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Spectrally degenerate graphs: Hereditary case

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ABSTRACT

It is well known that the spectral radius of a tree whose maximum degree is Δ cannot exceed $2\sqrt{\Delta-1}$. A similar upper bound holds for arbitrary planar graphs, whose spectral radius cannot exceed $\sqrt{8\Delta}+10$, and more generally, for all d-degenerate graphs, where the corresponding upper bound is $\sqrt{4d\Delta}$. Following this, we say that a graph G is *spectrally d-degenerate* if every subgraph H of G has spectral radius at most $\sqrt{d\Delta(H)}$. In this paper we derive a rough converse of the above-mentioned results by proving that each spectrally d-degenerate graph G contains a vertex whose degree is at most $4d\log_2(\Delta(G)/d)$ (if $\Delta(G)\geqslant 2d$). It is shown that the dependence on Δ in this upper bound cannot be eliminated, as long as the dependence on d is subexponential. It is also proved that the problem of deciding if a graph is spectrally d-degenerate is co-NP-complete.

1. Introduction

All graphs in this paper are finite and *simple*, i.e. no loops or multiple edges are allowed. We use standard terminology and notation. We denote by $\Delta(G)$ and $\delta(G)$ the maximum and the minimum degree of G, respectively. If H is a subgraph of G, we write $H\subseteq G$. For a graph G, let $\rho(G)$ denote its *spectral radius*, the largest eigenvalue of the adjacency matrix of G. More generally, if M is a square

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matrix, the spectral radius of M, denoted by $\rho(M)$, is the maximum modulus $|\lambda|$ taken over all eigenvalues λ of M.

If T is a tree, then it is a subgraph of the infinite $\Delta(T)$-regular tree. This observation implies that the spectral radius of T is at most $2\sqrt{\Delta(T)-1}$. Similar bounds have been obtained for arbitrary planar graphs and for graphs of bounded genus [8]. In particular, the following result holds.

Theorem 1.1. *(See Dvořák and Mohar [8].) If G is a planar graph, then*

$$\rho(G) \leqslant \sqrt{8\Delta(G)} + 10.$$

The proof in [8] uses the fact that the edges of every planar graph G can be partitioned into two trees of maximum degree at most $\Delta(G)/2$ and a graph whose degree is bounded by a small constant. A similar bound was obtained earlier by Cao and Vince [4].

Whenever a result can be proved for tree-like graphs and for graphs of bounded genus, it is natural to ask if it can be extended to a more general setting of minor-closed families. Indeed, this is possible also in our case, and a result of Hayes [11] (see Theorem 1.2 below) goes even further.

A graph is said to be *d-degenerate* if every subgraph of G has a vertex whose degree is at most d. This condition is equivalent to the requirement that G can be reduced to the empty graph by successively removing vertices whose degree is at most d.

A requirement that is similar to degeneracy is existence of an orientation of the edges of G such that each vertex has indegree at most d. Every such graph is easily seen to be $2d$-degenerate, and conversely, every d-degenerate graph has an orientation with maximum indegree d.

Theorem 1.2. *(See Hayes [11].) Any graph G that has an orientation with maximum indegree d (hence also any d-degenerate graph) and with $\Delta = \Delta(G) \geqslant 2d$ satisfies*

$$\rho(G) \leqslant 2\sqrt{d(\Delta - d)}.$$

It is well known that each planar graph G has an orientation with maximum indegree 3. Theorem 1.2 thus implies that $\rho(G) \leqslant \sqrt{12(\Delta-3)}$, which is slightly weaker than the bound of Theorem 1.1 (for large Δ).

The above results suggest the following definitions. We say that a graph G is *spectrally d-degenerate* if every subgraph H of G has spectral radius at most $\sqrt{d\Delta(H)}$. Hayes' Theorem 1.2 shows that d-degenerate graphs are spectrally $4d$-degenerate. The implication is clear for graphs G of maximum degree at least $2d$. On the other hand, if $\Delta(G) \leqslant 2d$, then $\rho(G) \leqslant \Delta(G) \leqslant \sqrt{2d\Delta(G)}$. The main result of this paper is a rough converse of this statement.

Theorem 1.3. *If G is a spectrally d-degenerate graph, then it contains a vertex whose degree is at most* $\max\{4d, 4d\log_2(\Delta(G)/d)\}$.

The proof is given in Section 3. If it were not for the annoying factor of $\log(\Delta)$, this would imply $f(d)$-degeneracy, which was our initial hope. However, in Section 4 we construct examples showing that the ratio between degeneracy and spectral degeneracy may be as large as (almost) $\log\log\Delta(G)$.

In the last section, we consider computational complexity questions related to spectral degeneracy. First we prove that for every integer $d \geqslant 3$, it is NP-hard to decide if the spectral degeneracy of a given graph G of maximum degree $d+1$ is at least d. Next we show that the problem of deciding if a graph is spectrally d-degenerate is co-NP-complete.

2. Spectral radius

We refer to [2,7,10] for basic results about the spectra of finite graphs and to [12] for results concerning the spectral radius of (non-negative) matrices. Let us review only the most basic facts that will be used in this paper. The spectral radius is monotone and subadditive. Formally this is stated in the following lemma.

Lemma 2.1.

(a) *If $H \subseteq G$, then $\rho(H) \leqslant \rho(G)$.*
(b) *If $G = K \cup L$, then $\rho(G) \leqslant \rho(K) + \rho(L)$.*

The application of Lemma 2.1(a) to the subgraph of G consisting of a vertex of degree $\Delta(G)$ together with all its incident edges gives a lower bound on the spectral radius in terms of the maximum degree.

Lemma 2.2. $\sqrt{\Delta(G)} \leqslant \rho(G) \leqslant \Delta(G)$.

A partition $V(G) = V_1 \cup \cdots \cup V_k$ of the vertex set of G is called an *equitable partition* if, for every $i, j \in \{1, \ldots, k\}$, there exists an integer b_{ij} such that every vertex $v \in V_i$ has precisely b_{ij} neighbors in V_j. The $k \times k$ matrix $B = [b_{ij}]$ is called the *quotient adjacency matrix* of G corresponding to this equitable partition.

Lemma 2.3. *Let B be the quotient adjacency matrix corresponding to an equitable partition of G. Then every eigenvalue of B is also an eigenvalue of G, and $\rho(G) = \rho(B)$.*

Proof. The first claim is well known (see [10] for details). To prove it, one just lifts an eigenvector y of B to an eigenvector x of G by setting $x_v = y_i$ if $v \in V_i$. By the Perron–Frobenius Theorem, the eigenvector corresponding to the largest eigenvalue of B is positive (if G is connected, which we may assume), so its lift is also a positive eigenvector of G. This easily implies (by using the Perron–Frobenius Theorem and orthogonality of eigenvectors of G) that this is the eigenvector corresponding to the largest eigenvalue of G. Thus, $\rho(G) = \rho(B)$. □

We will need an extension of Lemma 2.3. As above, let $V(G) = V_1 \cup \cdots \cup V_k$ be a partition of $V(G)$, and let $n_i = |V_i|$, $1 \leqslant i \leqslant k$. For every $i, j \in \{1, \ldots, k\}$, let e_{ij} denote the number of ordered pairs (u, v) such that $u \in V_i$, $v \in V_j$ and $uv \in E(G)$, i.e. e_{ij} is the number of edges between V_i and V_j if $i \neq j$, and is twice the number of edges between the vertices in V_i if $i = j$. Let $b_{ij} = e_{ij}/n_i$ and let $B = [b_{ij}]$ be the corresponding $k \times k$ matrix. This is a generalization from equitable to general partitions, so we say that B is the *quotient adjacency matrix* of G also in this case. If a matrix $B' = [b'_{ij}]_{i,j=1}^k$ satisfies $0 \leqslant b'_{ij} \leqslant b_{ij}$ for every pair i, j, then we say that B' is a *quotient sub-adjacency matrix* for the partition $V_1 \cup \cdots \cup V_k$.

Lemma 2.4. *If B' is a quotient sub-adjacency matrix corresponding to a partition of $V(G)$, then $\rho(G) \geqslant \rho(B')$.*

Proof. By the monotonicity of the spectral radius, $\rho(B') \leqslant \rho(B)$, where B is the quotient adjacency matrix. So we may assume that $B' = B$. The matrix B is element-wise non-negative. By the Perron–Frobenius Theorem, its spectral radius $\rho(B)$ is equal to the largest eigenvalue of B (which is real and positive) and the corresponding eigenvector y is non-negative. Let us define the vector $f \in \mathbb{R}^{V(G)}$ by setting $f_v = y_i$ if $v \in V_i$. Then

$$\|f\|^2 = \sum_{v \in V(G)} f_v^2 = \sum_{i=1}^k n_i y_i^2.$$

Furthermore, if A is the adjacency matrix of G, then

$$\begin{aligned}\langle f | Af \rangle &= 2 \sum_{uv \in E(H)} f_u f_v \\ &= \sum_{i=1}^k \sum_{j=1}^k e_{ij} y_i y_j\end{aligned}$$

$$= \sum_{i=1}^{k} n_i y_i \sum_{j=1}^{k} b_{ij} y_j$$

$$= \rho(B) \sum_{i=1}^{k} n_i y_i^2$$

$$= \rho(B)\|f\|^2.$$

Since the matrix A is symmetric, $\rho(A)$ is equal to the numerical radius of A. Thus, it follows from the above calculations that $\rho(G) \geqslant \frac{\langle f|Af\rangle}{\|f\|^2} = \rho(B)$, which we were to prove. □

3. Spectrally degenerate graphs are nearly degenerate

In this section we prove our main result, Theorem 1.3. For convenience we state it again (in a slightly different form).

Theorem 3.1. *Let G_0 be a spectrally d-degenerate graph with $r = \delta(G_0) > 4d$. Then $r \leqslant 4d \log_2(\Delta(G_0)/d)$.*

Proof. Suppose for a contradiction that $r > 4d \log_2(\Delta(G_0)/d) \geqslant 4d$. Let G be a subgraph of G_0 obtained by successively deleting edges xy for which $\deg(x) \geqslant \deg(y) > r$, as long as possible. Then G has the following properties:

(a) $\delta(G) = r > 4d \log_2(\Delta(G_0)/d) \geqslant 4d \log_2(\Delta(G)/d)$.
(b) G is spectrally d-degenerate.
(c) The set of vertices of G whose degree is bigger than r is an independent vertex set in G.

Our goal is to prove that $r \leqslant 4d \log_2(\Delta(G)/d)$. This will contradict (a) and henceforth prove the theorem.

Let us consider the vertex partition into the following vertex sets:

$$V_0 = \{v \in V(G) \mid \deg_G(v) = r\},$$

and for $i = 1, \ldots, l$,

$$V_i = \{v \in V(G) \mid 2^{i-1}r < \deg_G(v) \leqslant 2^i r\},$$

where $l = \lceil \log_2(\Delta(G)/r) \rceil \leqslant \log_2(\Delta(G)/d)$. Let $B = [b_{ij}]_{i,j=0}^{l}$ be the quotient adjacency matrix for the partition $V_0, V_1, \ldots, V_l$ of $V(G)$. Since all vertices in V_0 have the same degree r, it follows from the definitions of the entries of B that $r = \sum_{i=0}^{l} b_{0i}$. Thus it suffices to estimate the entries b_{0i} in order to bound r.

For $i = 0$, let $H \subseteq G$ be the induced subgraph of G on V_0. Since G is spectrally d-degenerate, we have that $\rho(H) \leqslant \sqrt{d\Delta(H)} \leqslant \sqrt{dr} \leqslant \sqrt{r^2/4} = \frac{r}{2}$. On the other hand, since H has average degree b_{00}, we have $\rho(H) \geqslant b_{00}$. Thus, $b_{00} \leqslant \frac{r}{2}$. This shows that $\sum_{i=1}^{l} b_{0i} = r - b_{00} \geqslant r/2$, and thus it suffices to prove that

$$\sum_{i=1}^{l} b_{0i} \leqslant 2d \log_2\bigl(\Delta(G)/d\bigr). \tag{1}$$

From now on we let B' be the matrix obtained from B by setting the entry b'_{00} to be 0. This is the quotient adjacency matrix of the subgraph G' of G obtained by removing edges between pairs of vertices in V_0.

We shall now prove that

$$\sum_{i=1}^{t} 2^{i-1} b_{0i} \leqslant 2^t d \tag{2}$$

for every $t = 1, \dots, l$. Let us consider the subgraph G_t of G' induced on $V_0 \cup V_1 \cup \dots \cup V_t$ and the corresponding matrix

$$B_t = \begin{bmatrix} 0 & b_{01} & \dots & b_{0t} \\ r & 0 & \dots & 0 \\ 2r & 0 & \dots & 0 \\ 4r & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 2^{t-1}r & 0 & \dots & 0 \end{bmatrix}.$$

Let us observe that the entries $2^{i-1}r$ $(i = 1, \dots, t)$ in the first column of B_t are smaller than the corresponding entries in B' because every vertex in V_i has degree more than $2^{i-1}r$. Therefore, B_t is a quotient sub-adjacency matrix for the subgraph G_t. By expanding the determinant of the matrix $\lambda I - B_t$, it is easy to see that

$$\rho(B_t)^2 = \sum_{i=1}^{t} 2^{i-1} r b_{0i}. \tag{3}$$

Using Lemma 2.4 and the fact that G_t is spectrally d-degenerate, we see that $\rho(B_t)^2 \leqslant \rho(G_t)^2 \leqslant d \cdot 2^t r$. This inequality combined with (3) implies (2).

We shall now prove by induction on s that

$$\sum_{i=1}^{s} b_{0i} \leqslant (s+1)d \tag{4}$$

for every $s = 1, \dots, l$. For $s = 1$, this is the same as the inequality (2) taken for $t = 1$. For $s \geqslant 2$, we apply inequalities (2) to get the following estimates:

$$2^{s-t} \sum_{i=1}^{t} 2^{i-1} b_{0i} \leqslant 2^s d \tag{5}$$

and henceforth

$$\sum_{t=1}^{s} 2^{s-t} \sum_{i=1}^{t} 2^{i-1} b_{0i} \leqslant s \cdot 2^s d. \tag{6}$$

Finally, inequalities (5) (taken with $t = s$) and (6) imply

$$\begin{aligned} 2^s \sum_{i=1}^{s} b_{0i} &= \sum_{i=1}^{s} \left(2^{i-1} + \sum_{j=i}^{s} 2^{j-1} \right) b_{0i} \\ &= \sum_{i=1}^{s} 2^{i-1} b_{0i} + \sum_{t=1}^{s} 2^{s-t} \sum_{i=1}^{t} 2^{i-1} b_{0i} \\ &\leqslant 2^s d + s \cdot 2^s d = 2^s (s+1) d. \end{aligned}$$

This proves (4). For $s = l$, this implies (1) and completes the proof of the theorem. □

4. A lower bound

In this section we show that the $\log(\Delta)$ factor in the bound given by Theorem 1.3 cannot be eliminated entirely.

Let $\alpha \in \mathbb{R}_+$. We say that a graph G is *α-log-sparse*, shortly *α-LS*, if every subgraph H of G has average degree at most $\alpha \log(\Delta(H))$. Observe that being α-LS is a hereditary property and that every α-LS graph G is $\alpha \log(\Delta(G))$-degenerate.

Pyber, Rödl, and Szemerédi [15, Theorem 2] proved that there exists a constant α_0 such that every graph G with average degree at least $\alpha_0 \log(\Delta(G))$ contains a 3-regular subgraph. On the other hand, they proved in the same paper [15] that there exists a constant $\beta > 0$ such that, for each $n \geqslant 1$, there is a bipartite graph of order n with average degree at least $\beta \log\log n$ which does not contain any 3-regular subgraph (and is hence α_0-LS). These results establish the following.

Theorem 4.1. *(See [15].) There exist constants $\alpha_0, \beta_0 > 0$ such that for every integer $\tau > 1$ there exists a bipartite graph G with bipartition $V(G) = A \cup B$ with the following properties:*

(a) *G is α_0-LS.*
(b) *$|A| \geqslant |B|$ and every vertex in A has degree τ.*
(c) *$\beta_0 \log\log|A| \leqslant \tau$.*

We will prove that graphs of Theorem 4.1 have small spectral degeneracy. The proof will use the Chernoff inequality in the following form (cf. [14, Theorem 7.2.1]):

Lemma 4.2. *Let $X_1, \ldots, X_n$ be independent random variables, each of them attaining value 1 with probability p, and having value 0 otherwise. Let $X = X_1 + \cdots + X_n$. Then, for any $r > 0$,*

$$\mathrm{Prob}\big[|X - np| \geqslant r\big] < \exp\left(-\frac{r^2}{2(np + r/3)}\right).$$

We can now prove the following lemma, showing that a bipartite graph whose bipartite parts are "almost" regular cannot be log-sparse.

Lemma 4.3. *Let $T \geqslant 10$ and $t > 0$ be integers such that*

$$6\alpha_0 \log(20T) \leqslant t \leqslant T.$$

Let H be a bipartite graph of maximum degree $\Delta \geqslant 2T$ with bipartition $V(H) = A \cup B$ satisfying the following properties:

(a) *$t \leqslant \deg v \leqslant T$ for each vertex $v \in A$.*
(b) *Each vertex $v \in B$ has degree at least $\Delta/2$.*

Then H is not α_0-LS.

Proof. Choose a subset A' of A by selecting each element uniformly independently with probability $p = 2T/\Delta$, and let H' be the subgraph of H induced by $A' \cup B$. The expected size of A' is $a' = 2T|A|/\Delta$. Note that $T|A| \geqslant |E(H)| \geqslant |B|\Delta/2$, thus $a' \geqslant |B|$. Furthermore, $|A| \geqslant \Delta/2$, and thus $a' \geqslant T \geqslant 10$. By Lemma 4.2,

$$\mathrm{Prob}\left[\left|A'\right| \leqslant \frac{1}{2}a'\right] < e^{-3a'/28} < \frac{1}{2}.$$

Therefore, we have $2|A'| \geqslant a' \geqslant |B|$ with probability greater than $\frac{1}{2}$.

Consider a vertex $v \in B$. The expected degree of v in H' is between T and $2T$, and the probability that v has degree greater than $2cT$ is less than e^{-cT} for any $c \geqslant 10$, by Lemma 4.2. Let $z = 0$ if $\deg_{H'} v \leqslant 20T$ and $z = \deg_{H'} v$ otherwise. The expected value of z is

$$\begin{aligned}\sum_{j=20T+1}^{\infty} \Pr(\deg_{H'} v = j) j &= \sum_{j=20T+1}^{\infty} \sum_{i=1}^{j} \Pr(\deg_{H'} v = j) \\ &= \sum_{i=1}^{20T} \sum_{j=20T+1}^{\infty} \Pr(\deg_{H'} v = j) + \sum_{i=20T+1}^{\infty} \sum_{j=i}^{\infty} \Pr(\deg_{H'} v = j) \\ &= 20T \Pr(\deg_{H'} v > 20T) + \sum_{i=20T+1}^{\infty} \Pr(\deg_{H'} v \geqslant i) \\ &= 20T \Pr(\deg_{H'} v > 20T) + \sum_{i=20T}^{\infty} \Pr(\deg_{H'} v > i) \\ &\leqslant 20Te^{-10T} + \sum_{i=20T}^{\infty} e^{-i/2}.\end{aligned}$$

We conclude that the expected number of edges of H' incident with vertices of degree greater than $20T$ is less than

$$|B|\left(20Te^{-10T} + \sum_{i=20T}^{\infty} e^{-i/2}\right) < |B|(20T + 3)e^{-10T}.$$

By Markov's inequality, it happens with positive probability that H' has less than $2|B|(20T + 3)e^{-10T}$ edges incident with vertices of degree greater than $20T$ and that $2|A'| \geqslant |B|$.

Let us now fix a subgraph H' with these properties. Let H'' be the graph obtained from H' by removing the vertices of degree greater than $20T$. Clearly, $\Delta(H'') \leqslant 20T$. Also, H'' has at most $3|A'|$ vertices and more than

$$\left|A'\right|t - 2|B|(20T + 3)e^{-10T} \geqslant \left|A'\right|\left(t - 4(20T + 3)e^{-10T}\right) \geqslant \frac{1}{2}\left|A'\right|t$$

edges, thus the average degree of H'' is greater than $t/6$. Since $t/6 \geqslant \alpha_0 \log(20T)$, this shows that H is not α_0-LS. □

Theorem 4.4. *Suppose that a bipartite graph G with bipartition $V(G) = A \cup B$ satisfies properties* (a)–(c) *of Theorem* 4.1*, where $\tau \geqslant 10$ and $6\alpha_0 \log(20\tau) \leqslant \tau$. Then G is spectrally d-degenerate, where*

$$d = 48(3 + 2\sqrt{2})\alpha_0 \log(20\tau).$$

Proof. Suppose for a contradiction that H is a subgraph of G with maximum degree $D = \Delta(H)$ whose spectral radius violates spectral d-degeneracy requirement,

$$\rho(H) > \sqrt{dD}. \tag{7}$$

We may assume that H is chosen so that D is minimum possible. Since G is α_0-LS, the same holds for its subgraph H. In particular, H is $\alpha_0 \log(D)$-degenerate and hence $\rho(H) \leqslant 2\sqrt{\alpha_0 \log(D) \cdot D}$. By (7) we conclude that

$$4\alpha_0 \log(D) > d. \tag{8}$$

This implies, in particular, that

$$D \geqslant 2\tau. \tag{9}$$

Let $\gamma = (3 - 2\sqrt{2})/8$. Let us partition the edges of H into three subgraphs, $H = H_0 \cup H_1 \cup H_2$, such that the following holds:

(a) Each vertex in $V(H_0) \cap B$ has degree in H_0 at least $D/2$.
(b) Each vertex in $V(H_0) \cap A$ has degree in H_0 at least γd.
(c) H_1 is γd-degenerate.
(d) $\Delta(H_2) \leqslant D/2$.

Such a partition can be obtained as follows. Let H_0 be a minimal induced subgraph of H such that $E(H) \setminus E(H_0)$ can be partitioned into graphs H_1 and H_2 satisfying the conditions (c) and (d) and $V(H_0) \cap V(H_1) \cap A = \emptyset$. We claim that H_0 satisfies (a) and (b). Indeed, suppose that H_0 violates (a). Then, there exists a vertex $v \in V(H_0) \cap B$ of degree at most $D/2$. Consider the graph H_2' obtained from H_2 by adding all edges of H_0 incident with v. Clearly, $\Delta(H_2') \leqslant D/2$, since v has degree at most $D/2$ and all vertices in $A \cap V(H_2')$ have degree at most $\tau \leqslant D/2$ by (9). Thus, there exists a partition of $E(H) \setminus E(H_0 - v)$ satisfying (c) and (d), which contradicts the minimality of H_0. Similarly, suppose that H_0 violates (b), so there exists $v \in V(H_0) \cap A$ of degree at most γd. Since $V(H_0) \cap V(H_1) \cap A = \emptyset$, $v \notin V(H_1)$, and thus the graph H_1' obtained from H_1 by adding all edges of H_0 incident with v is γd-degenerate. Furthermore, $V(H_0 - v) \cap V(H_1') \cap A = \emptyset$, so we again obtain a contradiction with the minimality of H_0.

Suppose that $H_0 \neq \emptyset$. Then we use properties (a)–(b) of H_0 and apply Lemma 4.3 to conclude that H_0 is not α_0-LS. This contradicts our assumption that G is α_0-LS and shows that H_0 must be empty.

Thus, $H = H_1 \cup H_2$. Since H was selected as a subgraph violating spectral degeneracy with its maximum degree smallest possible, we conclude that H_2 is spectrally d-degenerate. By applying Lemma 2.1(b) and using Theorem 1.2 on H_1, we obtain

$$\begin{aligned}
\rho(H) &\leqslant \rho(H_1) + \rho(H_2) \\
&\leqslant \sqrt{4\gamma d \Delta(H_1)} + \sqrt{d\Delta(H_2)} \\
&\leqslant \sqrt{4\gamma dD} + \sqrt{dD/2} \\
&\leqslant (\sqrt{4\gamma} + \sqrt{1/2})\sqrt{dD} = \sqrt{dD}.
\end{aligned}$$

This contradicts (7) and completes our proof. □

By Theorem 4.1, there exist constants β and n_0 such that we can apply Theorem 4.4 to graphs on n vertices with $\tau \geqslant \beta \log\log n$, for any $n \geqslant n_0$. Then, $d = O(\log\log\log n)$, and thus the ratio between the degeneracy and the spectral degeneracy is at least $\Omega(\log\log n/\log\log\log n) \geqslant \Omega(\log\log\Delta/\log\log\log\Delta)$.

Let us however remark that this does not exclude the possibility that the degeneracy is bounded by a function of the spectral degeneracy. Answering a question we posed in the preprint version of this paper, Alon [1] proved that that is not the case.

Theorem 4.5. *For every M, there exist spectrally 50-degenerate graphs with minimum degree at least M.*

5. Computational complexity remarks

Our results raise the problem of how hard it is to verify spectral degeneracy of a graph.

Spectral Degeneracy Problem
Input: A graph G and a positive rational number d.
Task: Decide if G is spectrally d-degenerate.

Below we prove that this problem is co-NP-complete. To demonstrate this, we need some preliminary results. First, we show that distinct roots of a polynomial cannot be too close to each other. For a polynomial $p(x) = \sum_{i=0}^{k} a_i x^i$ with integer coefficients, let $a(p) = \log\max_{0 \leqslant i \leqslant k} |a_i|$.

Lemma 5.1. *Let $p(x)$ be an integer polynomial of degree k. If $p(u) = p(v) = 0$ and $u \neq v$, then $-\log|u - v| = O(k^3(a(p) + \log k))$.*

Proof. Mahler [13] proved that if y and z are two roots of a polynomial $s(x)$ of degree d, then $-\log|y - z| = O(-\log|D| + d\log d + da(s))$, where D is the discriminant of s. To apply this result, we need to eliminate the roots of p with multiplicity greater than one. By Brown [3], there exists an integer polynomial $q(x)$ that is a greatest common divisor of $p(x)$ and $p'(x)$ such that $a(q) = O(k(a(p) + \log k))$. Let c be the leading coefficient of q and let $r(x) = c^k p(x)/q(x)$. Note that $r(x)$ is an integer polynomial, all of whose roots are simple, $r(u) = r(v) = 0$, and $a(r) = O(k^2(a(p) + \log k))$. Since r is an integer polynomial with simple roots, the absolute value of its discriminant is at least 1. Using the afore-mentioned result of Mahler [13], we conclude that $-\log|u - v| = O(k^3(a(p) + \log k))$. □

Cheah and Corneil [5] showed the following.

Theorem 5.2. *For any fixed integer $d \geqslant 3$, determining whether a graph of maximum degree $d + 1$ has a d-regular subgraph is NP-complete.*

We need an estimate on the spectral radius of graphs where the vertices of maximum degree are far apart.

Lemma 5.3. *Let G be a graph of maximum degree $d + 1$ such that the distance between every pair of vertices of degree $d + 1$ is at least three. Then*

$$\rho(G) \leqslant \sqrt[3]{(d + 1)\bigl(d^2 + 1\bigr)}.$$

Proof. We may assume that G is connected, since the spectral radius of a graph is the maximum of the spectral radii of its components. We use the fact that $\rho(G) = \limsup_{n\to\infty} \sqrt[n]{c_n}$, where c_n is the number of closed walks of length n starting at an arbitrary vertex v of G. For any vertex z of degree $d + 1$, G contains at most $(d + 1)[(d - 1)d + (d + 1)] = (d + 1)(d^2 + 1)$ walks of length 3 starting at z, including those whose second vertex is z as well. Similarly, the number of walks of length 3 from a vertex of degree at most d is at most $(d + 1)d^2$. We conclude that $c_n \leqslant [(d + 1)(d^2 + 1)]^{\lceil n/3\rceil}$, and the claim follows. □

We will also use the following result which shows that the spectral radius of a connected non-regular graph of maximum degree d cannot be arbitrarily close to d.

Lemma 5.4. *(See Cioabă [6].) Let G be a connected graph of maximum degree d and with diameter D. If G has a vertex of degree less than d, then*

$$\rho(G) < d - \frac{1}{D|V(G)|}.$$

We can now proceed with examining the complexity of spectral degeneracy computation.

Lemma 5.5. *The* Spectral Degeneracy Problem *is in co-NP.*

Proof. To verify that the spectral degeneracy of G is greater than d, guess a connected subgraph H of G (on $k \leqslant |V(G)|$ vertices) such that $\rho(H) > \sqrt{d\Delta(H)} = b$. To prove that H has this property, first compute the characteristic polynomial $p(x) = \det(xI - M)$, where M is the adjacency matrix of H. Note that the absolute value of each coefficient of p is at most $k!$ and that p can be computed in polynomial time using, for example, Le Verrier–Faddeev's algorithm [9]. Then, we need to show that p has a real positive root greater than b. This is the case if $p(b) < 0$ and this condition can be verified in a polynomial time, since b is a square root of a rational number. Hence, we may assume that $p(b) \geqslant 0$. Let us recall that $\rho(H)$ is a simple root of p. Hence, if $\rho(H) > b$, then there exists a root y of p such that $b \leqslant y < \rho(H)$ and $p(x) < 0$ when $y < x < \rho(H)$. To prove that $b < \rho(H)$, it suffices to guess a value $x > b$ such that $p(x) < 0$, say any value between y and $\rho(H)$. By Lemma 5.1, $-\log(\rho(H) - y) = O(k^4 \log k)$, and thus such a number x can be expressed in polynomial space. □

For the hardness part, let us first consider a related problem of deciding whether the spectral degeneracy is greater or equal to some given constant.

Theorem 5.6. *For any fixed integer $d \geqslant 3$, verifying whether the spectral degeneracy of a graph is at least d is NP-hard, even when restricted to graphs of maximum degree $d+1$.*

Proof. We give a reduction from the problem of finding a d-regular subgraph in a graph G of maximum degree $d+1$, which is NP-hard by Theorem 5.2. Let G' be the graph obtained from G by replacing each edge uv by a graph G_{uv} created from a clique on $d+1$ new vertices by removing an edge xy and adding the edges ux and vy. Consider a connected subgraph $H \subseteq G'$. If H is d-regular and $z \in V(H)$ belongs to $V(G_{uv}) \setminus \{u, v\}$, then $G_{uv} \subseteq H$. It follows that G' contains a d-regular subgraph if and only if G contains a d-regular subgraph.

Furthermore, if $\Delta(H) = d+1$, then by Lemma 5.3 we have

$$\rho(H) \leqslant \sqrt[3]{(d+1)\big(d^2+1\big)} < \sqrt{d\Delta(H)},$$

and if $\Delta(H) \leqslant d$, then $\rho(H) \leqslant \sqrt{d\Delta(H)}$, where the equality holds if and only if H is d-regular. Therefore, G has a d-regular subgraph if and only if the spectral degeneracy of G' is at least d. Since the size of G' is polynomial in the size of G, this shows that deciding whether the spectral degeneracy of a graph is at least d is NP-hard. □

A small variation of this analysis gives us the desired result.

Theorem 5.7. *The* Spectral Degeneracy Problem *is co-NP-complete.*

Proof. By Lemma 5.5, the problem is in co-NP, so it remains to exhibit a reduction from a co-NP-hard problem.

Consider the graph G' from the proof of Theorem 5.6 and its connected subgraph H. If H has maximum degree $d+1$, then the spectral radius of H is at most

$$\sqrt{\sqrt[3]{\frac{(d^2+1)^2}{d+1}}\Delta(H)}$$

by Lemma 5.3. If H has maximum degree at most $d-1$, then

$$\rho(H) \leqslant \sqrt{(d-1)\Delta(H)}.$$

Finally, if $\Delta(H) = d$ and H is not d-regular, then

$$\rho(H) \leqslant \sqrt{\big(d - \big|V(H)\big|^{-2}\big)^2} \leqslant \sqrt{\big(d - \big|V(H)\big|^{-2}\big)\Delta(H)}$$

by Lemma 5.4.

Let $n = |V(G')|$. Let b be a rational number such that

$$\max\left\{\sqrt[3]{\frac{(d^2+1)^2}{d+1}},\ d-1,\ d-n^{-2}\right\} \leqslant b < d.$$

We conclude that either G' has spectral radius at least d or at most b. Thus, deciding whether the spectral degeneracy of a graph is at most b (where b is part of the input) is co-NP-hard. □

However, this does not exclude the possibility that the spectral degeneracy could be approximated efficiently. Let $\varepsilon > 0$ be a constant.

Approximate spectral degeneracy
Input: A graph G and a rational number d.
Task: Either prove that G is spectrally $(1+\varepsilon)d$-degenerate, or show that it is not spectrally d-degenerate.

Does there exist ε such that this problem can be solved in a polynomial time? Or possibly, is it true that this question can be solved in a polynomial time for every $\varepsilon > 0$? Both of these questions are open.

References

[1] N. Alon, A note on degenerate and spectrally degenerate graphs, http://www.tau.ac.il/~nogaa/PDFS/degen.pdf.
[2] N.L. Biggs, Algebraic Graph Theory, 2nd ed., Cambridge Univ. Press, 1993.
[3] W.S. Brown, On Euclid's algorithm and the computation of polynomial greatest common divisors, J. ACM 18 (1971) 478–504.
[4] D. Cao, A. Vince, The spectral radius of a planar graph, Linear Algebra Appl. 187 (1993) 251–257.
[5] F. Cheah, D.G. Corneil, The complexity of regular subgraph recognition, Discrete Appl. Math. 27 (1990) 59–68.
[6] S.M. Cioabă, The spectral radius and the maximum degree of irregular graphs, Electron. J. Combin. 14 (2007), #R38.
[7] D.M. Cvetković, M. Doob, H. Sachs, Spectra of Graphs, 3rd ed., Johann Ambrosius Barth Verlag, 1995.
[8] Z. Dvorak, B. Mohar, Spectral radius of finite and infinite planar graphs and of graphs of bounded genus, J. Combin. Theory Ser. B 100 (2010) 729–739.
[9] D.K. Faddeev, V.N. Faddeeva, Computational Methods of Linear Algebra, Freeman, San Francisco, 1963.
[10] C. Godsil, G. Royle, Algebraic Graph Theory, Springer, 2001.
[11] T.P. Hayes, A simple condition implying rapid mixing of single-site dynamics on spin systems, in: Proceedings of the 47th Annual IEEE Symposium on Foundations of Computer Science (FOCS 2006), IEEE, 2006, pp. 39–46.
[12] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge Univ. Press, Cambridge, 1985.
[13] K. Mahler, An inequality for the discriminant of a polynomial, Michigan Math. J. 11 (1964) 257–262.
[14] J. Matousek, J. Vondrak, The probabilistic method, http://kam.mff.cuni.cz/~matousek/lectnotes.html, lecture notes.
[15] L. Pyber, V. Rödl, E. Szemerédi, Dense graphs without 3-regular subgraphs, J. Combin. Theory Ser. B 63 (1995) 41–54.