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On periodic matrix-valued Weyl–Titchmarsh functions

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Abstract

We consider a certain class of Herglotz–Nevanlinna matrix-valued functions which can be realized as the Weyl–Titchmarsh matrix-valued function of some symmetric operator and its self-adjoint extension. New properties of Weyl–Titchmarsh matrix-valued functions as well as a new version of the functional model for such realizations are presented. In the case of periodic Herglotz–Nevanlinna matrix-valued functions, we provide a complete characterization of their realizations in terms of the corresponding functional model. We also obtain properties of a symmetric operator and its self-adjoint extension which generate a periodic Weyl–Titchmarsh matrix-valued function. We study pairs of operators (a symmetric operator and its self-adjoint extension) with constant Weyl–Titchmarsh matrix-valued functions and establish connections between such pairs of operators and representations of the canonical commutation relations for unitary groups of operators in Weyl's form. As a consequence of such an approach, we obtain the Stone–von Neumann theorem for two unitary groups of operators satisfying the commutation relations as well as some extension and refinement of the classical functional model for generators of those groups. Our examples include multiplication operators in weighted spaces, first and second order differential operators, as well as the Schrödinger operator with linear potential and its perturbation by bounded periodic potential.

Keywords: Weyl–Titchmarsh function; Symmetric operator; Self-adjoint extension; Unitary group

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1. Introduction

In this paper we study a certain class of Herglotz–Nevanlinna matrix-valued functions which can be realized as the Weyl–Titchmarsh matrix-valued function $M_{\mathcal{H},H}(z)$ generated by the densely defined symmetric operator $\mathcal{H}$ and its self-adjoint extension H acting on some Hilbert space $\mathfrak{H}$ [5,7,8]. The new properties of these functions as well as a new version of the functional model for the pair $(\mathcal{H}, H)$ in terms of $M_{\mathcal{H},H}(z)$ are obtained. We introduce so-called (U, b)-periodic pair of operators $(\mathcal{H}, H)$ $(U\mathcal{H}U^* = \mathcal{H} - bI, UHU^* = H - bI, U$ is a unitary operator in $\mathfrak{H})$ and establish that the Weyl–Titchmarsh matrix-valued function is b-periodic $(M_{\mathcal{H},H}(z + b) = M_{\mathcal{H},H}(z))$ if and only if the corresponding pair of operators $(\mathcal{H}, H)$ generating this matrix-valued function is (U, b)-periodic. It is shown that any Weyl–Titchmarsh function $M_{\mathcal{H},H}(z)$ corresponding to symmetric operator $\mathcal{H}$ with the defect indices $(1, 1)$ which admits quasi-Hermitian extension $\mathcal{H}_v$ without spectrum is always $\pi/\operatorname{tr}(\Im\mathcal{H}_v^{-1})$-periodic. Each (U, b)-periodic symmetric operator $\mathcal{H}$ is associated with a group Γ of transformations of the set $U(m)$ of all $m \times m$ unitary matrices into itself. It turns out that the group Γ is cyclic if and only if an operator $\mathcal{H}$ admits periodic extension. We consider a pair of operators $(\mathcal{H}, H)$ with the constant Weyl–Titchmarsh matrix-valued functions and find connections between such pairs and representations of the canonical commutation relations for unitary groups of operators in Weyl's form. As a consequence of this approach we obtain the Stone–von Neumann theorem [17] for two unitary groups of operators satisfying the commutation relations as well as some extension and refinement of the classical functional model for generators of those groups. The examples of the Schrödinger operator with linear potential and its perturbation by a bounded periodic function are considered.

2. The Weyl–Titchmarsh function

Let $\mathfrak{H}$ be a Hilbert space, and let $\mathcal{H}$ be a prime symmetric operator in $\mathfrak{H}$, that is, $\mathfrak{H}$ does not contain a proper subspace that reduces $\mathcal{H}$, and in which $\mathcal{H}$ induces a self-adjoint operator. Let $\mathfrak{D}(\mathcal{H})$ denote the domain of $\mathcal{H}$. We assume that the defect index of $\mathcal{H}$ is (m, m), $m < \infty$. This means that for any nonreal z the defect subspace $\mathfrak{N}_z = [(\mathcal{H} - \bar{z}I)\mathfrak{D}(\mathcal{H})]^\perp$ has dimension m. Let H be a self-adjoint extension of $\mathcal{H}$ in $\mathfrak{H}$ (an orthogonal extension) with domain $\mathfrak{D}(H)$. The Weyl–Titchmarsh function of the pair $(\mathcal{H}, H)$, $M_{\mathcal{H},H}(z)$, is an operator-valued function whose values are operators on the m-dimensional space $\mathfrak{N}_i$. $M_{\mathcal{H},H}(z)$ is defined on the resolvent set $\rho(H)$ of the operator H by

$$M_{\mathcal{H},H}(z) = P_+(zH + I)(H - zI)^{-1}|_{\mathfrak{N}_i}, \tag{1}$$

where P_+ is the orthogonal projection from $\mathfrak{H}$ onto $\mathfrak{N}_i$. From the spectral representation of H, it follows that $M_{\mathcal{H},H}(z)$ can be written as

$$M_{\mathcal{H},H}(z) = \int_{\mathbb{R}} \frac{\lambda z + 1}{\lambda - z}\, d\sigma(\lambda). \tag{2}$$

Values of a nondecreasing function $\sigma(\lambda)$ are operators on $\mathfrak{N}_i$, and are defined by $\sigma(\lambda) = P_+E(\lambda)|_{\mathfrak{N}_i}$, where $E(\lambda)$ is the resolution of identity associated with H. We normalize

$E(\lambda)$ by the condition $E(\lambda) = 1/2(E(\lambda+0) + E(\lambda-0))$. It is evident that $M_{\mathcal{H},H}(z)$ is analytic on $\rho(H)$, particularly, for $\Im z \neq 0$, and from (2) it follows that $\Im M_{\mathcal{H},H}(z) \geqslant 0$ for $z \in \mathbb{C}_+$. Therefore, $M_{\mathcal{H},H}(z)$ belongs to the Herglotz–Nevanlinna class.

The function σ has the following properties:

$$\int_{\mathbb{R}} d\sigma(\lambda) = I_{\mathfrak{N}_i}, \tag{3}$$

$$\int_{\mathbb{R}} (1+\lambda^2)\big(d\sigma(\lambda)h, h\big) = \infty \quad \forall h \in \mathfrak{N}_i, \tag{4}$$

where $\sigma(\lambda) = 1/2(\sigma(\lambda+0) + \sigma(\lambda-0))$. Condition (3) is obvious, while condition (4) follows from the fact, that according to von Neumann's formulas, for vector $h \in \mathfrak{N}_i$, $h \notin \mathfrak{D}(H)$. Condition (3) provides a normalization condition for the Weyl–Titchmarsh function $M_{\mathcal{H},H}(i) = iI_{\mathfrak{N}_i}$. From condition (4) it follows that points of growth of σ form a noncompact set.

Upon selecting an orthonormal basis in $\mathfrak{N}_i$ we can identify the space $\mathfrak{N}_i$ with $\mathbb{C}^m$, and regard $M_{\mathcal{H},H}(z)$ and $\sigma(\lambda)$ as operators on $\mathbb{C}^m$. Matrices of these operators, with respect to the selected basis, are also denoted by $M_{\mathcal{H},H}(z)$ and $\sigma(\lambda)$.

An important property of the Weyl–Titchmarsh functions is given by the following theorem.

Theorem 1. *Let $\mathcal{H}$ and $\tilde{\mathcal{H}}$ be prime symmetric operators with equal defect numbers in Hilbert spaces $\mathfrak{H}$ and $\tilde{\mathfrak{H}}$, respectively, and H and $\tilde{H}$ be their self-adjoint extensions. Suppose that there is the unitary operator $W:\mathfrak{H} \to \tilde{\mathfrak{H}}$ such that $W\mathcal{H} = \tilde{\mathcal{H}}W$ and $WH = \tilde{H}W$. Then there is a unitary operator $W_0:\mathfrak{N}_i \to \tilde{\mathfrak{N}}_i$ such that $W_0 M_{\mathcal{H},H}(z) = M_{\tilde{\mathcal{H}},\tilde{H}}(z)W_0$.*

Proof. From the assumptions of the theorem it follows that $WE(\lambda) = \tilde{E}(\lambda)W$, where $E(\lambda)$ and $\tilde{E}(\lambda)$ are the resolutions of the identity, associated with H and $\tilde{H}$, respectively. From the assumption about $\mathcal{H}$ and $\tilde{\mathcal{H}}$ we have that $W\mathfrak{D}(\mathcal{H}) = \mathfrak{D}(\tilde{\mathcal{H}})$, and for $f \in \mathfrak{D}(\mathcal{H})$, $W(\mathcal{H} - zI)f = (\tilde{\mathcal{H}} - z\tilde{I})Wf$. In other words, $W\mathfrak{M}_z = \tilde{\mathfrak{M}}_z$, where $\mathfrak{M}_z = (\mathcal{H} - zI)\mathfrak{D}(\mathcal{H})$, $\tilde{\mathfrak{M}}_z = (\tilde{\mathcal{H}} - z\tilde{I})\mathfrak{D}(\tilde{\mathcal{H}})$. Since W is a unitary operator, we obtain that $W\mathfrak{N}_z = \tilde{\mathfrak{N}}_z$, and $WP_+ = \tilde{P}_+W$.

Put $W_0 = W|\mathfrak{N}_i$. Then W_0 is the unitary operator from $\mathfrak{N}_i$ onto $\tilde{\mathfrak{N}}_i$, $W_0^* = W^*|\tilde{\mathfrak{N}}_i$. For any $f \in \mathfrak{N}_i$ and $\tilde{g} \in \tilde{\mathfrak{N}}_i$ we have

$$\begin{aligned}\big(W_0 M_{\mathcal{H},H}(z)f, \tilde{g}\big) &= \big(WM_{\mathcal{H},H}(z)f, \tilde{g}\big) = \big(M_{\mathcal{H},H}(z)f, W^*\tilde{g}\big) \\ &= \int_{\mathbb{R}} \frac{\lambda z+1}{\lambda - z}\, d\big(P_+E(\lambda)f, W^*\tilde{g}\big) = \int_{\mathbb{R}} \frac{\lambda z+1}{\lambda - z}\, d\big(WP_+E(\lambda)f, \tilde{g}\big) \\ &= \int_{\mathbb{R}} \frac{\lambda z+1}{\lambda - z}\, d\big(\tilde{P}_+\tilde{E}(\lambda)Wf, \tilde{g}\big) = \big(M_{\tilde{\mathcal{H}},\tilde{H}}(z)W_0 f, \tilde{g}\big).\end{aligned}$$

These equalities show that W_0 possesses desired property. □

If $\{e_j\}_{j=1}^m$ is an arbitrary orthonormal basis in $\mathfrak{N}_i$, then $\{W_0 e_j\}$ is the orthonormal basis in $\tilde{\mathfrak{N}}_i$. With respect to these bases, matrices of $M_{\mathcal{H},H}(z)$ and $M_{\tilde{\mathcal{H}},\tilde{H}}(z)$ are equal. Therefore, Theorem 1 can be reformulated as follows:

If pairs $(\mathcal{H}, H)$ and $(\tilde{\mathcal{H}}, \tilde{H})$ are unitarily equivalent, then there are bases with respect to which matrices of their Weyl–Titchmarsh functions are equal.

The next theorem is a statement about realization. It provides the functional model of a pair with prescribed Weyl–Titchmarsh function.

Theorem 2. *Let $F(z)$ be a function whose values are linear operators on the m-dimensional space $\mathfrak{N}$, and which admits integral representation*

$$F(z) = \int_{-\infty}^{\infty} \frac{\lambda z + 1}{\lambda - z}\, d\sigma(\lambda),$$

where $\sigma(\lambda)$ is a nondecreasing function with values on the set of linear operators on $\mathfrak{N}$, and which satisfies (3) *and* (4)*. Then, there is a Hilbert space $\tilde{\mathfrak{H}}$ which contains $\mathfrak{N}$ as a subspace, prime symmetric operator $\tilde{\mathcal{H}}$ with defect index (m, m), and self-adjoint extension $\tilde{H}$ in $\tilde{\mathfrak{H}}$, such that $F(z) = M_{\tilde{\mathcal{H}},\tilde{H}}(z)$. If $(\hat{\mathfrak{H}}, \hat{\mathcal{H}}, \hat{H})$ is another realization of F, then there is a unitary operator $\Psi : \tilde{\mathfrak{H}} \to \hat{\mathfrak{H}}$ such that $\Psi\tilde{\mathcal{H}} = \hat{\mathcal{H}}\Psi$, and $\Psi\tilde{H} = \hat{H}\Psi$.*

Remark. Conditions (3) and (4) are understood now with $\mathfrak{N}$ instead of $\mathfrak{N}_i$.

Proof. Since $\sigma(\lambda)$ is a nondecreasing operator-valued function and satisfies (3), it is the generalized resolution of identity which acts in $\mathfrak{N}$. We use the following fundamental theorem by M.A. Najmark (see, for example, [1]):

Let $\sigma(\lambda)$ be the generalized resolution of identity which acts on the Hilbert space $\mathfrak{N}$. Then, there exists a Hilbert space $\tilde{\mathfrak{H}}$ which contains $\mathfrak{N}$ as a subspace and the orthogonal resolution of identity $\tilde{E}(\lambda)$, such that for any Borel set $\Delta \in \mathcal{B}(\mathbb{R})$ ($\mathcal{B}(\mathbb{R})$ is the Borel field of $\mathbb{R}$) $\sigma(\Delta) = P\tilde{E}(\Delta)|\mathfrak{N}$, where P is the orthogonal projection from $\tilde{\mathfrak{H}}$ onto $\mathfrak{N}$. The space $\tilde{\mathfrak{H}}$ can be selected to be minimal in that sense that c.l.h.$\{\tilde{E}(\Delta)h \mid \Delta \in \mathcal{B}(\mathbb{R}), h \in \mathfrak{N}\} = \tilde{\mathfrak{H}}$*, where* c.l.h. *means the closed linear hull. The orthogonal resolution of the identity $\tilde{E}(\lambda)$ defines the self-adjoint operator H in $\tilde{\mathfrak{H}}$. Under minimality condition the Hilbert space $\tilde{\mathfrak{H}}$ and the operator $\tilde{H}$ are defined uniquely up to unitary equivalence.*

In our situation this construction gives the Hilbert space $\tilde{\mathfrak{H}} = L^2(\mathbb{R}, \mathfrak{N}, d\sigma)$. Elements of $\tilde{\mathfrak{H}}$ are measurable functions $f(\lambda)$, $\lambda \in \mathbb{R}$, with values in $\mathfrak{N}$ such that

$$\int_{\mathbb{R}} \big(d\sigma(\lambda) f(\lambda), f(\lambda)\big)_{\mathfrak{N}} < \infty.$$

The space $\mathfrak{N}$ is identified with the subspace of $L^2(\mathbb{R}, \mathfrak{N}, d\sigma)$ which consists of constant functions. The orthogonal resolution of identity $\tilde{E}$ is defined as $\tilde{E}(\Delta) f(\lambda) = \chi_\Delta(\lambda) f(\lambda)$, where χ_Δ is the indicator function of the set Δ.

The self-adjoint operator $\tilde{H}$ is defined as follows:

$$\mathfrak{D}(\tilde{H}) = \left\{ f \in \tilde{\mathfrak{H}} \mid \int_{\mathbb{R}} (1+\lambda^2)\left(d\sigma(\lambda) f(\lambda), f(\lambda)\right)_{\mathfrak{N}} < \infty \right\}, \tag{5}$$

$$(\tilde{H} f)(\lambda) = \lambda f(\lambda), \quad f \in \mathfrak{D}(\tilde{H}). \tag{6}$$

From (4) it follows that $\tilde{H}$ is an unbounded operator.

Let

$$\mathfrak{D}(\tilde{\mathcal{H}}) = \left\{ f \in \mathfrak{D}(\tilde{H}) \mid \int_{\mathbb{R}} (\lambda + i)\, d\sigma(\lambda) f(\lambda) = 0 \right\} \tag{7}$$

and

$$(\tilde{\mathcal{H}} f)(\lambda) = \lambda f(\lambda), \quad f \in \mathfrak{D}(\tilde{\mathcal{H}}). \tag{8}$$

$\mathfrak{D}(\tilde{\mathcal{H}})$ is a linear manifold, dense in $\tilde{\mathfrak{H}}$ (this fact follows from (4)), and $(\tilde{\mathcal{H}} f, g) = (f, \tilde{\mathcal{H}} g)$ for $f, g \in \mathfrak{D}(\tilde{\mathcal{H}})$. Thus, $\tilde{\mathcal{H}}$ is a symmetric operator. Moreover, condition (7) implies, that $\mathfrak{N} = [(\tilde{\mathcal{H}} + iI)\mathfrak{D}(\tilde{\mathcal{H}})]^{\perp} = \mathfrak{N}_i$. Indeed, for $f \in L^2(\mathbb{R}, \mathfrak{N}, d\sigma)$ put $f_0 = \int d\sigma(\lambda) f$. Then we have $f = (\lambda + i)g + h$, where $g = (f - f_0)/(\lambda + i) \in \mathfrak{D}(\tilde{\mathcal{H}})$, $h = f_0 \perp (\lambda + i)g$. Therefore, one of the defect numbers of $\tilde{\mathcal{H}}$ is m. It is easily seen that $\mathfrak{N}_{-i} = \{(\lambda - i)(\lambda + i)^{-1}\xi \mid \xi \in \mathfrak{N}\}$, which means that $\dim \mathfrak{N}_{-i} = m$, and the defect index of $\tilde{\mathcal{H}}$ is (m, m). In general, for arbitrary nonreal z the defect subspace is $\mathfrak{N}_z = \{(\lambda - i)(\lambda - z)^{-1}\xi \mid \xi \in \mathfrak{N}\}$.

The Weyl–Titchmarsh function for the pair $(\tilde{\mathcal{H}}, \tilde{H})$ is

$$M_{\tilde{\mathcal{H}}, \tilde{H}} = P_+(zH + I)(H - zI)^{-1}|_{\mathfrak{N}_i} = \int_{\mathbb{R}} \frac{z\lambda + 1}{\lambda - z}\, d\sigma(\lambda)$$

and coincides with the given function F. Uniqueness of this realization up to unitary equivalence is provided by Najmark's theorem which was formulated above. □

Combining results of Theorems 1 and 2 we obtain the following statement (see [7,8]).

Corollary 1. *Let $\mathcal{H}$ be a prime symmetric operator on a Hilbert space $\mathfrak{H}$ with index of defect (m, m) $(m < \infty)$, and let H be a self-adjoint extension of $\mathcal{H}$ in $\mathfrak{H}$. Let $M_{\mathcal{H},H}(z)$ be the Weyl–Titchmarsh function of the pair $(\mathcal{H}, H)$. Let $(\tilde{\mathfrak{H}}, \tilde{\mathcal{H}}, \tilde{H})$ be the realization of $M_{\mathcal{H},H}$ described in Theorem 2. Then, there is a unitary operator $\Phi : \mathfrak{H} \to \tilde{\mathfrak{H}}$ such that*

$$\tilde{\mathcal{H}} = \Phi \mathcal{H} \Phi^* \tag{9}$$

and

$$\tilde{H} = \Phi H \Phi^*. \tag{10}$$

Let U be a unitary operator on $\mathfrak{H}$, and $\tilde{U} = \Phi U \Phi^*$ be its representation in the model space $\tilde{\mathfrak{H}}$. We say that the operator U is of *shift-type* (s-type) operator if for $f \in \tilde{\mathfrak{H}}$,

$$(\tilde{U} f)(\lambda) = D \frac{\lambda - i}{\lambda - i - b} f(\lambda - b), \tag{11}$$

where D is a unitary operator on $\mathfrak{N}$ which commutes with $\sigma(\lambda)$, and where b is a real number.

Often it is more convenient to use the following realization of F (see [7,8]). Let

$$d\tau(\lambda) = (1+\lambda^2)\,d\sigma(\lambda); \tag{12}$$

then,

$$F(z) = \int_{-\infty}^{\infty} \left[\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\right] d\tau(\lambda). \tag{13}$$

The mapping given by $W : L^2(\mathbb{R}, \mathfrak{N}, d\sigma) \to L^2(\mathbb{R}, \mathfrak{N}, d\tau)$, where $(Wf)(\lambda) = f(\lambda) \times (\lambda - i)^{-1}$ is then unitary. For the self-adjoint operator $\hat{H} = WHW^*$, we then have

$$\mathfrak{D}(\hat{H}) = \left\{ f \in L^2(\mathbb{R}, \mathfrak{N}, d\tau) \;\Big|\; \int_{\mathbb{R}} (1+\lambda^2)\big(d\tau(\lambda) f(\lambda), f(\lambda)\big)_{\mathfrak{N}} < \infty \right\}$$

and $\hat{H} f(\lambda) = \lambda f(\lambda)$.

For the symmetric operator $\hat{\mathcal{H}} = W\mathcal{H}W^*$ the following properties hold:

(i) $\mathfrak{D}(\hat{\mathcal{H}}) = \left\{ f \in \mathfrak{D}(\hat{H}) \;\Big|\; \int_{\mathbb{R}} f(\lambda)\, d\tau(\lambda) = 0 \right\}$;

(ii) $(\hat{\mathcal{H}} f)(\lambda) = \lambda f(\lambda)$;

(iii) $\mathfrak{N}_z = \left\{ \frac{1}{\lambda - z}\xi \;\Big|\; \xi \in \mathfrak{N} \right\}$.

In such a representation, the s-type unitary operator U acts in the following way:

$$(\hat{U} f)(\lambda) = Df(\lambda - b).$$

For further development of theory of the Weyl–Titchmarsh functions and their applications we refer readers to [2,4–13,15] and references therein.

3. Periodic operators

Let $\mathcal{H}$ be a prime symmetric operator with index of defect (m, m), $m < \infty$, and let H be its orthogonal self-adjoint extension. In this section we study pairs $(\mathcal{H}, H)$ for which the Weyl–Titchmarsh function is b-periodic, that is

$$M_{\mathcal{H},H}(z) = M_{\mathcal{H},H}(z + b), \tag{14}$$

where b is some real number.

We start from the following lemma.

Lemma 1. *Let $F(z)$ be a function whose values are linear operators on the m-dimensional space $\mathfrak{N}$, and which admits the integral representation*

$$F(z) = \int_{-\infty}^{\infty} \frac{\lambda z + 1}{\lambda - z}\, d\sigma(\lambda) = zI_{\mathfrak{N}} + (1+z^2)\int_{-\infty}^{\infty} \frac{1}{\lambda - z}\, d\sigma(\lambda),$$

where $\sigma(\lambda)$ *is a nondecreasing function with values on the set of linear operators on* $\mathfrak{N}$ *which satisfies conditions* (3) *and* (4). *The function* $F(z)$ *is* b*-periodic, if and only if*

$$\tau(\Delta + b) = \tau(\Delta) \tag{15}$$

for any $\Delta \in \mathcal{B}(\mathbb{R})$, *where* τ *is defined by* (12).

Proof. In order to prove the lemma we need the following generalization of the Stieltjes inversion formula due to M. Livsic (see [14, Lemma 2.1]):

Let $\sigma(\lambda) = 1/2(\sigma(\lambda+0) + \sigma(\lambda-0))$ $(-\infty < \lambda < \infty)$ *be some function of bounded variation on each finite interval, such that the integral*

$$\Phi(z) = \int_{-\infty}^{\infty} \frac{d\sigma(\lambda)}{\lambda - z}$$

converges absolutely. Let $\varphi(\lambda)$ *be some function analytic on the closed interval* $\Delta = [\alpha, \beta]$. *Denote by* Δ_ϵ *the broken path of integration consisting of directed segment* $[\alpha - i\epsilon, \beta - i\epsilon]$ *and antiparallel segment* $[\beta + i\epsilon, \alpha + i\epsilon]$. *Then*

$$\lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{\Delta_\epsilon} \varphi(z)\Phi(z)\, dz = -\int_{\alpha}^{\beta} \varphi(\lambda)\, d\sigma(\lambda).$$

Fix an orthonormal basis $\{e_j\}_{j=1}^m$ in the space $\mathfrak{N}$. The b-periodicity of the function $F(z)$ yields

$$b\delta_{jk} + \left(1 + (z+b)^2\right) \int_{-\infty}^{\infty} \frac{1}{\lambda - b - z}\, d\sigma_{jk}(\lambda) = (1 + z^2) \int_{-\infty}^{\infty} \frac{1}{\lambda - z}\, d\sigma_{jk}(\lambda), \tag{16}$$

$\Im z \neq 0$, and $\sigma_{jk}(\lambda) = (\sigma(\lambda)e_k, e_j)$. Since $\dim \mathfrak{N} = m < \infty$, of all functions $\sigma_{jk},\ j,k = 1,2,\dots,m,$ are of uniform bounded variation and (15) follows from the Livsic's lemma. Indeed, evaluating the integral of both sides of (16) along Δ_ϵ and then taking the limit as $\epsilon \to 0$ we obtain

$$\int_{\alpha}^{\beta} \left[1 + (\lambda+b)^2\right] d\sigma(\lambda+b) = \int_{\alpha}^{\beta} (1+\lambda^2)\, d\sigma(\lambda),$$

which is (15).

Suppose now that (15) is fulfilled. Then we have for $\Im z \neq 0$,

$$F(z+b) - F(z) = \int_{\mathbb{R}} \left[\frac{1}{\lambda - z - b} - \frac{1}{\lambda - z}\right] d\tau(\lambda) = c,$$

where $c = \int_{\mathbb{R}} [\lambda/(1+\lambda^2) - (\lambda+b)/(1+(\lambda+b)^2)]\, d\tau(\lambda)$, and where the integrals converge absolutely. We assume for simplicity that $m = 1$ (for case $m < \infty$ the proof can be done by componentwise arguments). Consider that for the difference

$$\begin{aligned}\left|F(iy+b)-F(iy)\right| &= \left|\int_{\mathbb{R}}\left[\frac{1}{\lambda-iy-b}-\frac{1}{\lambda-iy}\right]d\tau(\lambda)\right| \\ &\leqslant b\int_{\mathbb{R}}\frac{d\tau(\lambda)}{\sqrt{\lambda^2+y^2}\sqrt{(\lambda-b)^2+y^2}}.\end{aligned}$$

Then, for large y we note that

$$1\Big/\left(\sqrt{\lambda^2+y^2}\sqrt{(\lambda-b)^2+y^2}\right)\leqslant 1\Big/\left(\sqrt{\lambda^2+1}\sqrt{(\lambda-b)^2+1}\right);$$

therefore, for a given $\epsilon>0$, there is $A>0$ such that

$$\left(\int_{-\infty}^{-A}+\int_{A}^{\infty}\right)\frac{d\tau(\lambda)}{\sqrt{\lambda^2+y^2}\sqrt{(\lambda-b)^2+y^2}}<\frac{\epsilon}{2}$$

uniformly with respect to y. Now using the fact that $\int_{\mathbb{R}} d\tau(\lambda)/(1+\lambda^2)=1$, we see that for sufficiently large y,

$$\int_{-A}^{A}\frac{d\tau(\lambda)}{\sqrt{\lambda^2+y^2}\sqrt{(\lambda-b)^2+y^2}}\leqslant\frac{1+A^2}{y^2}<\frac{\epsilon}{2}.$$

It follows that $c=0$, and $F(z+b)=F(z)$; thereby, proving the lemma. □

Definition. An operator T acting on a Hilbert space $\mathfrak{H}$ with domain $\mathfrak{D}(T)$ is said to be *(U,b)-periodic,* if there is a unitary operator U such that

$$U\mathfrak{D}(T)\subset\mathfrak{D}(T), \tag{17}$$

$$UTU^*=T-bI \tag{18}$$

for some number $b\neq 0$.

Of course, a periodic operator cannot be bounded. One can easily see that if the operator T^* exists, then it is $(U,\bar{b})$-periodic.

We say that prime symmetric operator $\mathcal{H}$ in $\mathfrak{H}$ and its self-adjoint extension H form a *(U,b)-periodic pair*, if conditions (17) and (18) are fulfilled for both $\mathcal{H}$ and H with the same unitary operator U.

It is evident, that if $\mathcal{H}$ is a (U,b)-periodic operator, and $\mathfrak{N}_z$ is a defect subspace of $\mathcal{H}$, then $U\mathfrak{N}_z=\mathfrak{N}_{z+b}$.

Proposition 1. *Let $\mathcal{H}$ be a prime symmetric operator, and let $H\supset\mathcal{H}$ be its self-adjoint extension such that the pair $(\mathcal{H},H)$ is (U,b)-periodic and (V,b)-periodic. Then the unitary operator $W=V^*U$ has following properties*:

(1) *W commutes with H;*
(2) *each defect subspace $\mathfrak{N}_z$ reduces W;*

(3) *if $\mathcal{H}$ has defect index (m,m), $m<\infty$, then the spectrum of W consists of finite number of eigenvalues; number of distinct eigenvalues not greater than m.*

Indeed, properties (1) and (2) follow directly from the definitions above. The property (3) follows from the fact that the operator W commutes with the resolution of identity $E(\lambda)$ associated with H, c.l.h.$\{E(\Delta)\mathfrak{N} \mid \Delta \in \mathcal{B}(\mathbb{R})\} = \mathfrak{H}$, where $\mathfrak{N}$ is a defect subspace of $\mathcal{H}$, and the spectrum of $W|\mathfrak{N}$ consists of finite numbers of eigenvalues.

Theorem 3. *Let $\mathcal{H}$ be a prime symmetric operator on a Hilbert space $\mathfrak{H}$ with defect index (m,m) $(m<\infty)$, and let H be its self-adjoint extension in $\mathfrak{H}$. Then the following conditions are equivalent*:

(1) *the Weyl–Titchmarsh function $M_{\mathcal{H},H}(z)$ of the pair $(\mathcal{H},H)$ is b-periodic*;
(2) *the pair $(\mathcal{H},H)$ is (U,b)-periodic, where U is an s-type operator.*

Proof. Let the pair $(\mathcal{H},H)$ have a b-periodic Weyl–Titchmarsh function. Let $(\tilde{\mathfrak{H}},\tilde{\mathcal{H}},\tilde{H})$ be the realization of $(\mathfrak{H},\mathcal{H},H)$, described in Theorem 2. According to Lemma 1 the function $\sigma(\lambda)$ satisfies the periodicity condition

$$\bigl(1+(\lambda+b)^2\bigr)\,d\sigma(\lambda+b) = (1+\lambda^2)\,d\sigma(\lambda).$$

On the space $\tilde{\mathfrak{H}} = L^2(\mathbb{R},\mathfrak{N}_i,d\sigma)$ consider the operator $\tilde{U}: f\to \tilde{U}f$ defined by

$$(\tilde{U}f)(\lambda) = \frac{\lambda-i}{\lambda-b-i}\,f(\lambda-b). \tag{19}$$

The operator $\tilde{U}$ is a unitary operator in $L^2(\mathbb{R},\mathfrak{N}_i,d\sigma)$. Indeed,

$$\begin{aligned}(\tilde{U}f,\tilde{U}f) &= \int_{-\infty}^{\infty} \frac{\lambda^2+1}{1+(\lambda-b)^2}\bigl(d\sigma(\lambda)f(\lambda-b),\, f(\lambda-b)\bigr)\\ &= \int_{-\infty}^{\infty} \frac{1+(\lambda-b)^2}{1+(\lambda-b)^2}\, d\bigl(\sigma(\lambda-b)f(\lambda-b),\, f(\lambda-b)\bigr) = (f,f).\end{aligned}$$

The domain of the operator $\tilde{\mathcal{H}}$ is invariant under $\tilde{U}$. For $f\in\mathfrak{D}(\tilde{\mathcal{H}})$, that is

$$\int_{\mathbb{R}} (\lambda+i)\,d\sigma(\lambda)f(\lambda) = 0,$$

we have

$$\begin{aligned}\int_{-\infty}^{\infty} (\lambda+i)\,d\sigma(\lambda)(Uf)(\lambda) &= \int_{-\infty}^{\infty} \frac{\lambda^2+1}{\lambda-i-b}\,d\sigma(\lambda)f(\lambda-b)\\ &= \int_{-\infty}^{\infty} \frac{1+(\lambda-b)^2}{\lambda-b-i}\,d\sigma(\lambda-b)f(\lambda-b) = \int_{-\infty}^{\infty}(\lambda+i)\,d\sigma(\lambda)f(\lambda) = 0.\end{aligned}$$

Similar calculations show that if $f \in \mathfrak{D}(\tilde{H})$, then $\tilde{U} f \in \mathfrak{D}(\tilde{H})$, and $\tilde{U}\tilde{H} f = (\tilde{H} - bI)\tilde{U} f$. Therefore, $(\tilde{\mathcal{H}}, \tilde{H})$ is the $(\tilde{U}, b)$-periodic pair. Therefore, the pair $(\mathcal{H}, H)$ is (U, b)-periodic, and U is s-type operator.

Conversely, let $(\mathcal{H}, H)$ be a (U, b)-periodic pair, with operator U of s-type. Therefore, in the realization $(\tilde{\mathfrak{H}}, \tilde{\mathcal{H}}, \tilde{H})$ the pair $(\tilde{\mathcal{H}}, \tilde{H})$ is $(\tilde{U}, b)$-periodic, with $\tilde{U}$ of the form (11). From the equation $\tilde{U}\tilde{H}\tilde{U}^* = \tilde{H} - bI$ it follows that the resolution of identity $\tilde{E}(\lambda)$ of the operator $\tilde{H}$ satisfies the condition

$$\tilde{U}\tilde{E}(\lambda)\tilde{U}^* = \tilde{E}(\lambda + b). \tag{20}$$

If $\hat{\mathfrak{N}}_i$ is the defect subspace of the operator $\tilde{U}\tilde{H}\tilde{U}^*$, then $\hat{\mathfrak{N}}_i = \mathfrak{N}_{i+b}$. Let $\{e_j\}$ be an orthonormal basis in $\mathfrak{N}$. Then

$$\tilde{U} e_j = \frac{\lambda - i}{\lambda - i - b} D e_j, \quad j = 1, 2, \ldots, m,$$

is the orthonormal basis in $\hat{\mathfrak{N}}_i = \mathfrak{N}_{i+b}$. Now Theorem 1 gives

$$\sigma_{jk}(\lambda) = \bigl(\tilde{E}(\lambda) e_k, e_j\bigr) = \bigl(\tilde{E}(\lambda + b)\tilde{U} e_k, \tilde{U} e_j\bigr) = \int\limits_{-\infty}^{\lambda+b} \frac{1 + s^2}{1 + (s - b)^2}\, d\sigma(s),$$

from which we get $(1 + \lambda^2)\, d\sigma(\lambda) = (1 + (\lambda + b)^2)\, d\sigma(\lambda + b)$.

Therefore, the function σ satisfies the condition of Lemma 1, and $M_{\mathcal{H},H}(z)$ is the b-periodic function. The theorem is proved. □

Remark. It can be proved, that if $(\mathcal{H}, H)$ is a (U, b)-periodic pair, where index of defect of $\mathcal{H}$ is $(1, 1)$, then the unitary operator U is necessarily of s-type.

Lemma 2. *Let $\mathcal{H}$ be a (U, b)-periodic prime symmetric operator with finite and equal defect numbers, and let $(\mathcal{H}, H_0)$ is a (U, b)-periodic pair. Define operator functions $\mathcal{A}(z)$ and $\mathcal{B}(z)$ by the equations*

$$\mathcal{A}(z) = \int\limits_{\mathbb{R}} \frac{\lambda - i}{\lambda - z}\, d\sigma_0(\lambda), \tag{21}$$

$$\mathcal{B}(z) = \int\limits_{\mathbb{R}} \frac{\lambda + i}{\lambda - z}\, d\sigma_0(\lambda), \tag{22}$$

where $\sigma_0(\lambda) = P_+ E_0(\lambda)|_{\mathfrak{N}_i}$, and where $E_0(\lambda)$ is the resolution of identity for H_0. Then the functions $\mathcal{A}$ and $\mathcal{B}$ satisfy the following identities:

$$\mathcal{A}(z + b) = \frac{z + i}{z + b + i}\mathcal{A}(z), \tag{23}$$

$$\mathcal{B}(z + b) = \frac{z - i}{z + b - i}\mathcal{B}(z). \tag{24}$$

Proof. We prove identity for $\mathcal{A}$. Identity for $\mathcal{B}$ is proved similarly,

$$\begin{aligned}\mathcal{A}(z+b) &= \int \frac{\lambda - i}{\lambda - z - b}\, d\sigma_0(\lambda)\\ &= \frac{1}{z+b+i}\int\left[\frac{1}{\lambda - z - b} - \frac{1}{\lambda + i}\right](1+\lambda^2)\, d\sigma_0(\lambda).\end{aligned}$$

Since $(\mathcal{H}, H_0)$ is the (U, b)-periodic pair, the Weyl–Titchmarsh function $M_{\mathcal{H},H_0}(z)$ for the pair has period b, from which it follows, that the measure $d\tau_0(\lambda) = (1+\lambda^2)\, d\sigma_0(\lambda)$ also has period b. This condition provides that

$$\int\left[\frac{1}{\lambda - z - b} - \frac{1}{\lambda + i}\right] d\tau_0(\lambda) = \int\left[\frac{1}{\lambda - z} - \frac{1}{\lambda + i}\right] d\tau_0(\lambda),$$

and the statement regarding the function $\mathcal{A}(z)$ follows. $\square$

Corollary 2. *Let $\mathcal{H}$ be a prime symmetric operator in the Hilbert space $\mathfrak{H}$ with index of defect (m, m), and H_0 be its orthogonal self-adjoint extension such that the pair $(\mathcal{H}, H_0)$ is a (U, b)-periodic. Then for any other orthogonal self-adjoint extension H of the operator $\mathcal{H}$ the corresponding pair $(\mathcal{H}, H)$ is a (U', b)-periodic with some unitary operator U'.*

Proof. In light of Theorem 1 it is enough to show that periodicity of $M_{\mathcal{H},H_0}(z)$ implies periodicity of $M_{\mathcal{H},H}(z)$.

Let σ_0 be the nondecreasing operator valued function which provides the integral representation of the $M_{\mathcal{H},H_0}(z)$. Consider the functional model for the pair $(\mathcal{H}, H_0)$. Then, according to the von Neumann formulas, the domain $\mathfrak{D}(H)$ of the self-adjoint extension H of the operator $\mathcal{H}$ consists of the functions $f(\lambda) \in L^2(\mathbb{R}, \mathfrak{N}_i, d\sigma_0)$ which can be written as

$$f = g + (\varphi_i - V\varphi_{-i}), \tag{25}$$

where $g \in \mathfrak{D}(\mathcal{H})$, that is $\int_{\mathbb{R}}(\lambda + i)g(\lambda)\, d\sigma_0(\lambda) = 0$, $\varphi \in \mathfrak{N}_i$, $\varphi_{-i} \in \mathfrak{N}_{-i}$, $\|\varphi\| = \|\varphi_{-i}\|$, and V is a unitary operator in $\mathfrak{N}_{-i}$. We also have that for $f \in \mathfrak{D}(H)$ $Hf = \mathcal{H}g + i(\varphi_i + V\varphi_{-i})$.

From the definition of the Weyl–Titchmarsh function of the pair we have that

$$\frac{M_{\mathcal{H},H}(z) - M_{\mathcal{H},H_0}(z)}{1+z^2} = P_+\big[R(z) - R_0(z)\big]\big|_{\mathfrak{N}_i},$$

where R and R_0 are resolvents of H and H_0, respectively. Calculating the difference of resolvents, we get the following expression:

$$\frac{M_{\mathcal{H},H}(z) - M_{\mathcal{H},H_0}(z)}{1+z^2} = \mathcal{A}(z)(I - V)\big[(i+z)\mathcal{A}(z)V + (i-z)\mathcal{B}(z)\big]^{-1}\mathcal{B}(z), \tag{26}$$

where $\mathcal{A}(z)$ and $\mathcal{B}(z)$ are defined by (21) and (22). Now using formulas (23) and (24), we obtain that $M_{\mathcal{H},H}(z) - M_{\mathcal{H},H_0}(z) = M_{\mathcal{H},H}(z+b) - M_{\mathcal{H},H_0}(z+b)$, and the corollary is proved. $\square$

Let $\mathcal{H}$ be a (U, b)-periodic prime symmetric operator in a Hilbert space $\mathfrak{H}$ with index of defect (m, m) $(m < \infty)$. Fix orthonormal bases $\{\varphi_j\}_{j=1}^m$ in $\mathfrak{N}_i$ and $\{\psi_j\}_{j=1}^m$ in $\mathfrak{N}_{-i}$, and a unitary operator V_0 in $\mathfrak{N}_{-i}$. The matrix of this operator with respect to the basis $\{\psi_j\}_{j=1}^m$

we also denote by V_0. Denote by $\mathfrak{D}(H_0)$ the domain of self-adjoint extension H_0 of the operator $\mathcal{H}$ defined as

$$\mathfrak{D}(H_0) = \left\{ f \in \mathfrak{H} \;\middle|\; f = f_0 + \sum_j c_j(\varphi_j - V_0\psi_j),\ f_0 \in \mathfrak{D}(\mathcal{H}),\ c_j \in \mathbb{C} \right\}.$$

Since $U\mathfrak{D}(\mathcal{H}) = \mathfrak{D}(\mathcal{H})$ the set $U^n\mathfrak{D}(H_0)$ is the domain of another self-adjoint extension H_n of the operator $\mathcal{H}$. The extension H_n is defined by the pair of defect subspaces $\mathfrak{N}_{i+nb}$ and $\mathfrak{N}_{-i+nb}$, and by the unitary operator $V_0^{(n)}$ in the space $\mathfrak{N}_{-i+nb}$. This operator is defined by the condition that its matrix with respect to the basis $\{U^n\psi_j\}$ coincides with the matrix V_0. It is easily seen that $V_0^{(n)} = U^n V_0 U^{*n}|\mathfrak{N}_{-i+nb}$.

The von Neumann theory of self-adjoint extensions provides that the extension H_n can also be characterized in terms of the defect subspaces $\mathfrak{N}_i$ and $\mathfrak{N}_{-i}$; that is, for any unitary operator V_0 on $\mathfrak{N}_{-i}$ there exists unique unitary operator V_n on $\mathfrak{N}_{-i}$, such that

$$\text{l.h.}\{\varphi_j - V_n\psi_j \mid j = 1, 2, \dots, m\} = \text{l.h.}\{U^n\varphi_j - U^n V_0\psi_j \mid j = 1, 2, \dots, m\}. \quad (27)$$

Thus, the extension H_0 is (U^n, nb)-periodic if and only if $V_n = V_0$.

The unitary operator V_n which satisfies (27) can be found as the solution of the system of equations

$$\varphi_j - V_n\psi_j = \sum_{k=1}^{m} \alpha_{kj}(U^n\varphi_k - U^n V_0\psi_k), \quad j = 1, 2, \dots, m. \quad (28)$$

Our previous comments can then be reformulated as follows: *For any unitary V_0, system* (28) *has one and only one unitary solution V_n.*

Acting on both sides of (28) by $\mathcal{H}^* + iI$ and by $\mathcal{H}^* - iI$, we obtain that

$$2i\varphi_j = \sum_{k=1}^{m} \alpha_{kj}\big[(2i + nb)U^n\varphi_k - nbU^n V_0\psi_k\big] \quad (29)$$

and

$$2iV_n\psi_j = \sum_{k=1}^{m} \alpha_{kj}\big[nbU^n\varphi_k - (-2i + nb)U^n V_0\psi_k\big]. \quad (30)$$

If (29) and (30) are fulfilled then Eq. (28) is also fulfilled. Let P_+ and P_- be orthogonal projections onto subspaces $\mathfrak{N}_i$ and $\mathfrak{N}_{-i}$, respectively. Applying P_+ to the both sides of (29) we obtain that Eq. (29) can be written as

$$2i\varphi_j = \sum_{k=1}^{m} \alpha_{kj}\left[(2i + nb)\sum_{l=1}^{m}(U^n\varphi_k, \varphi_l)\varphi_l - nb\sum_{l=1}^{m}(U^n V_0\psi_k, \varphi_l)\varphi_l\right]. \quad (31)$$

Therefore the matrix $\alpha = [\alpha_{kj}]$, $k, j = 1, 2, \dots, m$, satisfies equation

$$2iI = [C_n V_0 + D_n]\alpha, \quad (32)$$

where matrices C_n and D_n are defined by

$$C_n = -nb\big[(U^n\psi_k, \varphi_l)\big]_{k,l=1}^{m}, \qquad D_n = (2i + nb)\big[(U^n\varphi_k, \varphi_l)\big]_{k,l=1}^{m}, \quad (33)$$

and where V_0 means the matrix of operator V_0 with respect to the basis $\{\psi_j\}_{j=1}^{m}$.

Applying the operator P_- to the both sides of (30), we obtain

$$2iV_n\psi_j = \sum_{k=1}^{m}\alpha_{kj}\left[nb\sum_{l=1}^{m}(U^n\varphi_k,\psi_l)\psi_l - (-2i+nb)\sum_{l=1}^{m}(U^nV_0\psi_k,\psi_l)\psi_l\right]. \quad (34)$$

With the introduction of the $m\times m$ matrices

$$A_n = (2i-nb)\big[(U^n\psi_k,\psi_l)\big]_{k,l=1}^{m}, \qquad B_n = nb\big[(U^n\varphi_k,\psi_l)\big]_{k,l=1}^{m}, \quad (35)$$

Eq. (34) can then be written as

$$2iV_n = [A_nV_0 + B_n]\alpha. \quad (36)$$

Thus, from (32) and (36), we deduce that with respect to the basis $\{\psi_j\}$, the matrix of the operator V_n is defined by the expression

$$V_n = T_n(V_0) = [A_nV_0 + B_n][C_nV_0 + D_n]^{-1}. \quad (37)$$

Conversely, suppose that unitary matrices V_0 and V_n are related by (37). Define matrix α as $\alpha = 2i[C_nV_0 + D_n]^{-1}$. With this α and V_n, (31) and (34) hold for all j. Therefore, (29) and (30) also hold.

Letting $T_0 = \mathrm{id}$ (the identity mapping), we obtain the family $\Gamma = \{T_n,\ n\in\mathbb{Z}\}$ of mappings of the set of $m\times m$ unitary matrices into itself. By construction, the mappings T_n possess the property $T_n(T_m(\cdot)) = T_{n+m}(\cdot)$. Therefore the family Γ is a group.

From Corollary 1, we observe that if the trajectory $\{T_k(V_0)\}_{k=-\infty}^{\infty}$ for some initial unitary matrix V_0 is periodic (that is, $T_n(V_0) = V_0$ for some positive integer n), then it is periodic for any other initial unitary matrix with the same period n. In such a situation, the operator $\mathcal{H}$ admits a (U,nb)-periodic self-adjoint extension, where n is the period of the trajectory of an initial unitary matrix V_0. We reformulate this property as a property of the group Γ.

Proposition 2. *Let $\mathcal{H}$ be a (U,b)-periodic prime symmetric operator with index of defect (m,m) and Γ be the associated group of mappings of the set $m\times m$ unitary matrices into itself, defined by* (33), (35), *and* (37). *Then the operator $\mathcal{H}$ admits a periodic self-adjoint extension if and only if the group Γ is of finite order.*

Examples. (a) Let $h(\lambda)$ be a nonnegative bounded function which has period b. Let $d\sigma(\lambda) = h(\lambda)/(1+\lambda^2)\,d\lambda$ and use definition (2). Then, the corresponding function has the period b. In particular, for $h(\lambda) = 1+\sin\lambda$,

$$F(z) = i + e^{iz} - e^{-1}.$$

The function $F(z)$ has the period 2π, and is the Weyl–Titchmarsh function of the pair $(\mathcal{H}, H)$ defined by the formulas (5)–(7).

(b) Let $\mathfrak{H} = L_m^2[0,l]$, and let the operator $\mathcal{H}$ be defined as follows: The domain of $\mathcal{H}$ is the set of all absolutely continuous functions $f(t) = \{f_k(t)\}_{k=1}^{m}\in\mathfrak{H}$, such that $f'\in\mathfrak{H}$, $f(0) = f(l) = 0$, and where

$$\mathcal{H}f(t) = i\frac{df}{dt}. \quad (38)$$

The operator $\mathcal{H}$ has defect index (m, m). The defect subspace $\mathfrak{N}_i$ is generated by the columns of the $m \times m$ matrix $\exp(t) I_m$. There is a one-to-one correspondence between the set of self-adjoint extensions of $\mathcal{H}$ and the $m \times m$ unitary matrices V.

Any self-adjoint extension H_V of $\mathcal{H}$ can then be obtained as follows: The domain of H_V is set of all absolutely continuous functions f from $L^2_m[0, l]$, such that $f' \in L^2_m[0, 1]$, and $f(0) = V f(l)$, where V is a unitary matrix in $\mathbb{C}^m$. For the pair $(\mathcal{H}, H_V)$ the Weyl–Titchmarsh function $M_{\mathcal{H}, H_V}$ is equal to

$$M_{\mathcal{H}, H_V}(z) = -i I_m + \frac{2i}{e^{2l} - 1} \left(e^{l(1-iz)} - 1\right)\left(I_m - e^{-izl} V\right)^{-1}\left(I_m - e^l V\right). \tag{39}$$

This function has period $2\pi/l$. Therefore, the operator (38) and H_V form a $2\pi/l$-periodic pair. The unitary operator U, such that $U\mathcal{H}U^* = \mathcal{H} - (2\pi/l)I$, and with similar equality holding for H_V, is the operator of multiplication by $\exp(-2\pi i t/l)$.

(c) More generally, consider the operator $\mathcal{H}_1 = i\, d/dt + h(t)$ on $L^2_m[0, l]$ with the same domain that above. h is a Hermitian, bounded measurable matrix function. Then the operator $\mathcal{H}_1$ is symmetric with index of defect (m, m). Let H_1 be its self-adjoint extension. Then the Weyl–Titchmarsh function $M_{\mathcal{H}_1, H_1}(z)$ has the period $2\pi/l$.

Finally, we observe that, according to a theorem of M. Livsic [16], a prime symmetric operator with index of defect $(1, 1)$ which admits a quasi-Hermitian extension $\hat{\mathcal{H}}$ without spectrum in the finite complex plane is unitarily equivalent to the operator described in example (b) with $m = 1$ for $l = 2\,\mathrm{tr}(\Im \hat{\mathcal{H}}^{-1}) > 0$. For the definition and some properties of quasi-Hermitian extensions of symmetric operators see [1]. Therefore, we have the following statement.

Theorem 4. *Let $\mathcal{H}$ be a prime symmetric operator with index of defect $(1, 1)$, and H be a self-adjoint extension of $\mathcal{H}$. Suppose that $\mathcal{H}$ admits a quasi-self-adjoint extension $\hat{\mathcal{H}}$ without spectrum. Then, the Weyl–Titchmarsh function $M_{\mathcal{H}, H}(z)$ of the pair $(\mathcal{H}, H)$ is periodic with period equal to $\pi/\mathrm{tr}(\Im \mathcal{H}_v^{-1})$.*

This theorem does not admit generalization to the case of larger defect numbers. Indeed, let $\mathfrak{H} = L^2[0, l]$, and let $0 < \xi < l$. Consider the symmetric operator $\mathcal{H}$ on $\mathfrak{H}$, defined as follows: The domain $\mathfrak{D}(\mathcal{H})$ is the set of all functions $f(t)$ which are absolutely continuous for $0 < t < \xi$ and $\xi < t < l$, $f' \in \mathfrak{H}$, and $f(0) = f(\xi) = f(l) = 0$. For $f \in \mathfrak{D}(\mathcal{H})$, $\mathcal{H} f = i\, df/dt$. The index of defect for $\mathcal{H}$ is equal to $(2, 2)$. This operator admits a quasi-self-adjoint extension $\hat{\mathcal{H}}$ without spectrum, and $\hat{\mathcal{H}}^{-1}$ is dissipative and unicellular (see [3] for definitions and proofs of these properties). The operator $\mathcal{H}$ is isomorphic to the direct sum $\mathcal{H}_1 \oplus \mathcal{H}_2$ of two first order differential operators with zero boundary conditions on $[0, \xi]$ and $[\xi, l]$, respectively. Let H be the self-adjoint extension of $\mathcal{H}_1 \oplus \mathcal{H}_2$ obtained by imposing the following conditions: $f(0) = \omega_1 f(\xi - 0)$, $f(\xi + 0) = \omega_2 f(l)$, where $|\omega_1| = |\omega_2| = 1$. The the Weyl–Titchmarsh function $M_{\mathcal{H}, H}(z)$ of the pair $(\mathcal{H}, H)$ is a 2×2 diagonal matrix

$$M_{\mathcal{H}, H}(z) = \begin{bmatrix} M_1(z) & 0 \\ 0 & M_2(z) \end{bmatrix},$$

where

$$M_1(z) = -i + 2i(e^{\xi(1-iz)} - 1)(1 - \omega_1 e^{\xi})\big/\big[(e^{2\xi} - 1)(1 - \omega_1 e^{-iz\xi})\big],$$
$$M_2(z) = -i + 2i(\omega_2 e^{l} - e^{\xi})(e^{l} e^{-(l-\xi)iz} - e^{\xi})\big/\big[(e^{2l} - e^{2\xi})(\omega_2 - e^{-iz(l-\xi)})\big]$$

(compare with (39)). M_1 has the period $2\pi/\xi$, function M_2 has the period $2\pi/(l-\xi)$. Therefore, if $\xi/(l-\xi)$ is an irrational number, the function $M_{\mathcal{H},H}$ is not periodic.

4. Operators with constant Weyl–Titchmarsh function

Let H be a self-adjoint operator, and let $W(t) = \exp(itH)$, $t \in \mathbb{R}$, be the one-parametric group of unitary operators generated by H. If H is a (U,b)-periodic operator, then the following commutative relation is fulfilled:

$$UW(t) = e^{-itb}W(t)U. \tag{40}$$

So far we have considered the Weyl–Titchmarsh functions which are invariant under some fixed shift b of the argument. Let $F(z)$ be a function whose values are operators on m-dimensional space $\mathfrak{N}$, which admits representation (13), and which is invariant under arbitrary real shift; that is, $F(z+s) = F(z)$ for any real s. In such a situation the function $F(z)$ is constant in each half-plane,

$$F(z) = \begin{cases} iI_{\mathfrak{N}}, & z \in \mathbb{C}_+, \\ -iI_{\mathfrak{N}}, & z \in \mathbb{C}_-. \end{cases} \tag{41}$$

These properties are fulfilled if and only if $d\tau(\lambda) = \pi^{-1}\, d\lambda I_{\mathfrak{N}}$.

We have $F(z) = M_{\tilde{\mathcal{H}},\tilde{H}}(z)$ for the pair $(\tilde{\mathcal{H}}, \tilde{H})$ acting in the Hilbert space $\tilde{\mathfrak{H}} = L^2(\mathbb{R}, \mathfrak{N}, \pi^{-1}d\lambda)$, where

$$\mathfrak{D}(\tilde{H}) = \left\{ f \in L^2(\mathbb{R}, \mathfrak{N}, \pi^{-1}\, d\lambda) \;\Big|\; \int_{\mathbb{R}} (1+\lambda^2)\|f(\lambda)\|^2_{\mathfrak{N}}\, d\lambda < \infty \right\}, \tag{42}$$

$$(\tilde{H}f)(\lambda) = \lambda f(\lambda), \tag{43}$$

$$\mathfrak{D}(\tilde{\mathcal{H}}) = \left\{ f \in \mathfrak{D}(H) \;\Big|\; \int_{\mathbb{R}} f(\lambda)\, d\lambda = 0 \right\}, \tag{44}$$

$$(\tilde{\mathcal{H}}f)(\lambda) = \lambda f(\lambda). \tag{45}$$

According to Theorem 3, for any real number s there is a unitary operator $\tilde{V}(s)$ on $L^2(\mathbb{R}, \mathfrak{N}, \pi^{-1}\, d\lambda)$ such that $\tilde{V}(s)\tilde{H}\tilde{V}^*(s) = \tilde{H} - sI$, and $\tilde{V}(s)\tilde{\mathcal{H}}V^*(s) = \tilde{\mathcal{H}} - sI$. Moreover, according to Theorem 3, operator $\tilde{V}(s)$ acts as follows: $(\tilde{V}(s)f)(\lambda) = f(\lambda - s)$. Therefore, the family $\{\tilde{V}(s)\}$ is a strongly continuous unitary group. If $\tilde{W}(t) = \exp(it\tilde{H})$, then

$$\tilde{V}(s)\tilde{W}(t) = e^{-ist}\tilde{W}(t)\tilde{V}(s), \tag{46}$$

which is Weyl's form of the canonical commutative relation.

Conversely, let $\mathcal{H}$ be a prime symmetric operator with index of defect (m, m), and let H be a self-adjoint extension of $\mathcal{H}$. Let $W(t) = \exp(itH)$. Suppose there is a unitary group $\{V(s)\}$, $s \in \mathbb{R}$, of shift-type operators such that

$$V(s)W(t) = e^{-ist}W(t)V(s). \tag{47}$$

Then the Weyl–Titchmarsh function $M_{\mathcal{H},H}(z)$ of the pair $(\mathcal{H}, H)$ is constant in each half-plane.

Indeed, from Eq. (47) it follows that for $f \in \mathfrak{D}(H)$, $V(s)f \in \mathfrak{D}(H)$ and $V(s)Hf = (H - sI)V(s)f$.

On the other hand, according to Theorem 2, the operator H is unitarily equivalent to the operator $\tilde{H}$ of multiplication by λ on the Hilbert space $L^2(\mathbb{R}, \mathfrak{N}_i, d\sigma(\lambda))$, where $\sigma(\lambda) = P_+E(\lambda)|\mathfrak{N}_i$, and P_+ is the orthogonal projection from $\mathfrak{H}$ onto $\mathfrak{N}_i$. The domain of $\tilde{H}$ in such a representation is the set

$$\mathfrak{D}(\tilde{H}) = \left\{ f \in L^2\big(\mathbb{R}, \mathfrak{N}_i, d\sigma(\lambda)\big) \;\Big|\; \int \lambda^2\big(d\sigma(\lambda)f(\lambda), f(\lambda)\big) < \infty \right\}.$$

The same unitary operator that transforms H to $\tilde{H}$ transforms $\mathcal{H}$ to the symmetric operator $\tilde{\mathcal{H}}$ with domain

$$\mathfrak{D}(\tilde{\mathcal{H}}) = \left\{ f \in \mathfrak{D}(\tilde{H}) \;\Big|\; \int (\lambda + i)\big(d\sigma(\lambda)f(\lambda), f(\lambda)\big) = 0 \right\}.$$

Operators $V(s)$, being of shift-type, are transformed to the operators $\tilde{V}(s)$ which, according to (11), act as follows:

$$\big(\tilde{V}(s)f\big)(\lambda) = D\frac{\lambda - i}{\lambda - i - s} f(\lambda - s), \quad s \in \mathbb{R}.$$

Since $V(s)$, $s \in \mathbb{R}$, are unitary operators on $L^2(\mathbb{R}, \mathfrak{N}_i, d\sigma(\lambda))$, the same arguments that were used in proof of Theorem 3, and statement of Theorem 1 give that the Weyl–Titchmarsh function $M_{\mathcal{H},H}(z)$ of the pair $(\mathcal{H}, H)$ is s-periodic for any real s and, therefore, constant. Thus, we have proven the following theorem.

Theorem 5.[2] *Let $\mathcal{H}$ be a prime symmetric operator with index of defect (m, m), $m < \infty$, $H \supset \mathcal{H}$ be its self-adjoint extension, and let $W(t) = \exp(itH)$ be the unitary group generated by H. Then the following conditions are equivalent*:

(1) *there exists a unitary group $V(s)$ of s-type operators such that $V(s)W(t) = e^{-its}W(t)V(s)$*;
(2) *the Weyl–Titchmarsh function $M_{\mathcal{H},H}(z) = iI_{\mathfrak{N}_i}$ for $z \in \mathbb{C}_+$, and $M_{\mathcal{H},H}(z) = -iI_{\mathfrak{N}_i}$ for $z \in \mathbb{C}_-$, where $\mathfrak{N}_i$, $\dim \mathfrak{N}_i = m$, is the defect subspace of $\mathcal{H}$.*

Let G be the self-adjoint operator such that $V(s) = \exp(isG)$. Then condition (1) means that on a dense subset of $\mathfrak{H}$, $[G, H] = iI$.

[2] K.A. Makarov drew our attention to possible connections between canonical commutation relations and the behavior of the corresponding Weyl–Titchmarsh functions.

Consider the case $m = 1$. The group $\tilde{W}(t) = \exp(it\tilde{H})$ is the group of multiplication by $\exp(i\lambda t)$ in the space $\mathfrak{H} = L^2(\mathbb{R}, \pi^{-1}\,d\lambda)$, and the group $\tilde{V}(s)$ can be selected as the group of shifts, $(\tilde{V}(s)f)(\lambda) = f(\lambda - s)$. This statement follows form the fact that for each s the operator $\tilde{V}(s)$ satisfies $\tilde{V}(s)H = (\tilde{H} - sI)\tilde{V}(s)$, from Proposition 1, and from the group property $(\tilde{V}(s_1 + s_2) = \tilde{V}(s_1)\tilde{V}(s_2))$. Thus, we obtain the statement of the Stone–von Neumann theorem for one degree of freedom (cf. [17]).

Let D be the self-adjoint operator, such that $\tilde{V}(s) = \exp(isD)$. Then

$$\mathfrak{D}(D) = \left\{f \in L^2(\mathbb{R}, \pi^{-1}\,d\lambda) \mid f \in AC(\mathbb{R});\ f' \in L^2(\mathbb{R}, \pi^{-1}\,d\lambda)\right\}, \tag{48}$$

$$(Df)(\lambda) = if'(\lambda). \tag{49}$$

The operator D is the self-adjoint extension of the operator $\mathcal{D}$ defined as follows:

$$\mathfrak{D}(\mathcal{D}) = \left\{f \in L^2(\mathbb{R}, \pi^{-1}\,d\lambda) \mid f \in AC(\mathbb{R});\ f' \in L^2(\mathbb{R}, \pi^{-1}d\lambda);\ f(0) = 0\right\}, \tag{50}$$

$$(\mathcal{D}f)(\lambda) = if'(\lambda). \tag{51}$$

Applying Theorem 5, we can show that the Weyl–Titchmarsh function of the pair $(\mathcal{D}, D)$ is constant: a fact that can be checked by direct calculation. If D_ω and $\tilde{H}_\theta$ are arbitrary self-adjoint extensions of $\mathcal{D}$ and $\tilde{\mathcal{H}}$, respectively, then according to Corollary 1, the Weyl–Titchmarsh functions $M_{\tilde{\mathcal{H}},\tilde{H}_\theta}(z)$ and $M_{\mathcal{D},D_\omega}(z)$ are constant. Therefore, $(\tilde{\mathcal{H}}, \tilde{H}_\theta)$ is unitarily equivalent to $(\tilde{\mathcal{H}}, \tilde{H})$, and $(\mathcal{D}, D_\omega)$ is unitarily equivalent to $(\mathcal{D}, D)$.

Now consider the pair $(\tilde{\mathcal{H}}, \tilde{H})$ defined by (42)–(45), and the pair $(\tilde{\mathcal{H}}, \tilde{H}_\theta)$. According to the von Neumann formulas, we see that

$$\mathfrak{D}(\tilde{H}_\theta) = \left\{f \;\middle|\; f(\lambda) = f_0 + \left(\frac{1}{\lambda - i} - \frac{\theta}{\lambda + i}\right)z\right\}, \tag{52}$$

where $f_0 \in \mathfrak{D}(\mathcal{H})$, $|\theta| = 1$, and $z \in \mathbb{C}$, where

$$(\tilde{H}_\theta f)(\lambda) = \lambda f_0(\lambda) + i\left[1/(\lambda - i) + \theta/(\lambda + i)\right]z, \tag{53}$$

and where $\tilde{H} = \tilde{H}_1$. As pointed out above, the pairs $(\tilde{\mathcal{H}}, \tilde{H})$ and $(\tilde{\mathcal{H}}, \tilde{H}_\theta)$ are unitarily equivalent. In what follows next, we define the unitary operator Γ_θ which transforms $(\tilde{\mathcal{H}}, \tilde{H})$ to $(\tilde{\mathcal{H}}, \tilde{H}_\theta)$; that is, $\Gamma_\theta\tilde{\mathcal{H}}\Gamma_\theta^* = \tilde{\mathcal{H}}$, $\Gamma_\theta\tilde{H}_1\Gamma_\theta^* = \tilde{H}_\theta$.

For $f \in L^2(\mathbb{R}, d\lambda)$ we have

$$f(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\lambda t}F(t)\,dt, \tag{54}$$

where $F \in L^2(\mathbb{R}, dt)$.

The unitary operator Γ_θ such that $\tilde{H}_\theta = \Gamma_\theta\tilde{H}_1\Gamma_\theta^*$ acts as follows:

$$(\Gamma_\theta f)(\lambda) = \theta\hat{f}_+(\lambda) + \hat{f}_-(\lambda), \tag{55}$$

where

$$\hat{f}_\pm(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\lambda t}F(t)\chi_\pm\,dt, \tag{56}$$

and where $\chi_\pm$ are indicators functions of the positive and negative semiaxes, respectively. It is clear that $\Gamma_\theta^* = \Gamma_{\bar\theta}$. For $f \in \mathfrak{D}(\tilde H)$ the function F in (54) satisfies $F' \in L^2(\mathbb{R}, dt)$, and if $f \in \mathfrak{D}(\tilde{\mathcal{H}})$, $F(0) = 0$. From (54)–(56) it follows now that the operator Γ_θ has the desired properties.

Next, consider pair $(\mathcal{D}, D)$ defined by (48)–(51), and the pair $(\mathcal{D}, D_\omega)$. For the operator D_ω we have

$$\mathfrak{D}(D_\omega) = \bigl\{ f \in L^2(\mathbb{R}, d\lambda) \mid f \in AC\bigl([-R,0]\bigr) \cap AC\bigl([0,R]\bigr)\ \forall R > 0; \\ f(0_-) = \omega f(0_+),\ |\omega| = 1;\ f' \in L^2(\mathbb{R}, d\lambda) \bigr\}, \tag{57}$$

$$(D_\omega f)(\lambda) = i f'(\lambda), \tag{58}$$

and $D = D_1$.

The unitary operator J_ω which transforms $(\mathcal{D}, D_1)$ to $(\mathcal{D}, D_\omega)$ is defined as follows:

$$(J_\omega f)(\lambda) = \bigl[\chi_-(\lambda) + \omega \chi_+(\lambda)\bigr] f(\lambda), \tag{59}$$

$J_\omega^* = J_{\bar\omega}$. From (55) and (59) it follows that $\Gamma_\theta J_\omega = J_\omega \Gamma_\theta$.

Let $\tilde W_\theta$ be the unitary group generated by $\tilde H_\theta$, and let $\tilde V_\omega(s)$ be the unitary group generated by D_ω. For example, the group $\tilde V_\omega(s)$ acts as follows: For $s > 0$,

$$\bigl(\tilde V_\omega(s) f\bigr)(\lambda) = \begin{cases} f_-(\lambda - s), & \lambda < 0, \\ \omega f_-(\lambda - s), & 0 \leqslant \lambda \leqslant s, \\ f_+(\lambda - s), & \lambda \geqslant s, \end{cases}$$

and for $s < 0$,

$$\bigl(\tilde V_\omega(s) f\bigr)(\lambda) = \begin{cases} f_-(\lambda - s), & \lambda < s, \\ \bar\omega f_+(\lambda - s), & s \leqslant \lambda < 0, \\ f_+(\lambda - s), & \lambda \geqslant 0. \end{cases}$$

It is clear, that $\Gamma_\theta D_1 = D_1 \Gamma_\theta$, and $J_\omega H_1 = H_1 J_\omega$.

Proposition 3. *Let $\tilde H_\theta$ and D_ω be the operators defined by* (52)–(53) *and* (57)–(58), *respectively. Then for the unitary groups $\tilde W_\theta(t)$ and $\tilde V_\omega(s)$ generated by $\tilde H_\theta$ and D_ω, respectively, the Weyl commutative relation* (46) *is fulfilled, that is*

$$\tilde V_\omega(s) \tilde W_\theta(t) = e^{-its} \tilde W_\theta(t) \tilde V_\omega(s).$$

The proposition follows from the following chain of equalities where above mentioned properties of the operators Γ_θ, J_ω, D_1, and $\tilde H_1$ are used:

$$\begin{aligned} \tilde V_\omega(s) \tilde W_\theta(t) &= J_\omega \tilde V_1(s) J_\omega^* \Gamma_\theta \tilde W_1(t) \Gamma_\theta^* = J_\omega \Gamma_\theta \tilde V_1(s) \tilde W_1(t) \Gamma_\theta^* J_\omega^* \\ &= e^{-ist} J_\omega \Gamma_\theta \tilde W_1(t) \tilde V_1(s) \Gamma_\theta^* J_\omega^* = e^{-ist} \Gamma_\theta \tilde W_1(t) \Gamma_\theta^* J_\omega \tilde V_1(s) J_\omega^* \\ &= e^{-ist} \tilde W_\theta(t) \tilde V_\omega(s). \end{aligned}$$

The last proposition admits reformulation in abstract form.

Proposition 4. *Let F_1 and G_1 be self-adjoint operators with simple spectra acting in a Hilbert space $\mathfrak{H}$. Let $V_1(s) = \exp(iF_1 s)$ and $W_1(t) = \exp(iG_1 t)$ denote the corresponding unitary groups which satisfy* (46). *Then,*

(1) *there are prime symmetric operators F_0 and G_0 which have index of defect* $(1,1)$ *such that $F_0 \subset F_1$ and $G_0 \subset G_1$;*
(2) *for any other self-adjoint extensions F_ω and G_θ of the operators F_0 and G_0, respectively, the corresponding unitary groups $V_\omega(s)$ and $W_\theta(t)$ also satisfy* (46);
(3) *there exists a unitary operator $U_{\theta\omega} : \mathfrak{H} \to L^2(\mathbb{R}, \pi^{-1} d\lambda)$ such that $F_\omega = U^*_{\theta\omega} D_\omega U_{\theta\omega}$, $G_\theta = U^*_{\theta\omega} \tilde{H}_\theta U_{\theta\omega}$, $F_0 = U^*_{\theta\omega} \mathcal{D} U_{\theta\omega}$, and $G_0 = U^*_{\theta\omega} \tilde{\mathcal{H}} U_{\theta\omega}$.*

This proposition follows from the Stone–von Neumann theorem and previous considerations. It also gives a refinement of the Stone–von Neumann theorem. The case $\omega = \theta = 1$ is the best known, and corresponds to momentum and coordinate operators in quantum mechanics.

We consider one more example of a pair with constant Weyl–Titchmarsh function. Let $\mathfrak{H} = L^2(\mathbb{R}, dt)$, and let the self-adjoint operator L be defined by the differential expression

$$Lf = -\frac{1}{\gamma}\frac{d^2 f}{dx^2} + xf, \tag{60}$$

where γ is a real constant. The corresponding self-adjoint operator describes a particle in uniform electrical field. This operator, via Fourier transform, is unitarily equivalent to the self-adjoint operator H defined by

$$\mathfrak{D}(H) = \left\{ f \in L^2(\mathbb{R}, dt) \mid f \in AC(\mathbb{R}),\ f' \in L^2(\mathbb{R}, dt),\ t^2 f(t) \in L^2(\mathbb{R}, dt) \right\},$$
$$(Hf)(t) = i\frac{df}{dt} + \frac{1}{\gamma} t^2 f(t).$$

The operator $\mathcal{H}$ is then defined as follows:

$$\mathfrak{D}(\mathcal{H}) = \left\{ f \in L^2(\mathbb{R}, dt) \mid f \in AC(\mathbb{R}),\ f(0) = 0,\ f' \in L^2(\mathbb{R}, dt),\ t^2 f(t) \in L^2(\mathbb{R}, dt) \right\},$$
$$(\mathcal{H}f)(t) = i\frac{df}{dt} + \frac{1}{\gamma} t^2 f(t).$$

The operator $\mathcal{H}$ is a symmetric operator with index of defect $(1,1)$, and H is the self-adjoint extension of $\mathcal{H}$. For any real s, define a unitary operator U_s on $\mathfrak{H}$ by $(U_s f)(t) = e^{ist} f(t)$. Then, we have $U_s \mathfrak{D}(\mathcal{H}) = \mathfrak{D}(\mathcal{H})$, $U_s \mathfrak{D}(H) = \mathfrak{D}(H)$, and $U_s H U_s^* = (H - sI)$; that is, the pair $(\mathcal{H}, H)$ is (U_s, s)-periodic. From Theorem 5, it follows now that the Weyl–Titchmarsh function of the pair $(\mathcal{H}, H)$ is constant in each half-plane. Therefore, the operator H is unitarily equivalent to the operator of multiplication by independent variable in $L^2(\mathbb{R}, dt)$.

Thus, the pair consisting of self-adjoint operator, generated by the differential expression (60) and its appropriate symmetric restriction has a constant Weyl–Titchmarsh function.

Consider the self-adjoint operator

$$L_1 = L + V,$$

where L is defined by (60), and V is a bounded, measurable, real-valued periodic function. Without loss of generality, we assume that the period of V is 2π. The Fourier series of V,

$$\sum_{k=-\infty}^{\infty} \hat{V}(k)e^{ikx},$$

converges to $V(x)$ a.e., where $\hat{V}(k)$ are the Fourier coefficients of the function V.

Then, using a Fourier transform, one can show that the operator L_1 is unitarily equivalent to the operator

$$H_1 f = i\frac{df}{dt} + \frac{1}{\gamma}t^2 f + \sum_k \hat{V}(k) f(t+k).$$

The operator H_1 is the self-adjoint extension of the symmetric operator $\mathcal{H}_1$with the same domain as the operator $\mathcal{H}$ above. Using the same operator U_s as defined above, we then have

$$U_s H_1 f - H_1 U_s f = -se^{ist} f + e^{ist} \sum_k \hat{V}(k)(1 - e^{isk}) f(t+k),$$

with a similar expression holding for $U_s\mathcal{H}_1 - \mathcal{H}_1 U_s$. Letting $s = 2\pi$, we see that $U_{2\pi} H_1 - H_1 U_{2\pi} = -2\pi U_{2\pi}$, with a similar equation holding for $\mathcal{H}_1$. Therefore, the pair $(\mathcal{H}_1, H_1)$ is 2π-periodic. As a consequence, a 2π-periodic Weyl–Titchmarsh function is possessed by the pair $(\mathcal{L}_1, L_1)$, where $\mathcal{L}_1$ is the symmetric restriction of the Schrödinger operator L_1 with index of defect $(1, 1)$ (the inverse Fourier transform of $\mathcal{H}_1$).

For an example of a Dirac-type operator with constant Weyl–Titchmarsh functions we refer to [4] and the references therein.

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