, considered below, are concerned with systems of first-order equations of the form

$$\frac{du_i}{dt} = f_i(t, u_j), \qquad i, j = 1, 2, \ldots, n; \quad t \in [a, b] \subset R, \tag{1.4}$$

where f_i is a map from $[a, b] \times R^n$ into R.

Simple counter-examples show that one's first concept of an extension where, say u, v and f in (1.1) are taken as n-vectors, will not do. For example, the functions $v_1 = \sin t$, $v_2 = \cos t$, $u_1 = -\frac{1}{2}$, $u_2 = +\frac{1}{2}$ satisfy $u_1(0) < v_1(0)$, $u_2(0) < v_2(0)$, $(du_1/dt) - u_2 < (dv_1/dt) - v_2$, $(du_2/dt) + u_1 < (dv_2/dt) + v_1$ on $(0, 2\pi)$, but at $3\pi/2$, $u_1 = -\frac{1}{2} > \sin(3\pi/2) = -1$, and $u_2 = \frac{1}{2} > \cos(3\pi/2) = 0$. However, an extension can be obtained by the concurrent use of upper and lower bounds in the formulation of the comparison theorems.

2. Comparison Theorems for Systems of Equations

Weak Comparison Theorem. *Suppose the n-vectors $u(t)$, $\bar{u}(t)$, $\underline{u}(t)$ are continuous on $[a, b]$ (an interval on the real line R) and differentiable on $(a, b]$, $f(t, u)$ is a continuous map from $[a, b] \times R^n$ to R^n, and*

$$
\begin{aligned}
&(1)\quad \underline{u}(a) < u(a) < \bar{u}(a) \qquad (\textit{component-wise});\\
&(2)\quad \frac{d}{dt}\underline{u} - \underline{f}(t, \bar{u}, \underline{u}) < \frac{du}{dt} - f(t, u) < \frac{d\bar{u}}{dt} - \bar{f}(t, \bar{u}, \underline{u}), \qquad \text{on } (a, b],
\end{aligned}
\tag{2.1}
$$

where

$$\underline{f}_i(t, \bar{u}, \underline{u}) \equiv \inf_{\theta} f_i(t, \theta) \qquad \text{when} \quad \theta_i = \underline{u}_i,$$

$$\bar{f}_i(t, \bar{u}, \underline{u}) \equiv \sup_{\theta} f_i(t, \theta) \qquad \text{when} \quad \theta_i = \bar{u}_i \text{ and } \underline{u}_j \leqq \theta_j \leqq \bar{u}_j \text{ for all } j \neq i.$$

Then

(3) $\underline{u} < u < \bar{u}$ on $[a, b]$.

Proof. If the inequality (2.1 (3)) is violated, there is an integer i of the set $(1, 2, \ldots, n)$ and a point c in $(a, b]$ such that

$$u_i(c) = \bar{u}_i(c) \quad (\text{or } \underline{u}_i(c)) \qquad \text{and} \qquad \underline{u} < u < \bar{u} \qquad \text{in } [a, c),$$

where u_i denotes the ith component of the vector u.

Suppose $u_i(c) = \bar{u}_i(c)$. (A similar argument holds if $u_i(c) = \underline{u}_i(c)$.) Then

$$\frac{du_i}{dt}(c) \geqq \frac{d\bar{u}_i}{dt}(c)$$

and

$$
\begin{aligned}
\frac{du_i}{dt}(c) - f_i(c, u(c)) &\geqq \frac{d\bar{u}_i}{dt}(c) - \sup f_i(c, \theta)\\
&\qquad \text{when} \quad \theta_i = \bar{u}_i \qquad \text{and} \qquad \underline{u}_j \leqq \theta_j \leqq \bar{u}_j, \quad j \neq i\\
&\geqq \frac{d\bar{u}_i}{dt}(c) - \bar{f}_i(c, \bar{u}, \underline{u}).
\end{aligned}
$$

Since this violates the inequality (2.1 (2)), we conclude that (2.1 (3)) holds on $[a, b]$.

This result may be sharpened in the case where $f(t, u)$ satisfies a Lipschitz condition of the form

$$\|f(t, u)-f(t, v)\| = \sup_i f_i(t, u)-f_i(t, v)| \leqq K\|u-v\|, \tag{2.2}$$

where $\|u-v\| = \sup_i |u_i - v_i|$, for some constant K, and t in $[a, b]$.

STRONG COMPARISON THEOREM. *Suppose the n-vectors $u(t)$, $\underline{u}(t)$, $\bar{u}(t)$ are continuous on the interval $[a, b)$ of the real line R, differentiable on $(a, b]$, while $f(t, u)$ is a Lipschitz continuous map (in the sense (2.2)) from $[a, b]\times R^n$ to R^n, and*

(1) $\underline{u}(a) < u(a) < \bar{u}(a)$;

(2) $$\frac{d}{dt}\underline{u} - \underline{f}(t, \bar{u}, \underline{u}) \leqq \frac{du}{dt} - f(t, u) \leqq \frac{d\bar{u}}{dt} - \bar{f}(t, \bar{u}, \underline{u}) \qquad \text{on } (a, b]. \tag{2.3}$$

Then

(3) $\underline{u} < u < \bar{u}$ on $[a, b]$.

Proof. Let $w \in R^n$ be such that $w > 0$, $2w \leqq \bar{u}(a) - u(a)$, and $2w \leqq u(a) - \underline{u}(a)$ component-wise. Define $\bar{u}^* = \bar{u} - e^{-K(t-a)}w$, $\underline{u}^* = \underline{u} + e^{-2K(t-a)}w$, so that $\underline{u}(a) < \underline{u}^*(a) < u(a) < \bar{u}^*(a) < \bar{u}(a)$. Consider the expression $\bar{E}$ defined by

$$\begin{aligned}
\bar{E} &= \frac{d\bar{u}^*}{dt} - \bar{f}(t, \bar{u}^*, \underline{u}^*) \\
&= \left\{\frac{d\bar{u}}{dt} - \bar{f}(t, \bar{u}, \underline{u})\right\} + 2Ke^{-2K(t-a)}w + \{\bar{f}(t, \bar{u}, \underline{u}) - \bar{f}(t, \bar{u}^*, \underline{u}^*)\} \\
&\geqq \left\{\frac{du}{dt} - f(t, u)\right\} + 2Ke^{-2K(t-a)}w + \{\bar{f}(t, \bar{u}, \underline{u}) - \bar{f}(t, \bar{u}^*, \underline{u}^*)\}.
\end{aligned}$$

Now

$$\begin{aligned}
\bar{f}_i(t, \bar{u}, \underline{u}) = \sup_{\underline{u} \leqq \theta \leqq \bar{u}} \{ & f_i(t, \theta_1, \theta_2, \ldots, \bar{u}_i, \theta_{i+1}, \ldots, \theta_n) \\
& - f_i(t, \theta_1, \theta_2, \ldots, \bar{u}_i^*, \ldots, \theta_n) + f_i(t, \theta_1, \theta_2, \ldots, \bar{u}_i^*, \ldots, \theta_n)\},
\end{aligned}$$

and if we choose θ so that $f_i(t, \theta_1, \theta_2, \bar{u}_i^*, \ldots, \theta_n)$ is maximized in the given θ

range instead of $f_i(t, \theta_1, \theta_2, \ldots, \bar{u}_i, \ldots, \theta_n)$, this term is reduced in value or at worst unchanged. Similarly

$$\bar{f}_i(t, \bar{u}^*, \underline{u}^*) = \sup_{\underline{u}^* \leqq \theta \leqq \bar{u}^*} f_i(t, \theta_1, \theta_2, \ldots, \bar{u}_i^*, \ldots, \theta_n),$$

and if the range of θ is increased to $\underline{u} \leqq \theta \leqq \bar{u}$, this term is increased or unchanged. Let θ^* denote this value of θ, so that

$$\begin{aligned} \bar{E}_i &\geqq \left\{\frac{du}{dt} - f(t, u)\right\}_i + 2Ke^{-2K(t-a)}w_i \\ &\quad + \{f_i(t, \theta_1^*, \ldots, \bar{u}_i, \ldots, \theta_n^*) - f_i(t, \theta_1^*, \ldots, \bar{u}_i^*, \theta_n^*)\} \\ &\geqq \left\{\frac{du}{dt} - f(t, u)\right\}_i + 2Ke^{-2K(t-a)}w_i - Ke^{-2K(t-a)}w_i \\ &> \left\{\frac{du}{dt} - f(t, u)\right\}_i. \end{aligned}$$

Thus

$$\frac{d\bar{u}^*}{dt} - \bar{f}(t, \bar{u}^*, \underline{u}^*) > \frac{du}{dt} - f(t, u).$$

In a similar fashion,

$$\frac{d}{dt}\underline{u}^* - \underline{f}(t, \bar{u}^*, \underline{u}^*) < \frac{du}{dt} - f(t, u),$$

and so, by the previous theorem,

$$\underline{u} < \underline{u}^* < u < \bar{u}^* < \bar{u} \qquad \text{in } [a, b].$$

3. Special Cases

In the case where the inequalities (2.3 (2)) are equalities and each term is zero on $(a, b]$, u is a solution of the differential equation

$$\frac{du}{dt} = f(t, u), \tag{3.1}$$

and $\bar{u}$ and $\underline{u}$ are solutions of the system of equations

$$\frac{d\bar{u}}{dt} = \bar{f}(t, \bar{u}, \underline{u}), \qquad \frac{d}{dt}\underline{u} = \underline{f}(t, \bar{u}, \underline{u}), \tag{3.2}$$

in $2n$ dependent variables. Each solution of (3.1) provides a solution $\bar{u}=u=\underline{u}$ of the system (3.2).

In the case where each $f_i(t, u_1, u_2,..., u_n)$ is a non-decreasing function of u_α, where $\alpha \in S_i$ a subset of the set $S=\{1, 2,..., n\}$ and $\alpha \neq i$ (see [1]), and non-increasing in u_β, $\beta \in S-S_i$, $\beta \neq i$, we see that

$$\bar{f}_i(t, \bar{u}, \underline{u})=f_i(t, \bar{u}_\alpha, \underline{u}_\beta, \bar{u}_i),$$

$$\underline{f}_i(t, \bar{u}, \underline{u})=f_i(t, \underline{u}_\alpha, \bar{u}_\beta, \underline{u}_i).$$

In the case where each $f_i(t, u)$ is non-decreasing in all u_j, $j \neq i$ (see [2]), the differential inequalities for $\bar{u}$ and $\underline{u}$ are uncoupled and of the form $d\bar{u}/dt \geqq f(t, \bar{u})$, $d\underline{u}/dt \leqq f(t, \underline{u})$.

Note that $\bar{f}_i(t, \bar{u}, \underline{u})$ is monotone increasing as $\bar{u}$ increases and $\underline{u}$ decreases, and $\underline{f}_i(t, \bar{u}, \underline{u})$ is monotone decreasing in the same sense. In a way, the original equation of first-order in n-dependent variables has been imbedded in a $2n$ variable system with nice comparison properties due to the monotone character of the new functions.

For a linear system of n equations

$$\frac{du}{dt}=A(t)\,u$$

the matrix A can be written

$$A(t)=A^+(t)+A^-(t)+D(t),$$

where D is diagonal with $D_{ii}=A_{ii}$, $A^+_{ij}(t)=A_{ij}(t)$ if $i \neq j$ and $A_{ij}(t)>0$, and is zero otherwise.

A^- is defined in like fashion to accomodate the non-diagonal, negative elements of A. Then

$$\bar{f}(t, \bar{u}, \underline{u})=A^+\bar{u}+A^-\underline{u}+D\bar{u},$$

$$\underline{f}(t, \bar{u}, \underline{u})=A^+\underline{u}+A^-\bar{u}+D\underline{u}.$$

For example, the harmonic equation

$$\frac{d^2y}{dt^2}+a^2y=0$$

may be written in first-order form as

$$\frac{du}{dt}=\begin{pmatrix} 0 & 1 \\ -a^2 & 0 \end{pmatrix} u$$

and the equation for the bounds is

$$\frac{d}{dt}\begin{pmatrix}\bar{u}_1\\ \bar{u}_2\\ \underline{u}_1\\ \underline{u}_2\end{pmatrix}=\begin{pmatrix}0 & 1 & 0 & 0\\ 0 & 0 & -a^2 & 0\\ 0 & 0 & 0 & 1\\ -a^2 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\bar{u}_1\\ \bar{u}_2\\ \underline{u}_1\\ \underline{u}_2\end{pmatrix}.$$

It is interesting to see how the solution $u_1=\sin(at)$, $u_2=a\cos(at)$ of the harmonic equation is bounded by solutions of the bound equation like $\bar{u}_1=\sin(at)+\varepsilon e^{at}$, $\bar{u}_2=a\cos(at)+\varepsilon a e^{at}$, $\underline{u}_1=\sin(at)-\varepsilon e^{at}$, $\underline{u}_2=a\cos(at)-\varepsilon a e^{at}$.

4. Comparison Theorems for Elliptic and Parabolic Systems

In an earlier paper [1], comparison and existence theorems were studied for coupled systems of diffusion and reaction equations of the form

$$\begin{aligned}\frac{\partial u_i}{\partial t}-L^i(u_i)&=f_i(x,t,u_j,w_k), &\quad i&=1,2,\ldots,N_1\\ \frac{\partial w_i}{\partial t}&=g_i(x,t,u_j,w_k), &\quad i&=1,2,\ldots,N_2,\end{aligned}\tag{4.1}$$

where $x\in B$, a bounded domain in R^n, L^i is a linear elliptic operator of the form

$$L^i(u)\equiv\sum_{\alpha,\beta=1}^{n}a^i_{\alpha\beta}(x,t)\frac{\partial^2 u}{\partial x_\alpha\,\partial x_\beta}+\sum_{\alpha=1}^{n}b^i_\alpha(x,t)\frac{\partial u}{\partial x_\alpha}+c^i(x,t)\,u,$$

satisfying conditions for uniform ellipticity and Hölder continuity (exponent α) in $\bar{D}_T=\{(x,t): x\in\bar{B},\ 0\leqq t\leqq T\}$. The functions f_i, g_i were also required to be Hölder continuous and satisfy a Lipschitz condition, but in addition needed to satisfy a more special monotone property, that f_i be a non-decreasing function of u_j for all $j\neq i$ and all w_k and g_i behave similarly for all u_j and all w_k, $k\neq i$.

Here we wish to derive similar results for Eqs. (4.1) without the monotoneity requirement. This is done by imbedding this system (4.1) of order N_1+N_2 with no monotone restrictions on f_i and g_i in one of twice the order, but with monotone properties required for the proofs in [1].

Consider the systems

$$
\left.
\begin{aligned}
&\left.
\begin{aligned}
&\frac{\partial \underline{u}_i}{\partial t} - L^i(\underline{u}_i) = \underline{F}_i(x, t, \bar{u}_j, \underline{u}_j, \bar{w}_k, \underline{w}_k)\\
&\frac{\partial \bar{u}_i}{\partial t} - L^i(\bar{u}_i) = \bar{F}_i(x, t, \bar{u}_j, \underline{u}_j, \bar{w}_k, \underline{w}_k)
\end{aligned}
\right\} \quad i = 1, 2, \ldots, N_1,\\
&\left.
\begin{aligned}
&\frac{\partial \underline{w}_i}{\partial t} = \underline{G}_i(x, t, \bar{u}_j, \underline{u}_j, \bar{w}_k, \underline{w}_k)\\
&\frac{\partial \bar{w}_i}{\partial t} = \bar{G}_i(x, t, \bar{u}_j, \underline{u}_j, \bar{w}_k, \underline{w}_k)
\end{aligned}
\right\} \quad i = 1, 2, \ldots, N_2,
\end{aligned}
\right. \tag{4.2}
$$

where

$$\underline{F}_i(x, t, \bar{u}_j, \underline{u}_j, \bar{w}_k, \underline{w}_k) = \inf f_i(x, t, \theta, \phi), \qquad \text{where} \quad \theta_i = \underline{u}_i,$$

$$\bar{F}_i(x, t, \bar{u}_j, \underline{u}_j, \bar{w}_k, \underline{w}_k) = \sup f_i(x, t, \theta, \phi), \qquad \text{where} \quad \theta_i = \bar{u}_i,$$

and $\underline{u}_j \leqq \theta_j \leqq \bar{u}_j$ for all $j \neq i$, and $\underline{w}_k \leqq \phi_k \leqq \bar{w}_k$ for all k, and similarly

$$\underline{G}_i(x, t, \bar{u}_j, \underline{u}_j, \bar{w}_k, \underline{w}_k) = \inf g_i(x, t, \theta, \phi), \qquad \text{where} \quad \phi_i = \underline{w}_i,$$

$$\bar{G}_i(x, t, \bar{u}_j, \underline{u}_j, \bar{w}_k, \underline{w}_k) = \sup g_i(x, t, \theta, \phi), \qquad \text{where} \quad \phi_i = \bar{w}_i,$$

and $\underline{u}_j \leqq \theta_j \leqq \bar{u}_j$ for all j and $\underline{w}_k \leqq \phi_k \leqq \bar{w}_k$ for all $k \neq i$. This coupling is of the form considered earlier in Section 2, and allows weak and strong comparison theorems to be formulated as in that section.

If we slightly modify the system (4.2) by defining new variables $y_i = \bar{u}_i$, $y_{N_1+i} = -\underline{u}_i$, $z_i = \bar{w}_i$, and $z_{N_2+i} = -\underline{w}_i$, and if we set $F_i^* = \bar{F}_i$ and $F_{N_1+i}^* = -\underline{F}_i$, $G_i^* = \bar{G}_i$ and $G_{N_2+i}^* = -\underline{G}_i$, then we obtain a new system like (4.2) for which F_i^* are non-decreasing functions of y_j for all $j \neq i$ and all z_j and G_i^* are non-decreasing functions of y_j for all j and z_j for all $j \neq i$. It can be shown that these new functions have the Hölder and Lipschitz conditions that were imposed on the original functions f_i, g_i.

Any solution u, w of (4.1) defines a solution $y_i = u_i$, $y_{N_1+i} = -u_i$, $z_j = w_j$, $z_{N_2+j} = -w_j$ of the new system, and a solution of the new system satisfying $y_i + y_{N_1+i} = 0$, $z_j + z_{N_2+j} = 0$ for all $i = 1, 2, \ldots, N_1$, and $j = 1, 2, \ldots, N_2$ gives rise to a solution of Eqs. (4.1).

The comparison theorems and existence proof given in [1] for monotone functions thus provide comparison theorems and an existence proof for the general system (4.1) without monotone requirements.

STRONG COMPARISON THEOREM. *Suppose* (a) $\underline{u}$, u, $\bar{u}$, $\underline{w}$, w, $\bar{w}$ *are continuous in* $\bar{D}_T$;

(b) *their second-order x_i-derivatives and first-order t-derivatives exist, are uniformly bounded in D_T and satisfy the inequalities*

$$L^i(\underline{u}_i)-\underline{F}_i(x,t,\underline{u},\bar{u},\underline{w},\bar{w}) \leqq L^i(u_i)-f_i(x,t,u,w) \leqq L^i(\bar{u}_i)-\bar{F}_i(x,t,\underline{u},\bar{u},\underline{w},\bar{w})$$

$$\frac{\partial \underline{w}_i}{\partial t}-\underline{G}_i(x,t,\underline{u},\bar{u},\underline{w},\bar{w}) \leqq \frac{\partial w_i}{\partial t}-g_i(x,t,u,w) \leqq \frac{\partial \bar{w}_i}{\partial t}-\bar{G}_i(x,t,\underline{u},\bar{u},\underline{w},\bar{w})$$

for $i=1,2,\ldots,N_1$ *in* D_T;

(c) $\underline{u} \leqq u \leqq \bar{u}$, $\underline{w} \leqq w \leqq \bar{w}$, on B at $t=0$;

$\underline{u} \leqq u \leqq \bar{u}$, on $\{x, t\colon x \in \partial B, 0 < t \leqq T\}$.

Then $\underline{u} \leqq u \leqq \bar{u}$ *and* $\underline{w} \leqq w \leqq \bar{w}$ *in* D_T.

5. Linear Boundary Value Problems

The comparison theorems of Section 2 concern initial value problems, and the question arises, are there analogous upper and lower functions satisfying differential inequalities for problems defined by more general boundary conditions.

Bellman's "Invariant Imbedding" technique [2] connects two point linear boundary value problems to initial value problems. Since these equivalent initial value problems have associated equations for upper and lower bounds, we have a means to extend this study to cover more general boundary conditions. We will carry through the details using the setting outlined in [3].

Let S be the linear vector space of solutions u of the linear differential equation

$$\frac{du}{dt}=Au, \qquad t \in (c,d), \tag{5.1}$$

where u is a mapping from points of the segment (c,d) of the real line into the real vector space E_n, and A is a regulated map from points on (c,d) into $L(E_n)$, the space of bounded $n \times n$ matrices mapping E_n into E_n. Associated with this equation is a general linear boundary condition of the form

$$Bu=\xi, \quad \xi \in E_n, \tag{5.2}$$

where $B \in L(S, E_n)$, the set of bounded linear operators from S into E_n.

and is non-singular in the sense that it is a bijection onto E_n. B may be imbedded in a differentiable family of boundary conditions:

$$B(\tau)\,u=\xi, \qquad \tau\in[a,b], \tag{5.3}$$

such that

$$\frac{d}{d\tau}B(\tau)=\mu B_0(\tau), \qquad B(b)=B, \quad B(a)=\nu B_0(a), \tag{5.4}$$

where $\mu, \nu\in L(E_n)$ and $B_0(\tau)$ is the linear operator defined by $B_0(\tau)\,v=v(\tau)$ for all $v\in S$, the space of regulated maps from (c, d) into E_n.

The solution $u(t, \tau)$ of Eq. (5.1) with boundary condition (5.3) can be expressed in the form:

$$u(t,\tau)=\phi(t)\,U(t,\tau)\,\xi, \tag{5.5}$$

where the auxiliary function $\phi(t)$ satisfies the Riccati initial value problem

$$\frac{d\phi}{dt}=A\phi-\phi\mu\phi, \qquad \phi(a)=\nu^{-1}, \tag{5.6}$$

and the fundamental operator $U(t, \tau)$ satisfies the equations

$$\frac{\partial U}{\partial\tau}(t,\tau)=-U(t,\tau)\,\mu(\tau)\,\phi(\tau), \qquad U(t,t)=I, \tag{5.7}$$

and I is the identity or unit matrix in $L(E_n)$ (see [3]).

Let $\bar\phi$, $\underline{\phi}$, $\bar U$, $\underline{U}$, $\bar u$, $\underline{u}$ satisfy the inequalities

$$\begin{aligned}
\frac{d\bar\phi}{dt}&\geqq A^+\bar\phi+A^-\underline{\phi}+(A^0)^-(\bar\phi-\underline{\phi})-\bar\phi\mu\bar\phi+(\bar\phi-\underline{\phi})[\mu\bar\phi-(\mu\bar\phi)^0]^+\\
&\quad+[\bar\phi\mu-(\bar\phi\mu)^0]^+(\bar\phi-\underline{\phi})-(\bar\phi-\underline{\phi})\,\mu^-(\bar\phi-\underline{\phi})\\
\frac{d\underline{\phi}}{dt}&\leqq A^+\underline{\phi}+A^-\bar\phi-(A^0)^-(\bar\phi-\underline{\phi})-\bar\phi\mu\bar\phi\\
&\quad+(\bar\phi-\underline{\phi})\{(\mu\bar\phi)^0+[\mu\bar\phi-(\mu\bar\phi)^0]^-\}\\
&\quad+\{(\bar\phi\mu)^0+[\bar\phi\mu-(\bar\phi\mu)^0]^-\}(\bar\phi-\underline{\phi})-(\bar\phi-\underline{\phi})\,\mu^+(\bar\phi-\underline{\phi}) \qquad (5.8)\\
\bar\phi(a)&>\nu^{-1}, \qquad \underline{\phi}(a)<\nu^{-1};\\
\frac{\partial\bar U}{\partial\tau}&\geqq-\bar U\mu\bar\phi+(\bar U-\underline{U})[\mu\bar\phi-(\mu\bar\phi)^0]^+\\
&\quad+[\bar U\mu-(\bar U\mu)^0]^+(\bar\phi-\underline{\phi})-(\bar U-\underline{U})\,\mu^-(\bar\phi-\underline{\phi})
\end{aligned}$$

$$\frac{\partial \underline{U}}{\partial \tau} \leqq -\bar{U}\mu\bar{\phi} + (\bar{U}-\underline{U})\{(\mu\bar{\phi})^0 + [(\mu\phi)-(\mu\phi)^0]^-\}$$
$$+\{(\bar{U}\mu)^0 + [\bar{U}\mu - (\bar{U}\mu)^0]^-\}(\bar{\phi}-\underline{\phi}) - (\bar{U}-\underline{U})\,\mu^+(\bar{\phi}-\underline{\phi}) \quad (5.9)$$
$$\bar{U}(t,t) > I > \underline{U}(t,t);$$
$$\bar{U} \geqq \bar{\phi}\bar{U}\xi - (\bar{\phi}-\underline{\phi})(\bar{U}\xi)^- + (\bar{\phi}-\underline{\phi})(\bar{U}-\underline{U})\,\xi^+$$
$$-\bar{\phi}^+(\bar{U}-\underline{U})\,\xi^- - \bar{\phi}^-(\bar{U}-\underline{U})\,\xi^+$$
$$\underline{U} \leqq \bar{\phi}\bar{U}\xi - (\bar{\phi}-\underline{\phi})(\bar{U}\xi)^+ + (\bar{\phi}-\underline{\phi})(\bar{U}-\underline{U})\,\xi^-$$
$$-\bar{\phi}^+(\bar{U}-\underline{U})\,\xi^+ - \bar{\phi}^-(\bar{U}-\underline{\phi})\,\xi\ . \quad (5.10)$$

If $M(t)$ is an $n\times n$ matrix of the set $L(E_n)$, M^0 is the diagonal matrix with $M^0_{ii} = M_{ii}$, M^+ is the projection of M defined by $M^+_{ij} = M_{ij}$ if $M_{ij} > 0$ and $M^+_{ij} = 0$ otherwise. M^- is the analogous projection onto the negative components, so that $M = M^+ + M^-$. For $\xi \in E_n$, ξ^+ is derived from ξ, so that $\xi^+_i = \xi_i$ if $\xi_i > 0$ and $\xi^+_i = 0$ otherwise, and $\xi = \xi^+ + \xi^-$.

We first show that the strong comparison theorem of Section 2 implies $\bar{\phi} > \phi > \underline{\phi}$ component-wise. Note that from Eq. (5.6)

$$\frac{d\phi}{dt} = A\bar{\phi} - A(\bar{\phi}-\phi) - \bar{\phi}\mu\bar{\phi} + (\bar{\phi}-\phi)\,\mu\bar{\phi} + \bar{\phi}\mu(\bar{\phi}-\phi) - (\bar{\phi}-\phi)\,\mu(\bar{\phi}-\phi)$$
$$\equiv f(t,\phi) \quad \text{say,}$$

and

$$A^+\bar{\phi} + A^-\underline{\phi} + (A^0)^-(\bar{\phi}-\underline{\phi}) - \bar{\phi}\mu\bar{\phi} + (\bar{\phi}-\underline{\phi})[\mu\bar{\phi} - (\mu\bar{\phi})^0]^+$$
$$+[\bar{\phi}\mu - (\bar{\phi}\mu)^0]^+(\bar{\phi}-\underline{\phi}) - (\bar{\phi}-\underline{\phi})\,\mu^-(\bar{\phi}-\underline{\phi}) \geqq \sup f(t,\theta),$$

where $\bar{\phi}_{ij} \geqq \theta_{ij} \geqq \underline{\phi}_{ij}$ for $i \neq j$. A similar observation concerning $\inf f(t,\theta)$ shows $\bar{\phi}$ and $\underline{\phi}$ satisfy the requirements of the strong comparison theorem. In the same way, Eqs. (5.9) imply $\bar{U} \geqq U \geqq \underline{U}$. Since these equations are linear, it is possible to find such functions $\bar{U}$ and $\underline{U}$ if bounded functions $\bar{\phi}$ and $\underline{\phi}$ exist. We finally note that

$$u = \phi U\xi = \bar{\phi}\bar{U}\xi - (\bar{\phi}-\phi)\,\bar{U}\xi + (\bar{\phi}-\phi)(\bar{U}-U)\,\xi - \bar{\phi}(\bar{U}-U)\,\xi,$$

and it follows from the inequalities (5.10) that $\bar{u} \geqq u \geqq \underline{u}$ on $[a, b]$. As a limiting case, $\bar{\phi} = \phi = \underline{\phi}$, $\bar{U} = U = \underline{U}$ satisfy the Eqs. (5.8) and (5.9).

The comparison theorem may be sharpened in many ways and extended in various directions. For example, in the strong comparison theorem, the inequality $<$ in (2.3), (1) and (3) may be replaced by $\leqq$ and the setting in most instances could have been a Banach space with suitably defined projections ξ^+, ξ^-, etc., but in keeping with the Bellman spirit it is perhaps better to establish the concepts with clarity, and leave something for others.—"A tour de force is a cul-de-sac."—R. Bellman.

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