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On the number of P-vertices of some graphs

Milica Anđelić [a,1], C.M. da Fonseca [b,*,2], Ricardo Mamede [b,2]

[a] Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal
[b] Department of Mathematics, University of Coimbra, 3001-454 Coimbra, Portugal

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We provide positive answers to some open questions presented recently by Kim and Shader on a continuity-like property of the P-vertices of nonsingular matrices whose graph is a path. A criterion for matrices associated with more general trees to have at most $n-1$ P-vertices is established. The cases of the cycles and stars are also analyzed. Several algorithms for generating matrices with a given number of P-vertices are proposed.

1. Introduction

For a given $n \times n$ real symmetric matrix $A = (a_{ij})$, we define the graph of A, and write $G(A)$, as the (simple) graph whose vertex set is $\{1, \ldots, n\}$ and edge set is $\{ij | i \neq j \text{ and } a_{ij} \neq 0\}$. We confine our attention to the set

* Corresponding author.
E-mail addresses: milica.andelic@ua.pt (M. Anđelić), cmf@mat.uc.pt (C.M. da Fonseca), mamede@mat.uc.pt (R. Mamede).

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$$\mathcal{S}(G) = \{A \in \mathbb{R}^{n\times n} | A \text{ is symmetric and } G(A) = G\},$$

i.e., the set of all symmetric matrices sharing a common graph G on n vertices. Nevertheless, all results can easily be extended to Hermitian matrices. We will often omit the mention of G if it is clear from context.

If G is a tree, a matrix $A \in \mathcal{S}(G)$ is called acyclic. In particular, if G is a path, we order the vertices of G such that $A \in \mathcal{S}(G)$ is a tridiagonal matrix.

Let us denote the (algebraic) multiplicity of the eigenvalue θ of a symmetric matrix $A(= A(G))$ by $m_A(\theta)$. By $A(i)$ we mean the $(n-1)\times(n-1)$ principal submatrix, formed by the deletion of row and column indexed by i, which is equivalent to removal of the vertex i from G, i.e., $A(G\backslash i)$. More generally, if S is a subset of the vertex set of G, then $A(S)$ is the submatrix of A resulting on deleting the rows and columns indexed by S. By $A[S]$ we mean the principal submatrix of A whose rows and columns are indexed by S.

As a consequence of Cauchy's Interlacing Theorem for the eigenvalues of symmetric matrices, one can deduce that

$$m_{A(G)}(\theta) - 1 \leqslant m_{A(G\backslash i)}(\theta) \leqslant m_{A(G)}(\theta) + 1. \tag{1.1}$$

In the case of $m_{A(G\backslash i)}(\theta) = m_{A(G)}(\theta) + 1$, the vertex i was designated by Jonhson et al. as a *Parter-vertex* of A for θ, and intensively studied in [16–18], just to cite a few, motivated by the striking research due to Parter [21], complemented and extended by Wiener in [22], on the location and multiplicity of eigenvalues of sign symmetric acyclic matrices. Note that this concept has also been considered by Godsil in the context of the matchings polynomial theory as a θ-positive vertex of G [7,13]. When $\theta = 0$, a Parter-vertex is simply called a P-vertex of A [19], and $P_\nu(A)$ denotes the number of P-vertices of A. The multiplicity of zero eigenvalue is an important issue in many areas of pure and applied matrix theory. For example, in [13] Godsil observed that the multiplicity of 0 as a root of the matchings polynomial of a graph coincides with the classical notion of deficiency. We recall that the matchings polynomial of a tree is equal to the characteristic polynomial of its 01-adjacency matrix.

In 2004, Johnson and Sutton showed in [18] that each singular acyclic matrix of order n has at most $n-2$ P-vertices. Recently, Kim and Shader proved in [20] that this does not hold for nonsingular acyclic matrices by constructing some examples for paths and stars.

Theorem 1.1 [20]**.** *Let T be a path on n vertices.*

(a) *If n is even, then there exists a nonsingular matrix $A \in \mathcal{S}(T)$ such that $P_\nu(A) = n$.*
(b) *If n is odd, then there exists a nonsingular matrix $A \in \mathcal{S}(T)$ such that $P_\nu(A) = n-1$.*

Moreover, when n is odd, they proved that $P_\nu(A) \leqslant n-1$ for any nonsingular matrix A in $\mathcal{S}(T)$. When n is even, obviously $P_\nu(A) \leqslant n$. These observations led Kim and Shader to present two open questions on the “continuity" of $P_\nu(A)$, when A runs over all (tridiagonal) matrices of $\mathcal{S}(T)$, with T a path.

Question 1 [20]**.** Let T be a path with an even number n of vertices. Is the equality

$$\{P_\nu(A) | A \text{ is a nonsingular matrix in } \mathcal{S}(T)\} = \{0, 1, \ldots, n\}$$

true?

Question 2 [20]**.** Let T be a path with an odd number n of vertices. Is the equality

$$\{P_\nu(A) | A \text{ is a nonsingular matrix in } \mathcal{S}(T)\} = \{0, 1, \ldots, n-1\}$$

true?

We construct tridiagonal matrices in order to give positive answers to both open questions. Before that, we provide an overview of some results of 2-Toeplitz matrices and, in Sections 3 and 4, we consider the maximum number of P-vertices for matrices when the underlying graph is a cycle. We also analyze some particular cases of trees, such as stars, 2-generalized starts, and brooms. Some criteria are established answering partially to other remained open questions. We point out that other families of matrices may satisfy the continuity property of the number of P-vertices proved here.

2. Symmetric tridiagonal 2-Toeplitz matrices

A symmetric tridiagonal matrix of the form

$$B = \begin{pmatrix} a_1 & b_1 & & \\ b_1 & \ddots & \ddots & \\ & \ddots & \ddots & b_{n-1} \\ & & b_{n-1} & a_n \end{pmatrix}, \tag{2.1}$$

where all the non-mentioned entries are null, is called a 2-Toeplitz matrix if

$$a_{i+2} = a_i \quad \text{and} \quad b_{i+2} = b_i, \text{ for } i = 1, \ldots, n-2,$$

i.e.,

$$B_n^{(2)} = \begin{pmatrix} a_1 & b_1 & & & \\ b_1 & a_2 & b_2 & & \\ & b_2 & a_1 & b_1 & \\ & & b_1 & \ddots & \ddots \\ & & & \ddots & \end{pmatrix}_{n\times n}. \tag{2.2}$$

These particular kind of Jacobi matrices, and the more general k-Toeplitz matrices have been thoroughly studied by da Fonseca and Petronilho in [9,10].

Recall that the sequence of Chebyshev polynomials of second kind, $\{U_k(x)\}_{k\geqslant 0}$, satisfies the three-term recurrence relation

$$2xU_k(x) = U_{k+1}(x) + U_{k-1}(x),$$

for all $k \geqslant 1$, with initial conditions $U_0(x) = 1$ and $U_1(x) = 2x$ (e.g., [5]). Each $U_k(x)$ is of degree k and satisfies

$$U_k(x) = \frac{\sin(k+1)\theta}{\sin\theta}, \quad \text{with } x = \cos\theta \quad (0 \leqslant \theta < \pi),$$

and, therefore, the zeros of $U_k(x)$ are

$$\lambda_\ell = \cos\left(\frac{\ell\pi}{k+1}\right), \quad \text{for } \ell = 1, \ldots, k.$$

Setting

$$p_2(x) = (x-a_1)(x-a_2),$$
$$P_k^*(x) = (b_1b_2)^k\, U_k\left(\frac{x-b_1^2-b_2^2}{2\,b_1b_2}\right),$$

and

$$P_k(x) = (b_1b_2)^k\left[U_k\left(\frac{x-b_1^2-b_2^2}{2\,b_1b_2}\right) + \beta U_{k-1}\left(\frac{x-b_1^2-b_2^2}{2\,b_1b_2}\right)\right],$$

with $\beta = b_2/b_1$, we define a new sequence of (orthogonal) polynomials, $\{Q_k(x)\}_{k\geqslant 0}$, such that

$$Q_{2k+1}(x) = (x-a_1)P_k^*(p_2(x))$$

and

$$Q_{2k}(x) = P_k(p_2(x)).$$

Each $Q_k(x)$ is of degree k and, according to [9,10],

$$\det B_n^{(2)} = (-1)^n Q_n(x). \tag{2.3}$$

Observe that, if n is odd and $a_1 = 0$, then $B_n^{(2)}$ is singular.

3. P-vertices of an even cycle

In this section, for each cycle C_n of even order n, we construct a nonsingular matrix in $\mathcal{S}(C_n)$ such that the number of P-vertices is equal to n. Consider a matrix C in $\mathcal{S}(C_n)$ of the form

$$C = \begin{pmatrix} a_1 & b_1 & & & b_n \\ b_1 & a_2 & b_2 & & \\ & b_2 & \ddots & \ddots & \\ & & \ddots & \ddots & b_{n-1} \\ b_n & & & b_{n-1} & a_n \end{pmatrix}. \tag{3.1}$$

A matrix of the form (3.1) is often called a (symmetric) *periodic Jacobi matrix* when each b_i is positive [1].

Let us assume that

$$b_1 = \cdots = b_n = 1, a_{2\ell-1} = 1 \quad \text{and} \quad a_{2\ell} = a, \;\; \text{for } \ell = 1, 2, \ldots, \lfloor n/2 \rfloor.$$

If i is odd, then

$$\det C(i) = \det \begin{pmatrix} a & 1 & & & & & \\ 1 & 1 & 1 & & & & \\ & 1 & a & 1 & & & \\ & & 1 & \ddots & \ddots & & \\ & & & \ddots & \ddots & 1 & \\ & & & & 1 & 1 & 1 \\ & & & & & 1 & a \end{pmatrix} = a\, U_{(n-2)/2}\left(\frac{a-2}{2}\right).$$

Otherwise

$$\det C(i) = \det \begin{pmatrix} 1 & 1 & & & & & \\ 1 & a & 1 & & & & \\ & 1 & 1 & 1 & & & \\ & & 1 & \ddots & \ddots & & \\ & & & \ddots & \ddots & 1 & \\ & & & & 1 & a & 1 \\ & & & & & 1 & 1 \end{pmatrix} = U_{(n-2)/2}\left(\frac{a-2}{2}\right).$$

Therefore, $\det C(i) = 0$, for any $i \in \{1, \ldots, n\}$, if and only if

$$a = 2 + 2\cos\left(\frac{2\ell\pi}{n}\right), \quad \text{for } \ell = 1, \ldots, \frac{n-2}{2}. \tag{3.2}$$

On the other hand, for C defined in (3.1), we have [4]

$$\det C = (a-2)U_{(n-2)/2}\left(\frac{a-2}{2}\right) - 2U_{(n-4)/2}\left(\frac{a-2}{2}\right) - 2. \tag{3.3}$$

Hence, we can always find an a such that $\det C(i) = 0$, for any $i \in \{1, \ldots, n\}$, and $\det C \neq 0$. Note that, if a is chosen as in (3.2), then (3.3) becomes

$$\det C = -2U_{(n-4)/2}\left(\frac{a-2}{2}\right) - 2.$$

Example 3.1. Suppose that $n = 12$ and choose $a = 3$, i.e., $\ell = 1$. Then $\det C = -4 \neq 0$ and $\det C(i) = 0$, for any $i \in \{1, \ldots, 12\}$.

We remark that this approach cannot be used for cycles of odd order. For example, for $n = 3$, C is always singular and, for $n = 5$, $C(2)$ is nonsingular.

4. P-vertices of an odd cycle

For the case of an odd cycle, our approach is different. We begin computing the determinants of a particular family of even order tridiagonal matrices. Since we are using elementary row and column operations, we leave the details for the reader.

Let us consider the tridiagonal matrix of order $2k$, for $k \geqslant 2$, block partitioned as

$$\left(\begin{array}{cccc|ccc|cccc}
0 & 1 & & & & & & & & & \\
1 & \ddots & \ddots & & & & & & & & \\
 & \ddots & \ddots & 1 & & & & & & & \\
 & & 1 & 0 & 1 & & & & & & \\
\cline{1-7}
 & & & 1 & a & 1 & & & & & \\
 & & & & 1 & a & 1 & & & & \\
 & & & & & 1 & a & 1 & & & \\
\cline{5-11}
 & & & & & & 1 & 0 & 1 & & \\
 & & & & & & & 1 & \ddots & \ddots & \\
 & & & & & & & & \ddots & \ddots & 1 \\
 & & & & & & & & & 1 & 0
\end{array}\right) \tag{4.1}$$

or as, with all blocks of even order,

$$\left(\begin{array}{cccc|cc|cccc}
0 & 1 & & & & & & & & \\
1 & \ddots & \ddots & & & & & & & \\
 & \ddots & \ddots & 1 & & & & & & \\
 & & 1 & 0 & 1 & & & & & \\
\cline{1-6}
 & & & 1 & a & 1 & & & & \\
 & & & & 1 & a & 1 & & & \\
\cline{5-10}
 & & & & & 1 & 0 & 1 & & \\
 & & & & & & 1 & \ddots & \ddots & \\
 & & & & & & & \ddots & \ddots & 1 \\
 & & & & & & & & 1 & 0
\end{array}\right). \tag{4.2}$$

We assume, with some conventions, that the sum of the orders of the blocks gives $2k$.

Lemma 4.1. *The determinant of each tridiagonal matrix of the form (4.1) or (4.2) is $(-)^k(1-a^2)$.*

Remark 4.1. Lemma 4.1 is still true for the tridiagonal matrix of order $2k$ of the form

$$\begin{pmatrix} a & 1 & & & & \\ 1 & 0 & 1 & & & \\ & 1 & \ddots & \ddots & & \\ & & \ddots & \ddots & 1 & \\ & & & 1 & 0 & 1 \\ & & & & 1 & a \end{pmatrix}. \tag{4.3}$$

Taking into account the expression for the determinant of a matrix whose graph is a cycle (cf., e.g. [4]), and Lemma 4.1, the determinant of the matrix of order n (odd)

$$C^{(a)} = \begin{pmatrix} a & 1 & & & & & 1 \\ 1 & a & 1 & & & & \\ & 1 & a & 1 & & & \\ & & 1 & 0 & \ddots & & \\ & & & \ddots & \ddots & 1 & \\ & & & & 1 & 0 & 1 \\ 1 & & & & & 1 & 0 \end{pmatrix}, \tag{4.4}$$

is

$$\det C^{(a)} = \begin{cases} 2 + 3a - a^3, & \text{if } n = 4\ell + 1 \\ 2 - 3a + a^3, & \text{if } n = 4\ell - 1 \end{cases}.$$

Again from Lemma 4.1, if $n = 4\ell + 1$, then $\det C^{(1)} \neq 0$ and $P_\nu(C^{(1)}) = n$; else, $n = 4\ell - 1$, $\det C^{(-1)} \neq 0$ and $P_\nu(C^{(-1)}) = n$.

Theorem 4.2. *Given a cycle C_n of order n, there exist a nonsingular matrix in $\mathcal{S}(C_n)$ such that the number of P-vertices is n.*

5. The continuity of the P-vertices of a path

In this section, we return to the P-vertices of a path, establishing algorithms to answer Questions 1 and 2 both affirmatively. First, we recall some notation and facts about sign patterns.

A matrix whose entries are $+$, $-$, or 0 is called a sign pattern. For each $n \times n$ sign symmetric pattern A there is a natural class of real symmetric matrices whose entries have the signs indicated by A, i.e.,

$$Q_{SYM}(A) = \{B \in \mathbb{R}^{n \times n} | B \text{ is symmetric and sign } B = A\}.$$

Recall that we define the inertia of an $n \times n$ real symmetric matrix H as the triple $\mathrm{In}(H) = (n_+, n_-, n_0)$, where n_+ is the number of positive eigenvalues, n_- is the number of negative eigenvalues and $n_0 (= n - n_+ - n_-)$is the number of the zero eigenvalues. For a symmetric sign pattern A, we define the inertia (set) of A to be $\mathrm{In}(A) = \{\mathrm{In}(B) | B \in Q_{SYM}(A)\}$. We say the sign pattern A requires unique inertia and is sign nonsingular if every real matrix in $Q(A)$ has the same inertia and is nonsingular, respectively.

Using some techniques on congruences between Hermitian matrices explored in [2,3], da Fonseca presented in [6] a concise study of the inertia sets of tridiagonal sign patterns, latter extended to some other matrices, by da Fonseca and Mamede in [8], bringing together the known results until then (e.g. [11,12,14]).

By the same time, Kim and Shader [20] studied independently the singularity of a irreducible tridiagonal matrix (2.1) with a certain pattern of the zero main diagonal entries, namely $a_{2\ell-1} = 0$, for $\ell = 1, 2, \ldots, \lfloor n/2 \rfloor$. Note that such a matrix can be seen as belonging to the symmetric tridiagonal sign pattern class of

$$A=\begin{pmatrix} 0 & \pm & & & & \\ \pm & * & \pm & & & \\ & \pm & 0 & \pm & & \\ & & \pm & * & \pm & \\ & & & \pm & 0 & \ddots \\ & & & & \ddots & \ddots \end{pmatrix}_{n\times n} \tag{5.1}$$

where each $*$ assumes the value $+$, $-$, or 0. From [6, Proposition 2.1] A is sign singular if and only if n is odd, and in this case $\mathrm{In}(A)=\left(\frac{n-1}{2},\frac{n-1}{2},1\right)$; otherwise, $\mathrm{In}(A)=\left(\frac{n}{2},\frac{n}{2},0\right)$. This fact implies immediately [20, Proposition 1].

In order to simplify the notation, for $b_1=\cdots=b_{n-1}=1$, we identify the tridiagonal matrix B defined in (2.1) with its main diagonal $(a_1,\ldots,a_n)$.

Suppose that n is even. The 2-Toeplitz matrix (2.2) defined by

$$A_n^{(n)}=(1,-1,\ldots,1,-1)$$

has inertia $\left(\frac{n}{2},\frac{n}{2},0\right)$. Note also that, from (2.3) [10]

$$\det A_n^{(n)}=U_{\frac{n}{2}}\left(-\frac{3}{2}\right)+U_{\frac{n-2}{2}}\left(-\frac{3}{2}\right).$$

Remark 5.1. From [6, Theorem 2.3(a)], if one replaces any subset of diagonal entries by zeros, then the inertia remains unchanged, and is always equal to $\mathrm{In}(A_n^{(n)})=\left(\frac{n}{2},\frac{n}{2},0\right)$. Therefore, such tridiagonal matrices are always nonsingular.

Since the inertia of each principal submatrix is either $\left(\frac{n}{2},\frac{n-2}{2},0\right)$ or $\left(\frac{n-2}{2},\frac{n}{2},0\right)$, we have $P_\nu(A_n^{(n)})=0$. Setting $A_{n-1}^{(n)}=(0,-1,\ldots,1,-1)$, the only principal matrix with determinant equal to zero is $A_{n-1}^{(n)}(2)$. Therefore, $P_\nu(A_{n-1}^{(n)})=1$. Next we set, $A_{n-2}^{(n)}=(0,-1,\ldots,1,0)$, and the null principal minors are $A_{n-2}^{(n)}(2)$ and $A_{n-2}^{(n)}(n-1)$, and thus $P_\nu(A_{n-2}^{(n)})=2$. We proceed now repeating the two previous steps to $A_{n-2}^{(n)}$ replacing first the 3-entry (which is 1) by zero, and then replacing the $(n-2)$-entry (which is -1) on the resultant matrix by zero.

If $n/2$ is odd, on the step $n/2+2$, we get the nonsingular tridiagonal matrix

$$A_{n/2-1}^{(n)}=(\underbrace{0,-1,0,\ldots,-1,0}_{n/2\text{ entries}},\underbrace{0,1,0,\ldots,1,0}_{n/2\text{ entries}}).$$

Note that

$$P_\nu\left(A_{n/2-1}^{(n)}\right)=\frac{n}{2}+1.$$

Otherwise, $n/2$ is even and, on the step, $n/2+1$ we get the nonsingular tridiagonal matrix

$$A_{n/2}^{(n)}=(\underbrace{0,-1,0,\ldots,-1}_{n/2\text{ entries}},\underbrace{1,0,\ldots,1,0}_{n/2\text{ entries}}).$$

In both cases, the algorithm proceeds replacing alternately -1 and 1 by zeros as before, but now from "inside" to "outside", obtaining in the end (the $(n+1)$th step), the matrix

$$A_0^{(n)}=(0,\ldots,0),$$

where, obviously, $P_\nu(A_0^{(n)})=n$.

In general,

$$P_\nu\left(A_k^{(n)}\right) = n - k \quad \text{for } k = n, n-1, \ldots, 1, 0,$$

providing the following theorem.

Theorem 5.1. *Let T be a path with an even number n of vertices. Then*

$$\{P_\nu(A) | A \text{ is a nonsingular matrix in } \mathcal{S}(T)\} = \{0, 1, \ldots, n\}.$$

Example 5.1. For $n = 8$ we have:

k	$A_k^{(8)}$	$P_\nu\left(A_k^{(8)}\right)$
8	$(1, -1, 1, -1, 1, -1, 1, -1)$	0
7	$(0, -1, 1, -1, 1, -1, 1, -1)$	1
6	$(0, -1, 1, -1, 1, -1, 1, 0)$	2
5	$(0, -1, 0, -1, 1, -1, 1, 0)$	3
4	$(0, -1, 0, -1, 1, 0, 1, 0)$	4
3	$(0, -1, 0, 0, 1, 0, 1, 0)$	5
2	$(0, -1, 0, 0, 0, 0, 1, 0)$	6
1	$(0, 0, 0, 0, 0, 0, 1, 0)$	7
0	$(0, 0, 0, 0, 0, 0, 0, 0)$	8

Example 5.2. For $n = 10$ we have:

k	$A_k^{(10)}$	$P_\nu\left(A_k^{(10)}\right)$
10	$(1, -1, 1, -1, 1, -1, 1, -1, 1 - 1)$	0
9	$(0, -1, 1, -1, 1, -1, 1, -1, 1 - 1)$	1
8	$(0, -1, 1, -1, 1, -1, 1, -1, 1, 0)$	2
7	$(0, -1, 0, -1, 1, -1, 1, -1, 1, 0)$	3
6	$(0, -1, 0, -1, 1, -1, 1, 0, 1, 0)$	4
5	$(0, -1, 0, -1, 0, -1, 1, 0, 1, 0)$	5
4	$(0, -1, 0, -1, 0, 0, 1, 0, 1, 0)$	6
3	$(0, -1, 0, 0, 0, 0, 1, 0, 1, 0)$	7
2	$(0, -1, 0, 0, 0, 0, 0, 0, 1, 0)$	8
1	$(0, 0, 0, 0, 0, 0, 0, 0, 1, 0)$	9
0	$(0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$	10

The case when n is odd is divided in two parts. For $k = 1, \ldots, (n-1)/2$, we define the tridiagonal matrix

$$B_k^{(n)} = (\underbrace{0, \ldots, 0, 1, 1}_{2k \text{ entries}}, 0, \ldots, 0),$$

and observe that $\det B_k^{(n)}(2\ell - 1) \neq 0$, for $\ell = 1, \ldots, k$, and the remaining minors are zero.

For the second part of our algorithm, we define the tridiagonal matrix

$$B_k^{(n)} = (\underbrace{1, 0, \ldots, 0, 1}_{2k-n \text{ entries}}, 0, \ldots, 0),$$

for $k = (n+1)/2, \ldots, n$. Now, $\det B_k^{(n)}(n + 1 - 2\ell) = 0$, for $\ell = 1, \ldots, n-k$, and the remaining minors are nonzero.

Therefore

$$P_\nu(B_k^{(n)}) = n - k, \quad \text{for } k = 1, 2, \ldots, n.$$

Since $\mathrm{In}(B_k^{(n)}) = \left(\frac{n+1}{2}, \frac{n-1}{2}, 0\right)$, for any $k \in \{1, \ldots, n\}$, according [6, Theorem 2.3(b)], we can answer to Question 2.

Theorem 5.2. *Let T be a path with an odd number n of vertices. Then*

$$\{P_\nu(A) | A \text{ is a nonsingular matrix in } \mathcal{S}(T)\} = \{0, 1, \ldots, n-1\}.$$

Example 5.3. For $n = 7$ we have:

k	$B_k^{(7)}$	$P_\nu(B_k^{(7)})$
1	(1, 1, 0, 0, 0, 0, 0)	6
2	(0, 0, 1, 1, 0, 0, 0)	5
3	(0, 0, 0, 0, 1, 1, 0)	4
4	(1, 0, 0, 0, 0, 0, 0)	3
5	(1, 0, 1, 0, 0, 0, 0)	2
6	(1, 0, 0, 0, 1, 0, 0)	1
7	(1, 0, 0, 0, 0, 0, 1)	0

6. The case of the stars

In [20], the case when the tree is a star was also added to Question 2. Here we will answer affirmatively to this question, producing a new algorithm. If S_n is a star on n vertices, we assume that each matrix in $\mathcal{S}(S_n)$ is of the form

$$D = \begin{pmatrix} a_1 & b_1 & b_2 & \cdots & b_{n-1} \\ b_1 & a_2 & & & \\ b_2 & & a_3 & & \\ \vdots & & & \ddots & \\ b_{n-1} & & & & a_n \end{pmatrix}. \tag{6.1}$$

In our context, we may assume that all b_i's are positive, and say that A belongs the the sign class of

$$\begin{pmatrix} * & + & + & \cdots & + \\ + & * & & & \\ + & & * & & \\ \vdots & & & \ddots & \\ + & & & & * \end{pmatrix}_{n\times n},$$

where each diagonal entry is 0, + or −. The inertia sets of such sign class were analyzed in [6,8,14].

For each $k \in \{1, \ldots, n\}$, define

$$a_\ell = b_{\ell-1} = 1, \quad \text{for } \ell = k+1, \ldots, n,$$

$$a_\ell = b_{\ell-1} = 2, \quad \text{for } \ell = 2, \ldots, k,$$

and

$$a_1 = a_2 + \cdots + a_n - 1,$$

getting in (6.1) the matrix $S_k^{(n)}$. Now, we only have to point out that, for $\ell = k+1, \ldots, n$,

$$\det S_k^{(n)}(\ell) = 0,$$

and is nonzero, otherwise, to conclude that

$$P_\nu\left(S_k^{(n)}\right) = n - k.$$

Since $\mathrm{In}(S_k^{(n)}) = (n-1, 1, 0)$, for any $k \in \{1, \ldots, n\}$, we have the following.

Theorem 6.1. *Let T be a star with n vertices. Then, for each $k \in \{0, 1, \ldots, n-1\}$, there exists a nonsingular matrix A in $\mathcal{S}(T)$, such that, $P_\nu(A) = k$.*

Remark 6.1. Note that $\det S_k^{(n)} = -2^{k-1}$, for $k = 1, \ldots, n$.

Evaluating the inertia of D in (6.1) as in [6, Theorem 2.3], if we do not consider the first diagonal entry, when there exists exactly one diagonal entry zero, we find that there are no zero eigenvalues [20, Proposition 6(a)].

7. P-vertices of a tree

In this section, we establish criteria for the existence of nonsingular acyclic matrices, such that the number of P-vertices is equal to the order of the underlying tree.

Recall that a vertex of a graph is said to be *pendant* if its neighborhood contains exactly one vertex and a *center* of a tree is a vertex with degree greater than or equal to 3. A path connected to a vertex to some center is called a *pendant-path*.

Theorem 7.1. *Let T be a tree on n vertices, with more than one pendant vertex adjacent to some center, and let $A \in \mathcal{S}(T)$. If $P_\nu(A) = n$, then A is singular.*

Proof. Suppose, without loss of generality, that $S = \{1, \ldots, k\}$ is the set of pendant vertices of T adjacent to the center $k+1$. Since $P_\nu(A) = n$, we have $\det A(\ell) = 0$, for $\ell = 1, \ldots, k+1$. Suppose now that A is nonsingular. Then

$$0 \neq \det A = -a_{1,k+1}^2 a_{2,2} \cdots a_{k,k} \det A(1, \ldots, k+1).$$

Then, since

$$0 = \det A(k+1) = a_{11} a_{2,2} \cdots a_{k,k} \det A(1, \ldots, k+1),$$

we have $a_{11} = 0$. But

$$0 \neq \det A = -a_{2,k+1}^2 a_{1,1} a_{3,3} \cdots a_{k,k} \det A(1, \ldots, k+1),$$

which is impossible. Hence, A is singular. $\square$

Analogously, we may prove the following corollary.

Corollary 7.2. *Let B be a nonsingular tridiagonal matrix of the form (2.1), with $n > 1$. If $\det B(1) = \det B(2) = 0$ (resp., $\det B(n) = \det B(n-1) = 0$), then $a_1 = 0$ (resp., $a_n = 0$).*

Observe that in the previous corollary, B cannot be positive (or negative) semidefinite.

Corollary 7.3 [20]**.** *Let B be a nonsingular matrix of the form (2.1), with n even. Then $P_\nu(B) = n$ if and only if the main diagonal of B is zero.*

Corollary 7.4 [20]**.** *Let B be a nonsingular matrix of the form (2.1), with n odd. Then $P_\nu(B) \leqslant n-1$.*

We observe that the two matrices constructed by Kim and Shader [20], where both limits are attained, are different from the examples presented here.

The next corollary [20, Proposition 6(c)] is immediate from Theorem 7.1.

Corollary 7.5. *Let T be a star on $n \geqslant 3$ vertices. For any nonsingular matrix $A \in \mathcal{S}(T)$, $P_\nu(A) < n$.*

Recall that a *broom* is a tree consisting of a star and a path attached to an arbitrary pendant vertex of the star. Brooms have particulary minimal properties. For example, a broom is the unique tree which minimizes even spectral moments and the Estrada index [15], and has minimal energy among the trees with fixed diameter or fixed number of pendant vertices [23,24].

Corollary 7.6. *Let T be a broom on $n \geqslant 4$ vertices. For any nonsingular matrix $A \in \mathcal{S}(T)$, $P_\nu(A) < n$.*

Theorem 7.7. *Let T' be a pendant-path on a tree T. Suppose that $\{1, \ldots, k\}$ is the set of vertices of T'. Then if $A = (a_{ij})$ is a nonsingular matrix in $\mathcal{S}(T)$ such that $P_\nu(A) = n$, then $a_{2\ell-1,2\ell-1} = 0$, for $\ell = 1, \ldots, \lfloor k/2 \rfloor$.*

Proof. Let us choose a pendant-path T' with 2 or more vertices, and suppose, without loss of generality, that k is the vertex of T' adjacent to the center $k+1$. Let us prove that $a_{11} = 0$. In fact, since

$$0 \neq \det A = -a_{12}^2 \det A(1,2)$$

and

$$0 = \det A(2) = a_{11} \det A(1,2),$$

the result follows. If T' has up to 2 vertices, the procedure stops here. Otherwise, a similar argument is applied to prove $a_{33} = 0$. We stop when the kth vertex is reached. $\square$

Theorem 7.7 turns to be a very useful tool in the characterization of trees having maximal P-vertex value. For example, let us consider a 2-generalized star, i.e., a star where to each pendant vertex we add a new vertex [8,12].

Proposition 7.8. *Let T be a 2-generalized star on $2n+1$ vertices. The central vertex of T cannot be a P-vertex of any nonsingular matrix $A \in \mathcal{S}(T)$.*

Proof. If S is the set of pendant vertices, then $a_{ii} = 0$, for any $i \in S$. Let us assume that 1 is the central vertex of T. Then $\det A(1)$ is the product of determinants of 2×2 matrices with pattern

$$\begin{pmatrix} 0 & \pm \\ \pm & * \end{pmatrix},$$

which are nonsingular. $\square$

8. A last question

We end with one more question posed Kim and Shader in [20].

Question 3 [20]**.** If $n \geqslant 3$ is odd, is there a tree T on n vertices which has a nonsingular matrix A in $\mathcal{S}(T)$ such that $P_\nu(A) = n$?

For a small n, the answer to Question 3 is negative. Before we show that fact, let us establish the following proposition.

Proposition 8.1. *Let B be a nonsingular tridiagonal matrix of the form (2.1), with $n > 1$. If $P_\nu(B) = n$, then $a_{2\ell-1} = a_{n-2\ell+2} = 0$, for $\ell = 1, 2, \ldots, \lceil n/2 \rceil$.*

Proof. We have already seen $a_1 = a_n = 0$. Next we prove $a_3 = 0$. First, we observe that $\det B(123) = 0$, since $\det B(3) = 0$ and the inertia of $B[12]$ is $(1, 1, 0)$. Therefore

$$0 \neq \det B = -b_3^2 \det B(34),$$

and since $\det B(4) = 0$, we have $\det B[123] = 0$, which implies $a_3 = 0$. We prove $a_{n-2} = 0$ similarly. Using a similar procedure we prove the remaining equalities. $\square$

Now, for $n = 3$, the only tree is the path with 3 vertices, and Proposition 8.1 provides the negative answer. For $n = 5$, from Corollary 7.5 and, again, from Proposition 8.1 we get the same answer for the star and for the path, respectively. The remaining case is a broom, also called here *fork* or *chair*. But Corollary 7.6 gives a negative answer for that tree as well.

Other questions in [20] remain open, and we leave them for a future work.

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