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What makes a space have large weight? *

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Abstract

In Section 2 of this paper we formulate several conditions (two of them are necessary and sufficient) which imply that a space of small character has large weight. In Section 3 we construct a ZFC example of a 0-dimensional space X of size 2^ω with $w(X)=2^\omega$ and $\chi(X)=\mathrm{nw}(X)=\omega$, we show that CH implies the existence of a 0-dimensional space Y of size ω_1 with $w(Y)=\mathrm{nw}(Y)=\omega_1$ and $\chi(Y)=R(Y)=\omega$, and we prove that it is consistent that 2^ω is as large as you wish and there is a 0-dimensional space Z of size 2^ω such that $w(Z)=\mathrm{nw}(Z)=2^\omega$ but $\chi(Z)=R(Z^\omega)=\omega$.

Key words: Weight; Net weight; Weakly separated; Irreducible base

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1. Introduction

Since $\chi(X)\geqslant|X|$ implies $w(X)=\chi(X)$, one possible answer to the question in the title is that having large character will make a space have large weight. Thus we arrive at the following more interesting problem: What makes a space have weight larger than its character? Discrete spaces give examples of such spaces but the Sorgenfrey line is first countable, has weight 2^ω but it has no uncountable discrete subspace. The reason for the latter space to have weight 2^ω is that it is weakly separated, i.e., one can assign to every point x a neighbourhood U_x such

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that $x \neq y$ implies either $x \notin U_y$ or $y \notin U_x$. So we may ask now whether every first countable space of "large" weight has a "large" weakly separated subspace? This question was the actual starting point of our investigations, and while we found a negative answer to it we also succeeded in finding successively more and more general conditions that ensure having large weight for spaces of small character.

In Section 2 we introduce the notion of *irreducible base of a space* (see Definition 2.3) and investigate its basic properties. This notion is a weakening of weakly separatedness but the existence of such a base still implies that the weight of the space cannot be smaller than its cardinality. The main advantage of this notion, in contrast to weakly separatedness, lies in the fact that, as we will see in Section 3, a large space with an irreducible base might have small net weight.

This leads to the formulation of the following problem:

Problem 1.1. Does every first countable space of uncountable weight contain an uncountable subspace with an irreducible base?

In Section 3 we construct examples. First a ZFC example is given of a space Y with $|Y| = w(Y) = 2^\omega$ and $\chi(Y) = R(Y) = \mathrm{nw}(Y) = \omega$. After seeing that $\chi(Y)R(Y) < w(Y)$ but $\chi(Y)R(Y) \geqslant \mathrm{nw}(Y)$ in the above mentioned example, we asked whether $\mathrm{nw}(X) \leqslant R(X)\chi(X)$ or just $\mathrm{nw}(X) \leqslant R(X^\omega)\chi(X)$ are provable for every T_2 or regular space X. Using a CH a 0-dimensional counterexample is given to the first question and using a ccc forcing argument we disprove the second inequality in Section 3. However we don't know ZFC counterexamples.

Problem 1.2. Is there a ZFC example of a space X satisfying $R(X^\omega) = \chi(X) = \omega$ but $\mathrm{nw}(X) > \omega$?

We know that under MA the cardinality of such a space must be at least 2^ω (see [4]). In [6, p. 30] Todorčevič introduced the axiom (W):

(W) *If X is a regular space with $R(X)^\omega = \omega$ then* $\mathrm{nw}(X) = \omega$,

and he claimed that PFA implies (W).

We use standard topological notation and terminology throughout, cf. [3].

2. Conditions ensuring large weight

Definition 2.1. Given a topological space $\langle X, \tau\rangle$ and a subspace $Y \subset X$ a function f is called a *neighbourhood assignment on* Y iff $f: Y \to \tau$ and $y \in f(y)$ for each $y \in Y$.

The notion of weakly separated spaces and the cardinal function R were introduced by Tkačenko in [5].

Definition 2.2. A space Y is *weakly separated* if we can find a neighbourhood assignment f on Y such that

$$(\forall y \neq z \in Y)[y \notin f(z) \vee z \notin f(y)],$$

moreover

$$R(X) = \sup\{|Y|: Y \subset X \text{ is weakly separated}\}.$$

Obviously $R(X) \leqslant \mathrm{nw}(X)$. Tkačenko asked whether $R(X) = \mathrm{nw}(X)$ is provable for regular spaces. Hajnal and Juhász, in [2], gave several consistent counterexamples using CH and some ccc forcing arguments. However, their spaces were not first countable.

If one wants to construct a first countable space on ω_1 without uncountable weakly separated subspaces a natural idea is to force with finite approximations of a base of such a space. The space X given by a generic filter satisfies $R(X) = \omega$, but without additional assumptions standard density arguments give $w(X) = \omega$, too. To ensure large weight of the generic space we actually needed that the base should satisfy a certain property. As it turned out this notion proved to be useful not only in the special forcing construction. Its definition is now given below.

Definition 2.3. Let X be a topological space. A base $\mathscr{U}$ of X is called *irreducible* if it has an *irreducible decomposition* $\mathscr{U} = \bigcup\{\mathscr{U}_x: x \in X\}$, i.e., (i) and (ii) below hold:

(i) $\mathscr{U}_x$ is a neighbourhood base of x in X for each $x \in X$.

(ii) For each $x \in X$ the family $\mathscr{U}_x^- = \bigcup_{y \neq x} \mathscr{U}_y$ is not a base of X, hence it does not contain a neighbourhood base of x in X.

Let $\mathscr{U}$ be an irreducible base with the irreducible decomposition $\{\mathscr{U}_x: x \in X\}$. Then for each $x \in X$, since $\bigcup_{y \neq x} \mathscr{U}_y$ does not contain a neighbourhood base of x in X, we can fix an open neighbourhood U_x such that

$$(\forall y \in X \setminus \{x\})(\forall V \in \mathscr{U}_y)[x \in V \to V \setminus U_x \neq \emptyset].$$

Let $\mathscr{U}_x^* = \{U \in \mathscr{U}_x: U \subset U_x\}$. Then $\mathscr{U}^* = \bigcup\{\mathscr{U}_x^*: x \in X\}$ is an irreducible base of X and its irreducible decomposition $\{\mathscr{U}_x^*: x \in X\}$ has the following property $(*)$:

$$(\forall x \neq y \in X)(\forall U \in \mathscr{U}_x^*)(\forall V \in \mathscr{U}_y^*)[x \in V \wedge y \in U] \to V \setminus U \neq \emptyset \qquad (*)$$

To simplify our notation we will say that a base $\mathscr{U}$ has property $(*)$ if it has a decomposition $\mathscr{U} = \bigcup\{\mathscr{U}_x: x \in X\}$ satisfying (i) and $(*)$ above. Obviously, any base with property $(*)$ is irreducible. So we established the following lemma:

Lemma 2.4. *A space X has an irreducible base iff it has a base with property* $(*)$.

The next two lemmas establish the basic connection between weakly separatedness, existence of irreducible base and the requirement $w(X) \geqslant |X|$.

Lemma 2.5. *If X is weakly separated, then X has an irreducible base.*

Proof. Let f be a neighbourhood assignment of X witnessing that it is weakly separated. Take $\mathscr{U}_x = \{G \in \tau_X: x \in G \subset f(x)\}$ and $\mathscr{U} = \bigcup\{\mathscr{U}_x: x \in X\}$. Then $\{\mathscr{U}_x: x \in X\}$ is an irreducible decomposition of the base $\mathscr{U}$. □

The converse of this lemma fails as we will see it later (Theorem 3.1).

Lemma 2.6. *If X has an irreducible base, then $w(X) = \chi(X)|X|$.*

Proof. If $\chi(X) \geqslant |X|$ this is trivial, so assume that $\lambda = \chi(X) < |X|$. Consider an irreducible base $\mathscr{U}$ with irreducible decomposition $\{\mathscr{U}_x: x \in X\}$. We can assume that $|\mathscr{U}| = w(X)$ and $|\mathscr{U}_x| \leqslant \lambda$ for each $x \in X$. If $\mathscr{W} \subset \mathscr{U}$ with $|\mathscr{W}| < |X|$, then there is $x \in X$ with $\mathscr{W} \cap \mathscr{U}_x = \emptyset$, so $\mathscr{W}$ can't be a base by the irreducibility of $\mathscr{U}$. Thus $w(X) = |\mathscr{U}| \geqslant |X|$. □

Definition 2.7. Given a topological space X, a subspace $Y \subset X$, a neighbourhood assignment f on Y and a set $N \subset X$ let

$$D_N^f = \{y \in Y: y \in N \subset f(y)\}.$$

The following results show that both weakly separatedness and having an irreducible base may be characterized with the existence of a neighbourhood assignment f such that D_G^f is "small" in some sense for each open G. For example we have the following easy result whose proof we leave to the reader.

Theorem 2.8. *Given a topological space X, a subspace $Y \subset X$ is weakly separated iff there is a neighbourhood assignment f on Y such that $|D_G^f| \leqslant 1$ for each open $G \subset X$.*

Lemma 2.9. *If a space X has an irreducible base, then there is a neighbourhood assignment f on X such that D_G^f is closed and discrete in G for all open $G \subset X$.*

Proof. Let $\mathscr{U}$ be a base of X having a decomposition $\{\mathscr{U}_x: x \in X\}$ with property $(*)$ and fix a neighbourhood assignment f with $f(x) \in \mathscr{U}_x$. Assume on the contrary that $x \in G$ is an accumulation point of D_G^f for some open $G \subset X$. Choose $U \in \mathscr{U}_x$ with $x \in U \subset G$. Pick $y \in D_G^f \cap U$, $y \neq x$. Then $y \in U \in \mathscr{U}_x$, $x \in G \subset f(y) \in \mathscr{U}_y$, and $U \subset f(y)$, which contradicts property $(*)$ of $\mathscr{U}$. □

Theorem 2.10. *The following statements are equivalent for any regular space $\langle X, \tau \rangle$:*

(1) *X has an irreducible base.*

(2) *There is a neighbourhood assignment f on X such that D_G^f is closed and discrete in G for all open $G \subset X$.*

(3) *There is a neighbourhood assignment f on X such that D_G^f is a discrete subspace of X for each open $G \subset X$.*

Proof. (1) → (2). This is just Lemma 2.9.

(2) → (3) Straightforward.

(3) → (1) Fix a neighbourhood assignment on X witnessing (3). Since X is regular we can assume that $f(x)$ is regular open for each $x \in X$ and that $f(x) = \{x\}$ provided x is isolated. Given an open $G \subset X$ set

$$U(G, x) = \left(G \setminus \overline{D_G^f}\right) \cup \{x\}$$

and put

$$\mathscr{U}_x = \left\{U(G, x):\ x \in D_G^f \wedge G \text{ is regular open}\right\}.$$

Since D_G^f is discrete and $x \in D_G^f$, we have that $U(G, x)$ is also open, and $\mathscr{U}_x$ is a neighbourhood base of x because $x \in U(G, x) \subset G$. We claim that $\mathscr{U} = \bigcup\{\mathscr{U}_x:\ x \in X\}$ is an irreducible base because the decomposition $\{\mathscr{U}_x:\ x \in X\}$ has property $(*)$. Assume on the contrary that $x \neq y \in X$, $U(G, x) \in \mathscr{U}_x$, $U(H, y) \in \mathscr{U}_y$ with $\{x, y\} \subset U(G, x) \cap U(H, y)$ and $U(G, x) \subset U(H, y)$. Since $|G| > 1$ and $|f(x)| = 1$ whenever x is isolated in X, it follows that D_G^f cannot contain isolated points from X. But this set is also discrete, so $\overline{D_G^f}$ is nowhere dense in X. Since H is regular open, $U(G, x) = (G \setminus \overline{D_G^f}) \cup \{x\} \subset H$ implies $G \subset H$. Thus $y \in D_G^f$ for $y \in U(G, x) \subset G \subset H \subset f(y)$ and so $y \notin U(G, x)$, which is impossible. □

We don't know if the assumption on the regularity of X is essential in Theorem 2.10.

Next we show that the existence of an f with D_G^f "small" for all open sets G already implies that the weight of our space is large.

Definition 2.11. A topological space Y is *pseudo weakly separated* if it contains a weakly separated subspace Z with $|Z| = |Y|$.

Theorem 2.12. *Let X be a topological space, $Y \subset X$, f be a neighbourhood assignment on Y and $\lambda \leqslant |Y|$ be a regular cardinal. If D_G^f is the union of $< \lambda$ many pseudo weakly separated subspaces for each open $G \subset X$, then $w(X) \geqslant |Y|$.*

Proof. Assume on the contrary that $\mathscr{B}$ is a base with $|\mathscr{B}| < |Y|$ and let $\kappa = |\mathscr{B}|^+ + \lambda$. Since $Y = \bigcup_{G \in \mathscr{B}} D_G^f$, there is a $G \in \mathscr{B}$ with $|D_G^f| \geqslant \kappa$. But D_G^f is the union of $< \lambda$ many pseudo weakly separated subspaces, so one of them has cardinality $\geqslant \kappa$. Thus D_G^f contains a weakly separated subspace Z with $|Z| \geqslant \kappa$. Hence $w(X) \geqslant w(Y) \geqslant w(Z) \geqslant \kappa > |\mathscr{B}| \geqslant w(X)$, which is impossible. □

Since weakly separated spaces have not just large weight but also large net weight, if we assume that D_N^f is like in Theorem 2.12 for all subsets $N \subset X$, the same argument yields that even the net weight of X is large.

Theorem 2.13. *Let f be a neighbourhood assignment on a topological space X and $\lambda \leqslant |X|$ be a regular cardinal. If D_N^f is the union of $<\lambda$ many pseudo weakly separated subspaces for each $N \subset X$, then* $\mathrm{nw}(X) \geqslant |X|$.

The following results show that the above type of "smallness" assumptions on D_G^f can actually be used to characterize spaces of small character and large weight!

Theorem 2.14. *Let κ be a cardinal and X a topological space with $\chi(X) < \kappa$. (If κ is regular then the assumption $\chi(p, X) < \kappa$ for each $p \in X$ would suffice.) Then the following are equivalent:*

(a) $w(X) \geqslant \kappa$.

(b) *There is a subspace $Y \subset X$ of size κ and a neighbourhood assignment f on Y such that D_G^f is right separated for each open $G \subset X$.*

(c) *There is a subspace $Y \subset X$ of size κ, a neighbourhood assignment f on Y and a regular cardinal $\lambda \leqslant \kappa$ such that D_G^f is the union of $<\lambda$ many pseudo weakly separated subspaces for each open $G \subset X$.*

Proof. (a) $\to$ (b). For each $x \in X$ fix a neighbourhood base $\mathscr{B}_x$ of x in X with minimal cardinality. Since $w(X) \geqslant \kappa$ we can construct a sequence $\{y_\eta: \eta < \kappa\} \subset X$ such that for each $\eta < \kappa$ the family $\bigcup_{\xi<\eta} \mathscr{B}_{y_\xi}$ does not contain a base of y_η in X and we can pick an open set $f(\eta) \in \tau_X$ which witnesses this, i.e., $y_\eta \in f(\eta)$ and there is no $U \in \bigcup_{\xi<\eta} \mathscr{B}_{y_\xi}$ with $y_\eta \in U \subset f(\eta)$. We claim that the neighbourhood assignment f on $Y = \{y_\eta: \eta < \kappa\}$ has the property that D_G^f is right separated in its natural order for each open G. Assume on the contrary that there is an open G and $\xi < \kappa$ such that $y_\xi \in D_G^f \cap \overline{\{y_\eta \in D_G^f: \eta > \xi\}}$. Since $H \subset G$ implies $D_G^f \cap H \subset D_H^f$ we can assume that $G \in \mathscr{B}_{y_\xi}$. Then there is $\eta > \xi$ with $y_\eta \in G \cap D_G^f$. Hence $y_\eta \in G \subset f(\eta)$ and $G \in \mathscr{B}_{y_\xi}$, contradicting the choice of y_η and $f(\eta)$.

(b) $\to$ (c) Straightforward.

(c) $\to$ (a) This is immediate from Theorem 2.12. □

Conditions (b) and (c) in Theorem 2.14 have the (perhaps just aesthetic) drawback that the requirements on the subspace Y are external in nature, i.e., they do not only depend on Y. This drawback is eliminated in the following result, which however works only for regular spaces and regular cardinals.

Theorem 2.15. *Let X be a regular topological space and κ a regular cardinal with $\chi(p, X) < \kappa$ for each $p \in X$. Then the following are equivalent:*

(a) $w(X) \geqslant \kappa$.

(b) *There is a subspace $Y \subset X$ of size κ and a neighbourhood assignment $f: Y \to \tau_Y$ such that $|D_G^f| < \kappa$ for each open $G \in \tau_Y$.*

(c) *There is a subspace $Y \subset X$ of size κ and a neighbourhood assignment $f: Y \to \tau_Y$ such that D_G^f is the union of $<\kappa$ many pseudo weakly separated subspaces for each open $G \in \tau_Y$.*

Proof. (a) → (b) If there is a weakly separated subspace $Y \subset X$ with $|Y| = \kappa$ then we are done by Theorem 2.8. Otherwise we have $h(X) < \kappa$ and $d(X) < \kappa$, hence we can pick a dense $D \subset X$ with $|D| < \kappa$. For each $x \in X$ fix a neighbourhood base $\mathscr{B}_x$ of x in X with minimal cardinality containing only regular open sets. Since $w(X) \geqslant \kappa$ we can construct a sequence $\{y_\eta : \eta < \kappa\} \subset X \setminus D$ such that for each $\eta < \kappa$ writing $D_\eta = \{y_\xi : \xi < \eta\}$ the family $\bigcup\{\mathscr{B}_y : y \in D \cup D_\eta\}$ does not contain a base of y_η in X and we can pick a regular open set $f'(\eta)$ which witnesses this, i.e., $y_\eta \in f'(\eta)$ and there is no $U \in \bigcup\{\mathscr{B}_y : y \in D \cup D_\eta\}$ with $y_\eta \in U \subset f'(\eta)$. Let $Y = D \cup \{y_\eta : \eta < \kappa\}$ and define the neighbourhood assignment f on Y by $f(y_\eta) = f'(y_\eta) \cap Y$ for $\eta < \kappa$ and $f(d) = Y$ for $d \in D$. We claim that the space $\langle Y, \tau_Y \rangle$ and the neighbourhood assignment f have the property that $Z_G = D_G^f \setminus D$ is right separated in the order inherited from the indexing for each open G. Assume on the contrary that there is an open G and $\xi < \kappa$ such that $y_\xi \in Z_G \cap \overline{\{y_\eta \in Z_G : \eta > \xi\}}$. Since $H \subset G$ implies $D_G^f \cap H \subset D_H^f$ we can assume that there is $G' \in \mathscr{B}_{y_\xi}$ with $G = G' \cap Y$. Then there is $\eta > \xi$ with $y_\eta \in G \cap Z_G$. But $D \subset Y$ is dense in X and G' is regular open, so $y_\eta \in G \subset f(\eta)$ implies $G' \subset f'(\eta)$ which contradicts the choice of y_η and $f'(\eta)$. Thus Z_G is right separated and so $|Z_G| < \kappa$. Therefore $|D_G^f| = |Z_G| + |D| < \kappa$, which means that $\langle Y, \tau_Y \rangle$ and the neighbourhood assignment f satisfy (b).

(b) → (c) Straightforward.

(c) → (a) $w(X) \geqslant w(Y) \geqslant \kappa$ by Theorem 2.12. □

3. Examples of spaces with large weight and small character

Denote by $\langle \mathscr{N}, \varepsilon \rangle$ the space of irrational numbers endowed with the Euclidean topology. For $x \in \mathscr{N}$ and $\eta > 0$ write $U(x, \eta) = (x - \eta, x + \eta) \cap \mathscr{N}$.

Theorem 3.1. *There is a set $X \subset \mathscr{N}$ of size 2^ω and a 0-dimensional first countable refinement τ of ε on X such that*

(i) *$X = Y \cup Z$, where $\tau_Y = \varepsilon_Y$ and $\tau_Z = \varepsilon_Z$,*

(ii) *$\langle X, \tau \rangle$ has an irreducible base.*

Thus $\chi(X) = \mathrm{nw}(X) = \omega$ but $w(X) = 2^\omega$.

Proof. Let $Z = \{z_n : n \in \omega\} \subset \mathscr{N}$ be dense. Fix a nowhere dense closed set $Y \subset \mathscr{N} \setminus Z$ of size 2^ω. Let $X = Y \cup Z$. For each $y \in Y$ choose a strictly increasing sequence of pairwise disjoint intervals with rational endpoints, $\mathscr{I}^y = \{I_n^y : n \in \omega\}$, such that $\mathscr{I}^y$ converges to y and $J^y = \bigcup \mathscr{I}^y$ is disjoint from Y. This can be done because Y is nowhere dense. Set $J^z = \emptyset$ for $z \in Z$. For $x \in Y$ and $\eta > 0$ let $V(x, \eta) = (U(x, \eta) \setminus J^x) \cap X$. Let the neighbourhood base of $x \in Y$ in τ be

$$\mathscr{B}_x = \{V(x, \eta) : \eta > 0\}.$$

If $z = z_n \in Z$ then pick $\eta_n > 0$ such that $U(z_n, \eta_n)$ is disjoint from $Y \cup \{z_i: i < n\}$ and put

$$\mathscr{B}_{z_n} = \{U(x, \eta): \eta_n > \eta > 0\}.$$

Since

$$(\forall \eta > 0)(\forall x \in V(y, \eta) \setminus \{y\})(\exists \delta > 0)[U(x, \delta) \cap X \subset V(y, \eta)] \qquad (\dagger)$$

it follows that $\mathscr{B} = \bigcup\{\mathscr{B}_x: x \in X\}$ is a base of a topology. We claim that $\bigcup\{\mathscr{B}_x: x \in X\}$ is an irreducible decomposition of $\mathscr{B}$ because it has property $(*)$. So let $u, v \in X$, $U \in \mathscr{B}_u$, $V \in \mathscr{B}_v$ with $\{u, v\} \subset U \cap V$. Then $u, v \in Y$ because $W \in \mathscr{B}_{z_n}$ implies $W \cap X \subset \{z_k: k \geqslant n\}$. The density of Z in $\mathscr{N}$ implies

$$(\forall \eta > 0)(\forall \delta > 0)[U(y, \delta) \cap X \not\subset V(y, \eta)] \qquad (+)$$

for each $y \in Y$. But $v \in U$ implies that there is some η with $U(v, \eta) \subset U$, so $U \setminus V \neq \emptyset$. Thus the base $\mathscr{B}$ is irreducible. On the other hand, $\tau_Y = \varepsilon_Y$ and $\tau_Z = \varepsilon_Z$, because $U(y, \eta) \cap Y = V(y, \eta) \cap Y$ for $y \in Y$ and $U(z, \eta) \cap Z = V(z, \eta) \cap Z$ for $z \in Z$. □

Definition 3.2. Let $Y \subset \mathscr{N}$. We say that a topological space $\langle Y, \tau \rangle$ is a *standard refinement of* $\langle Y, \varepsilon \rangle$ provided that for each $y \in Y$ we can choose a sequence of pairwise disjoint intervals with rational endpoints, $\mathscr{I}^y = \{I_n^y: n \in \omega\}$, which converges to y such that taking $J^y = \bigcup \mathscr{I}^y$ the family

$$\mathscr{B}_y = \{U(y, \eta) \setminus J^y: \eta > 0\}$$

is a neighbourhood base of y in τ.

Theorem 3.3. *If CH holds, then there is a 0-dimensional first countable standard refinement τ of ε on $\mathscr{N}$ such that*

(i) $R(\langle \mathscr{N}, \tau \rangle) = \omega$,

(ii) $\mathrm{nw}(\langle \mathscr{N}, \tau \rangle) = 2^\omega$,

(iii) $\langle \mathscr{N}, \tau \rangle$ *has an irreducible base.*

Proof. First observe that for each $D \subset \mathscr{N}$ the set $\{x \in \overline{D}: x \notin \overline{(\infty, x) \cap D}\}$ is at most countable. Applying CH and this observation for each $y \in \mathscr{N}$ we can choose a sequence of pairwise disjoint intervals with rational endpoints, $\mathscr{I}^y = \{I_n^y: n \in \omega\}$, which is strictly increasing and converges to y, such that taking $J^y = \bigcup \mathscr{I}^y$ the assumptions (A) and (B) below are satisfied:

(A) $(\forall D \in [\mathscr{N}]^\omega)[\,|\{y \in \overline{D}^\varepsilon: y \notin \overline{D \cap J^y}^\varepsilon\}| \leqslant \omega]$.

To formulate property (B) we need the following notation: for $D \in [\mathscr{N}]^\omega$ and $y \in \mathscr{N}$ write $D[y] = \{d \in D: y \notin J^d\}$.

(B) $(\forall D \in [\mathscr{N}]^\omega)[\,|\{y \in \mathscr{N}: y \in \overline{D[y]}^\varepsilon \wedge D[y] \subset J^y\}| \leqslant \omega]$.

Write $V(y, \eta) = U(y, \eta) \setminus J^y$ for $\eta > 0$. Let the neighbourhood base of y in τ be

$$\mathscr{B}_y = \{V(y, \eta): \eta > 0\}.$$

Since

$$(\forall \eta > 0)(\forall x \in V(y, \eta)\setminus\{y\})(\exists \delta > 0)[U(x, \delta) \subset V(y, \eta)] \qquad (\dagger)$$

it follows that $\mathscr{B} = \{\mathscr{B}_y: y \in \mathscr{N}\}$ is a base of a topology. Since

$$(\forall \eta > 0)(\forall \delta > 0)[U(y, \delta) \not\subset V(y, \eta)] \qquad (+)$$

it follows that the base $\mathscr{B}$ is irreducible.

It is not hard to see that (A) implies that $\mathrm{nw}(\langle \mathscr{N}, \tau\rangle) > \omega$. Indeed, assume on the contrary that $\{M_m: m < \omega\}$ is a network. Pick countable sets $K_m \subset M_m$ with $\overline{K_m}^{\varepsilon} = \overline{M_m}^{\varepsilon}$. Then, by (A), there is $y \in \mathscr{N}$ such that for each $m \in \omega$ either $y \notin \overline{K_m}^{\varepsilon} = \overline{M_m}^{\varepsilon}$ or $y \in \overline{K_m \cap J^y}^{\varepsilon} \subset \overline{M_m \cap J^y}^{\varepsilon}$. Thus there is no $m \in \omega$ with $y \in M_m \subset \mathscr{N}\setminus J^y$.

We will show that (B) implies $R(\langle \mathscr{N}, \tau\rangle) = \omega$. Assume on the contrary that X is an uncountable weakly separated subspace of $\langle \mathscr{N}, \tau\rangle$. Since $\langle \mathscr{N}, \varepsilon\rangle$ has countable weight, we can assume that $x \in J^y$ or $y \in J^x$ holds for each $x \neq y \in X$.

Claim. $(\forall D \in [X]^{\omega})[\,|\{x \in X: x \in \overline{D[x]}^{\varepsilon}\}| \leqslant \omega]$.

Proof. If the above defined set is uncountable, then, by (B), there is $x \in X$ with $D[x] \not\subset J^x$. Let $d \in D[x]\setminus J^x$ be arbitrary. Then $d \notin J^x$ and $x \notin J^d$ which contradicts our assumption on X.

Using this claim, we can find an uncountable subset $Y = \{y_\mu: \mu < \omega_1\}$ of X such that $y_\mu \notin \overline{Y_\mu[y_\mu]}^{\varepsilon}$, where $Y_\mu = \{y_\nu: \nu < \mu\}$. So for each $\mu < \omega_1$ we have an interval K_μ with rational endpoints such that $y_\mu \in K_\mu$ and for each $\nu < \mu$ if $y_\nu \in K_\mu$ then $y_\nu \notin Y_\mu[y_\mu]$, that is, $y_\mu \in J^{y_\nu}$. Since there are only countable many intervals with rational endpoints, we can assume that $K_\nu = K$ for each $\nu < \omega_1$. Since $\mathscr{N}$ does not contain uncountable decreasing sequences, there are $\nu < \mu < \omega_1$ with $y_\nu < y_\mu$. But $J^y \subset (-\infty, y)$ by the construction, which contradicts $y_\mu \in J^{y_\nu}$. So $R(\langle \mathscr{N}, \tau\rangle) = \omega$. □

Let us remark that Todorčević, in [6], proved earlier that CH implies the existence of a 0-dimensional space Y of size ω_1 with $w(Y) = \mathrm{nw}(Y) = \omega_1$ and $\chi(Y) = R(Y) = \omega$.

The next theorem shows that some set theoretic assumption is necessary to construct a standard refinement having the above described properties. To start with let us recall the Open Coloring Axiom (OCA) (see [6] and [1]).

Open Coloring Axiom. *For each second countable T_3 space X and open $H \subset [X]^2$ either* (i) *or* (ii) *below holds*:

(i) *$X = \bigcup_{n\in\omega} X_n$ where X_n is H-independent,*
(ii) *X contains an uncountable H-complete subset.*

Theorem 3.4 (OCA). *If $Y \subset \mathscr{N}$ and $\langle Y, \tau\rangle$ is a standard refinement of $\langle Y, \varepsilon\rangle$ then either $R(Y) > \omega$ or $\langle Y, \tau\rangle$ is σ-second countable.*

Proof. For each $y \in Y$ choose a sequence of pairwise disjoint intervals with rational endpoints, $\mathscr{I}^y = \{I_n^y: n \in \omega\}$, which witnesses that $\langle Y, \tau \rangle$ is a standard refinement. Let $J^y = \bigcup \mathscr{I}^y$.

Unfortunately the set $E' = \{\langle y, y' \rangle \in Y \times Y: y \in J_{y'} \text{ or } y' \in J_y\}$ is not open in $Y \times Y$, so we need some extra work before applying OCA.

Fix an enumeration $\{K_k: k < \omega\}$ of the intervals with rational endpoints. For $y \in Y$ let us define the function $f_y : \omega \to 2$ by taking $f_y(k) = 1$ iff $K_k \subset J_y$. Consider the second countable space $Z = \{\langle y, f_y \rangle: y \in Y\} \subset Y \times D(2)^\omega$ and define the set of edges E on Z as follows:

$$\{\langle y, f_y \rangle, \langle y', f_{y'} \rangle\} \in E \quad \Leftrightarrow \quad (y \in J_{y'} \text{ or } y' \in J_y).$$

It is easy to see that E is open. So OCA implies that either there is an uncountable E-complete $Z' \subset Z$ or Z is the union of countable many E-independent subsets $\{Z_n: n \in \omega\}$.

By the definition of E, if Z' is E-complete, then $Y' = \{y \in Y: \langle y, f_y \rangle \in Z'\}$ is weakly separated. On the other hand, if Z_n is E-independent, then τ and ε agree on $Y_n = \{y \in Y: \langle y, f_y \rangle \in Z_n\}$. □

Theorem 3.5. *For each uncountable cardinal κ there is a ccc poset $\mathscr{P}^\kappa$ of cardinality κ such that in $V^{\mathscr{P}^\kappa}$ there is a 0-dimensional first countable topological space $X = \langle \kappa, \tau \rangle$ and there are ccc posets Q_0 and Q_1 satisfying the following conditions:*

(a) $V^{\mathscr{P}^\kappa} \models$ "*X has an irreducible base*",

(b) $V^{\mathscr{P}^\kappa * Q_0} \models$ "*X is σ-discrete*",

(c) $V^{\mathscr{P}^\kappa * Q_1} \models$ "*X is σ-second countable*".

So, in $V^{\mathscr{P}^\kappa}$, $w(X) = \kappa$ by (a), $\mathrm{nw}(X) = \kappa$ *by* (b) *and $R(X^\omega) = \omega$ by* (c).

Proof. We say that a quadruple $\langle A, n, f, g \rangle$ is in P_0^κ provided (1)–(5) below hold:

(1) $A \in [\kappa]^{<\omega}$,

(2) $n \in \omega$,

(3) f and g are functions,

(4) $f : A \times A \times n \to 2$,

(5) $g : A \times n \times A \times n \to 3$.

For $p \in P_0^\kappa$ we write $p = \langle A^p, n^p, f^p, g^p \rangle$. If $p, q \in P_0^\kappa$ we set $p \leqslant q$ iff $f^p \supseteq f^q$ and $g^p \supseteq g^q$. If $p \in P_0^\kappa$, $\alpha \in A^p$, $i < n^p$ set $U(\alpha, i) = U^p(\alpha, i) = \{\beta \in A^p: f^p(\beta, \alpha, i) = 1\}$.

A quadruple $\langle A, n, f, g \rangle \in P_0^\kappa$ is in P^κ iff (i)–(iv) below are also satisfied.

(i) $(\forall \alpha \in A)(\forall i < n)[\alpha \in U(\alpha, i)]$,

(ii) $(\forall \alpha \in A)(\forall i < j < n)[U(\alpha, j) \subset U(\alpha, i)]$,

(iii) $(\forall \alpha \neq \beta \in A)(\forall i, j < n)[U(\alpha, i) \subset U(\beta, j)$ iff $g(\alpha, i, \beta, j) = 0$, $U(\alpha, i) \cap U(\beta, j) = \emptyset$ iff $g(\alpha, i, \beta, j) = 1]$.

(iv) $(\forall \alpha \neq \beta \in A)(\forall i, j < n)$ [if $\alpha \in U(\beta, j)$ and $\beta \in U(\alpha, i)$ then $g(\alpha, i, \beta, j) = 2$].

We claim that $\mathscr{P}^\kappa = \langle P^\kappa, \leqslant \rangle$ satisfies the requirements.

Definition 3.6. Assume that $p_i = \langle A_i, n_i, f_i, g_i \rangle \in P_0^\kappa$ for $i \in 2$. We say that p_0 and p_1 are *twins* iff $n_0 = n_1$, $|A_0| = |A_1|$ and taking $n = n_0$ and denoting by σ the unique $<$-preserving bijection between A_0 and A_1 we have

(1) $\sigma \lceil A_0 \cap A_1 = \mathrm{id}_{A_0 \cap A_1}$,

(2) σ is an isomorphism between p_0 and p_1, i.e., $(\forall \alpha, \beta \subset A_0)(\forall i, j < n)$,

(a) $f_0(\alpha, \beta, i) = f_1(\sigma(\alpha), \sigma(\beta), i)$,

(b) $g_0(\alpha, i, \beta, j) = g_1(\sigma(\alpha), i, \sigma(\beta), j)$.

We say that σ is the *twin function* of p_0 and p_1. Define the *smashing function* $\bar{\sigma}$ of p_0 and p_1 as follows: $\bar{\sigma} = \sigma \cup \mathrm{id}_{A_1}$. The function σ^* defined by the formula $\sigma^* = \sigma \cup \sigma^{-1} \lceil A_1$ is called the *exchange function* of p_0 and p_1.

Definition 3.7. Assume that p_0 and p_1 are twins and $\varepsilon : A^{p_1} \setminus A^{p_0} \to 2$. A common extension $q \in P^\kappa$ of p_0 and p_1 is called an *ε-amalgamation* of the twins provided

$$(\forall \alpha \in A^{p_0} \Delta A^{p_1})[f^q(\alpha, \sigma^*(\alpha), i) = \varepsilon(\bar{\sigma}(\alpha))].$$

Lemma 3.8. *If $p_0, p_1 \in \mathscr{P}^\kappa$ are twins and $\varepsilon : A^{p_1} \setminus A^{p_0} \to 2$, then p_0 and p_1 have an ε-amalgamation in P^κ.*

Proof. Write $A = A_0 \cup A_1$, $f^- = f_0 \cup f_1$, $g^- = g_0 \cup g_1$. Let B and C be disjoint subsets of $\kappa \setminus A$ of size $|A|$ and let $\rho : B \to A$ and $\eta : C \to A$ be 1–1. Put $q = \langle A \cup B \cup C, n, f, g \rangle$ where

(1) $f^- \subset f$, $g^- \subset g$.

(2) $(\forall \alpha \neq \beta \in A)(\forall i, j < n)$

$$f(\alpha, \beta, i) = \begin{cases} f_1(\bar{\sigma}(\alpha), \bar{\sigma}(\beta), i) & \text{if } \sigma^*(\alpha) \neq \beta, \\ \varepsilon(\bar{\sigma}(\alpha)) & \text{if } \sigma^*(\alpha) = \beta, \end{cases}$$

and

$$g(\alpha, i, \beta, j) = \begin{cases} g_1(\bar{\sigma}(\alpha), i, \bar{\sigma}(\beta), j) & \text{if } \sigma^*(\alpha) \neq \beta, \\ 2 & \text{if } \sigma^*(\alpha) = \beta. \end{cases}$$

(3) $(\forall \beta \in B \cup C)(\forall i < n)[U^q(\beta, i) = \{\beta\}]$.

(4) $(\forall \alpha \in A)(\forall i, j < n)(\forall \beta \in B)$

$$\beta \in U^q(\alpha, i) \quad \text{iff} \quad (\exists l < n)[g(\rho(\beta), l, \alpha, i) = 0]$$

and

$$g(\beta, j, \alpha, i) = \begin{cases} 0 & \text{if } \rho(\beta) \in U^q(\alpha, i), \\ 1 & \text{if } \rho(\beta) \notin U^q(\alpha, i). \end{cases}$$

(5) $(\forall \alpha \in \mathrm{A})(\forall i, l < \mathrm{n})(\forall \gamma \in \mathrm{C})$

$$\gamma \in U^q(\alpha, i) \quad \text{iff} \quad \bar{\sigma}(\eta(\gamma)) \in U^q(\bar{\sigma}(\alpha), i).$$

and

$$g(\gamma, l, \alpha, i) = \begin{cases} 0 & \text{if } \eta(\gamma) \in U^q(\alpha, i), \\ 1 & \text{if } \eta(\gamma) \notin U^q(\alpha, i). \end{cases}$$

Let us remark that (4) and (5) contain no circularity because (1) and (2) define g on A. Obviously $q \in P_0^\kappa$ and $q \leqslant p_0, p_1$, so we have to show that $q \in P^\kappa$. (i) is straightforward. Before checking (ii)–(iv) we need some preparation. If $\alpha, \beta \in A$ and $i,j < n$ write $\langle \alpha, i\rangle \lhd \langle \beta, j\rangle$ iff $g(\alpha, i, \beta, j) = 0$.

Claim 3.8.1. *The relation* $\lhd$ *is transitive on* $A \times n$.

Proof. Assume that $\langle \alpha, i\rangle \lhd \langle \beta, j\rangle \lhd \langle \gamma, l\rangle$. Then, by (2), $\langle \bar\sigma(\alpha), i\rangle \lhd \langle \bar\sigma(\beta), j\rangle \lhd \langle \bar\sigma(\gamma), l\rangle$, so $\langle \bar\sigma(\alpha), i\rangle \lhd \langle \bar\sigma(\gamma), l\rangle$. Thus $\langle \alpha, i\rangle \lhd \langle \gamma, l\rangle$ provided $\sigma^*(\alpha) \neq \gamma$. Assume that $\sigma^*(\alpha) = \gamma$. Then $U_1(\bar\sigma(\alpha), i) \subset U_1(\bar\sigma(\beta), j) \subset U_1(\bar\sigma(\alpha), l)$ and so $\bar\sigma(\alpha) \in U_1(\bar\sigma(\beta), j)$ and $\bar\sigma(\beta) \in U_1(\bar\sigma(\alpha), l)$. Thus $g_1(\bar\sigma(\beta), j, \bar\sigma(\alpha), l) = 2$ because p_1 satisfies (iv). This contradiction proves that $\sigma^*(\alpha) \neq \gamma$.

Claim 3.8.2. $(\forall \alpha \neq \beta \in A)(\forall i, j < n)$ [*if* $g(\alpha, i, \beta, j) = 0$ *then* $U^q(\alpha, i) \subset U^q(\beta, j)$].

Proof. We have $g_1(\bar\sigma(\alpha), i, \bar\sigma(\beta), j) = 0$. Thus

$$U_1(\bar\sigma(\alpha), i) \subset U_1(\bar\sigma(\beta), j). \tag{†}$$

Let $\gamma \in A \cap U^q(\alpha, i)$. If $\sigma^*(\gamma) \neq \beta$ then (†) implies $\gamma \in U^q(\beta, j)$ by (2). Assume now that $\sigma^*(\gamma) = \beta$. Then $\bar\sigma(\beta) = \bar\sigma(\gamma) \in U_1(\bar\sigma(\alpha), i)$ and, on the other hand, $\bar\sigma(\alpha) \in U_1(\bar\sigma(\beta), j)$ by (†). Thus $g_1(\bar\sigma(\alpha), i, \bar\sigma(\beta), j) = 2$ because p_1 satisfies (iv). Contradiction, $\sigma^*(\gamma) \neq \beta$. So we have shown $U^q(\alpha, i) \cap A \subset U^q(\beta, j) \cap A$. Next we can see that $U^q(\alpha, i) \cap B \subset U^q(\beta, j) \cap B$ by Claim 3.8.1 and by (4). Finally let $\gamma \in U^q(\alpha, i) \cap C$. Then $\bar\sigma(\eta(\gamma)) \in U_1(\bar\sigma(\alpha), i)$, so $\bar\sigma(\eta(\gamma)) \in U_1(\bar\sigma(\beta), j)$. Thus $\gamma \in U^q(\beta, j)$ by (5).

Claim 3.8.3. $(\forall \alpha \neq \beta \in A)(\forall i, j < n)$[*if* $g(\alpha, i, \beta, j) = 1$ *then* $U^q(\alpha, i) \cap U^q(\beta, j) = \emptyset$].

Proof. Since $g(\alpha, i, \beta, j) = 1$ we have $\sigma^*(\alpha) \neq \beta$, so $U_1(\bar\sigma(\alpha), i) \cap U_1(\bar\sigma(\beta), j) = \emptyset$ implies $U^q(\alpha, i) \cap U^q(\beta, j) \cap A = \emptyset$. Assume now that $\gamma \in U^q(\alpha, i) \cap U^q(\beta, j) \cap B$. Then $\rho(\gamma) \in U^q(\alpha, i) \cap U^q(\beta, j) \cap A$, contradiction. Finally assume that $\gamma \in U^q(\alpha, i) \cap U^q(\beta, j) \cap C$. Then $\bar\sigma(\eta(\gamma)) \in U_1(\bar\sigma(\alpha), i) \cap U_1(\bar\sigma(\beta), j) = \emptyset$, which is impossible.

Claim 3.8.4. $(\forall \alpha \neq \beta \in A \cup B \cup C)(\forall i, j < n)$ [*if* $g(\alpha, i, \beta, j) = 2$ *then* $\emptyset \neq U^q(\alpha, i) \setminus U^q(\beta, j) \neq U^q(\alpha, i)$].

Proof. The assumption implies $\alpha, \beta \in A$. If $\alpha \neq \sigma^*(\beta)$ then $g(\bar\sigma(\alpha), i, \bar\sigma(\beta), j) = 2$, which implies the statement. So we can assume that $\alpha = \sigma^*(\beta)$. Let $\gamma = \rho^{-1}(\alpha)$. Then $\gamma \in U^q(\alpha, i)$ and, on the other hand, $\gamma \notin U^q(\beta, j)$, because $g(\alpha, l, \beta, j) = 2$ for each $l < n$ by (2). So $\gamma \in U^q(\alpha, i) \setminus U^q(\beta, j)$. Let $\delta = \eta^{-1}(\alpha)$. Then $\bar\sigma(\eta(\delta)) = \bar\sigma(\beta) = \bar\sigma(\alpha) \in U_1(\bar\sigma(\alpha), i) \cap U_1(\bar\sigma(\beta), j)$, so $\delta \in U^q(\alpha, i) \cap U^q(\beta, j)$.

So we have proved (iii) for q, which implies (ii). To check (iv) assume that $\alpha \neq \beta \in A \cup B \cup C$, $i,j < n$ with $\alpha \in U^q(\beta, j)$ and $\beta \in U^q(\alpha, i)$. Then $\alpha, \beta \in A$. If $\sigma^*(\alpha) \neq \beta$, then $\bar\sigma(\sigma) \in U_1(\bar\sigma(\beta), j)$ and $\bar\sigma(\beta) \in U_1(\bar\sigma(\alpha), i)$ implies $g_1(\bar\sigma(\alpha), i, \bar\sigma(\beta), j) = 2$. Hence $g(\alpha, i, \beta, j) = 2$. If $\sigma^*(\alpha) = \beta$, then $g(\alpha, i, \beta, j) = 2$ by definition. The lemma is proved. □

The previous lemma implies that $\mathscr{P}^\kappa$ satisfies ccc because among uncountably many elements of $\mathscr{P}^\kappa$ there are always two twins. Let $\mathscr{G}$ be the $\mathscr{P}^\kappa$ generic filter and let $F=\bigcup\{f^p\colon p\in\mathscr{G}\}$. For each $\alpha<\kappa$ and $n\in\omega$ let $V(\alpha, i)=\{\beta<\kappa\colon F(\beta, \alpha, i)=1\}$. Put $\mathscr{B}_\alpha=\{V(\alpha, i)\colon i<\kappa\}$ and $\mathscr{B}=\bigcup\{\mathscr{B}_\alpha\colon \alpha<\kappa\}$. By standard density arguments we can see that $\mathscr{B}$ is base of a first countable topological space $X=\langle\kappa, \tau\rangle$. Since $\mathscr{P}^\kappa$ satisfies (iv), $\bigcup\{\mathscr{B}_\alpha\colon \alpha<\kappa\}$ is an irreducible decomposition of $\mathscr{B}$.

We are now ready to define the posets Q_0 and Q_1 in $V^{\mathscr{P}^\kappa}$.

A triple $\langle B, d\ e\rangle$ is in Q_0 iff

(a) $B\in[\kappa]^{<\omega}$,

(b) $d: B\to\omega$,

(c) $e: B\to\omega$,

(d) $(\forall\alpha\neq\beta\in B)$[if $d(\alpha)=d(\beta)$ then $\alpha\notin \mathrm{V}(\beta, e(\beta))$].

A quadruple $\langle B, d, m, e\rangle$ is in Q_1 iff

(A) $B\in[\kappa]^{<\omega}$,

(B) $d: B\to\omega$,

(C) $m\in\omega$,

(D) $e: \mathrm{B}\times m\to\omega$,

(E) $(\forall\alpha, \beta, \gamma\in B)(\forall i, j<m)$ [if $d(\alpha)=d(\beta)=d(\gamma)$ and $e(\alpha, i)=e(\beta, j)$ then $\gamma\in V(\alpha, i)$ iff $\gamma\in V(\beta, j)$].

The orderings on Q_0 and Q_1 are defined in the straightforward way. If q and r are compatible elements of Q_i, then denote by $q\wedge r$ their greatest lower bound in Q_i.

Lemma 3.9. *$\mathscr{P}^\kappa * Q_0$ satisfies ccc.*

Proof. Let $\langle\langle p_\nu, q_\nu\rangle\colon \nu<\omega_1\rangle\subset\mathscr{P}^\kappa * Q_0$. We can assume that p_ν decides q_ν. Write $p_\nu=\langle A^\nu, n^\nu, f^\nu, g^\nu\rangle$ and $q_\nu=\langle B^\nu, d^\nu, e^\nu\rangle$. By standard density arguments we can assume that $A^\nu\supset B^\nu$. Applying standard Δ-system and counting arguments we can find $\nu<\mu<\omega_1$ such that p_ν and p_μ are twins and denoting by σ the twin function of p_ν and p_μ we have $d^\nu(\alpha)=d^\mu(\sigma(\alpha))$ and $e^\nu(\alpha)=e^\mu(\sigma(\alpha))$ for each $\alpha\in B^\nu$.

Define the function $\varepsilon^0: A^\mu\setminus A^\nu\to 2$ by the equation $\varepsilon^0(\alpha)=0$. By Lemma 3.8 the conditions p^ν and p^μ have an ε^0-amalgamation p. We claim that

$$p\Vdash q_\nu\wedge q_\mu\in Q_0.$$

It is enough to check that if $\alpha\in B^\nu$, $\beta\in B^\mu$, $d^\nu(\alpha)=d^\mu(\beta)$, then $p\Vdash\alpha\notin V(\beta, e^\mu(\beta))$. If $\sigma^*(\alpha)\neq\beta$, then (2) implies this. If $\sigma^*(\alpha)=\beta$, then $p\Vdash\alpha\notin V(\beta, e^\mu(\beta))$ because p is an ε^0-amalgamation. □

Lemma 3.10. *$P^\kappa * Q_1$ satisfies ccc.*

Proof. Let $\langle\langle p_\nu, q_\nu\rangle\colon \nu<\omega_1\rangle\subset\mathscr{P}^\kappa * Q_1$. We can assume that p_ν decides q_ν. Write $p_\nu=\langle A^\nu, n^\nu, f^\nu, g^\nu\rangle$ and $q_\nu=\langle B^\nu, d^\nu, m^\nu, e^\nu\rangle$. By standard density arguments we can assume that $A^\nu \supset B^\nu$. Applying standard arguments we can find $\nu < \mu < \omega_1$ such that p_ν and p_μ are twins and denoting by σ the twin function of p_ν and p_μ we have $m^\nu = m^\mu$ and $d^\nu(\alpha) = d^\mu(\sigma(\alpha))$ and $e^\nu(\alpha, i) = e^\mu(\sigma(\alpha), i)$ for each $\alpha \in A^\nu$ and $i < m^\mu$.

Define the function $\varepsilon^1 : A^\mu \setminus A^\nu \to 2$ by the equation $\varepsilon^1(\alpha) = 1$. By Lemma 3.8 the conditions p^ν and p^μ have an ε^1-amalgamation p. We claim that

$$p \Vdash q^\nu \wedge q^\mu \in Q_1.$$

Set $d = d^\nu \cup d^\mu$ and $e = e^\nu \cup e^\mu$. Let $\alpha, \beta, \gamma \in B^\nu \cup B^\mu$ with $d(\alpha) = d(\beta) = d(\gamma)$ and $e(\alpha, i) = e(\beta, j)$. We have to show that

$$p \Vdash \text{“}\gamma \in V(\alpha, i) \text{ iff } \gamma \in V(\beta, j)\text{”}. \qquad (*)$$

We know that $p \Vdash$ “$\bar\sigma(\gamma) \in V(\bar\sigma(\alpha), i)$ iff $\bar\sigma(\gamma) \in V(\bar\sigma(\beta), j)$”, so we can assume that $p \Vdash$ “$\bar\sigma(\gamma) \in V(\bar\sigma(\alpha), i)$ and $\bar\sigma(\gamma) \in V(\bar\sigma(\beta), j)$”. So if $\sigma^*(\gamma)$ is different from α and β, then we are done. Assume finally that $\sigma^*(\gamma) = \alpha$. Then $p \Vdash$ “$\bar\sigma(\gamma) \in V(\bar\sigma(\alpha), i)$”, so $p \Vdash$ “$\bar\sigma(\gamma) \in V(\bar\sigma(\beta), j)$”, thus $p \Vdash$ “$\gamma \in V(\beta, j)$” by (2). But p is an ε^1-amalgamation, so $p \Vdash$ “$\sigma^*(\alpha) \in V(\alpha, i)$”, i.e., $p \Vdash$ “$\gamma \in V(\alpha, i)$”. □

Lemma 3.11.

$V^{\mathscr{P}^\kappa * Q_0} \Vdash$ “*X is σ-discrete*”.

Proof. Let $\mathscr{H}$ be the Q_0-generic filter over $V^{\mathscr{P}^\kappa}$. Set $d = \bigcup\{d^q : q \in \mathscr{H}\}$ and $e = \bigcup\{e^q : q \in \mathscr{H}\}$. By standard density arguments the domains of the functions d and e are κ. We have $V(x, e(x)) \cap d^{-1}\{d(x)\} = \{x\}$ by (d), so $d^{-1}(n)$ is discrete for each $n \in \omega$. □

Lemma 3.12.

$V^{\mathscr{P}^\kappa * Q_1} \Vdash$ “*X is σ-second countable*”.

Proof. Let $\mathscr{H}$ be the Q_1-generic filter over $V^{\mathscr{P}^\kappa}$. Take $d = \bigcup\{d^q : q \in \mathscr{H}\}$ and $e = \bigcup\{e^q : q \in \mathscr{H}\}$. By standard density arguments $\operatorname{dom}(d) = \kappa$ and $\operatorname{dom}(e) = \kappa \times \omega$. Fix $n \in \omega$ and let $X_n = d^{-1}\{n\}$. We claim that $w(X_n) = \omega$. Indeed, $\bigcup\{V(\alpha, i) : \alpha \in X_n, i < \omega\}$ is a base of X_n and by (E), if $\alpha, \beta \in X_n$, $i, j < \omega$ and $e(\alpha, i) = e(\beta, j)$ then $V(\alpha, i) \cap X_n = V(\beta, j) \cap X_n$. □

This completes the proof of Theorem 3.5. □

We have shown in [4] that every first countable T_2 space satisfying $R(X^\omega) = \omega$ becomes σ-second countable in a suitable ccc extension. Thus (c) of Theorem 3.5 may be considered as the natural way to insure $R(X^\omega) = \omega$.

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