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J. Math. Anal. Appl. 293 (2004) 108–124
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

A note on singular nonlinear boundary value problems ☆

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Received 30 May 2002
Submitted by P.G.L. Leach

Abstract

In this paper we study the existence and multiplicity of solutions of the following operator equation in Banach space E:

$$u = \lambda A u, \quad 0 < \lambda < +\infty, \; u \in P \setminus \{\theta\},$$

where λ is a parameter and P a cone of Banach space E. Under certain conditions on the operator A we find λ^* such that the operator equation has at least two solutions for $0 < \lambda < \lambda^*$, at least one solution for $\lambda = \lambda^*$ and no solutions for $\lambda > \lambda^*$. As an application, we investigate the existence and multiplicity of positive solutions of a singular second order boundary value problem. In addition, we briefly outline an application of our results which simplifies a previous theorem that appeared in the literature.

Keywords: Singular boundary value problem; Fixed point index; Positive solution

1. Introduction

Consider the following singular boundary value problem

$$\begin{cases} y'' + \lambda f(t) g(y) = 0, & 0 < t < 1, \\ y(0) = 0 = y(1), \end{cases} \tag{1.1}$$

☆ This paper is supported by the Natural Science Foundation of the Jiangsu Education Committee (03KJB110137).
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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2003.12.017

where λ is a parameter, $f \in C((0,1),(0,+\infty))$ and $g \in C(R^+, R^+)$, $R^+ = [0,+\infty)$, $I = [0,1]$.

The boundary value problem (1.1) has been studied by many authors (see [1–6] and some of the references therein). Choi [1] studied the particular case where $g(y) = e^y$, $f \in C^1(0,1]$, $f > 0$ on $(0,1)$ and f can be singular at $t = 0$, but at most $O(t^{\delta-2})$as $t \to 0^+$ for some $\delta > 0$. Using the shooting method he established the following result.

Theorem A [1]. *There exists $\lambda_0 > 0$ such that the BVP* (1.1) *has a solution in $C^2(0,1] \cap C[0,1]$ for $0 < \lambda < \lambda_0$, while there is no solution for $\lambda > \lambda_0$.*

Ha and Lee [2] also studied the BVP (1.1). They proved that if $f > 0$ on $(0,1)$ and $\int_0^1 sf(s)\,ds < +\infty$, g is increasing on R^+ and $g(y) \geqslant e^y$ for all $y \in R^+$, then there exists $\lambda_0 > 0$ such that the BVP (1.1) has no solution for $\lambda > \lambda_0$, but at least one solution for $\lambda = \lambda_0$, and at least two solutions for $0 < \lambda < \lambda_0$.

Let $(E, \|\cdot\|)$ be a real Banach space which is ordered by a cone P, θ the zero element of E. Motivated by [1–6], in this paper we will study the following operator equation in Banach space E:

$$u = \lambda Au, \quad 0 < \lambda < +\infty,\ u \in P \setminus \{\theta\}, \tag{1.2}$$

where λ is a positive parameter.

Our purpose here is to give some existence results for solutions of the operator equation (1.2). The main results of this paper is a generalization of some results in [1–6]. This paper also serves to provide an unified treatment to a variety of results for the existence of multiple solutions of integral equations in $C[I,R]$ and $C[I,R^2]$.

This paper is organized as follows. In Section 2 we shall prove the main results. In Section 3, we apply these results to prove the existence of positive solutions of nonlinear singular boundary value problems. Finally, in Section 4, to illustrate the applications of our main results, we give a different proof of a result in [5].

2. Main results

Let E be a real Banach space, P a total cone of E, i.e., $E = \overline{P - P}$. Let $A = KF : P \mapsto P$ be a completely continuous operator, where $K : P \mapsto P$ is a completely continuous linear operator and $F : P \mapsto P$ is a continuous bounded increasing operator. We will study the existence of solutions of the operator equation (1.2). Assume that there exists a completely continuous linear operator $K_0 : P \mapsto P$ such that $Ku \geqslant K_0 u$ for all $u \in P$, and $r(K_0) > 0$ ($r(K_0)$ denotes the spectral radius of K_0). It follows from the well-known Krein–Rutman theorem [10] that there exist $\phi \in P \setminus \{\theta\}$ and $h \in P^* \setminus \{\theta\}$ such that

$$K_0\phi = r(K_0)\phi, \qquad K_0^* h = r(K_0)h,$$

where K_0^* is the conjugated operator of K_0, P^* is the dual cone of P.

Let τ be a positive number and $P_0 = \{u \in P \mid h(u) \geqslant \tau\|u\|\}$. It is easy to see that P_0 is also a cone of E.

Definition 2.1 [11]. Let $T : D \mapsto E$ be an operator and ζ a nonzero element of P. Let $x_0 \in D$ be fixed. If for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$-\varepsilon\zeta < Tx - Tx_0 < \varepsilon\zeta$$

for all $x \in D$ with $\|x - x_0\| < \delta$, then T is called ζ-continuous at x_0. If for any $x \in D$, T is ζ-continuous at x, then T is called ζ-continuous on D.

Let us list some conditions for convenience.

(H_1) There exists a cone Q of E such that $Q \subset P_0$, $KP \subset Q$, and $K_0 P \subset Q$.

(H_2) F is $F\theta$-continuous on Q, $h(F\theta) > 0$ and

$$\lim_{u \in Q,\, \|u\| \to +\infty} \frac{h(Fu)}{h(u)} = +\infty.$$

(H_3) There exists a positive number m such that

$$h(Fu) \geqslant mh(u), \quad \forall u \in Q,$$

and

$$\lim_{u \in Q,\, u \to \theta} \frac{h(Fu)}{h(u)} = +\infty.$$

Throughout this section, we will assume that (H_1) always holds.

By Lemma 2.3.1 and Corollary 2.3.1 in [8], we have

Lemma 2.1. *Let Ω_1 and Ω_2 be two bounded open sets in E such that $\theta \in \Omega_1$ and $\bar{\Omega}_1 \subset \Omega_2$. Let the operator $A : P \cap \bar{\Omega}_2 \mapsto P$ be completely continuous, and $u_0 \in P \setminus \{\theta\}$. Suppose that*

$$x - Ax \neq tu_0, \quad \forall x \in P \cap \partial\Omega_1,\ t \geqslant 0,$$

and

$$\mu Ax \neq x, \quad \forall x \in P \cap \partial\Omega_2,\ \mu \in [0, 1].$$

Then A has at least one fixed point in $P \cap (\Omega_1 \setminus \bar{\Omega}_2)$.

Lemma 2.2. *Let $\Sigma = \{(\lambda, u) \mid \lambda > 0,\ u \in P \setminus \{\theta\},\ (\lambda, u)$ is a solution of (1.2)$\}$. Assume (H_2) or (H_3) holds. Then $\Sigma \neq \emptyset$.*

Proof. Let $R_0 > 0$ be fixed. Choose $\lambda_0 > 0$ such that

$$\lambda_0 \sup_{u \in \bar{B}_{R_0} \cap Q} \|Au\| < R_0, \tag{2.1}$$

where $B_{R_0} = \{u \in E \mid \|u\| < R_0\}$. Then (2.1) implies that

$$\mu\lambda_0 Au \neq u, \quad \forall u \in \partial B_{R_0} \cap Q,\ \mu \in [0, 1]. \tag{2.2}$$

On the other hand, if (H_2) holds, then we have

$$\lim_{u \in Q,\, u \to \theta} \frac{h(Fu)}{h(u)} \geqslant \lim_{u \in Q,\, u \to \theta} \frac{h(F\theta)}{h(u)} = +\infty,$$

and so

$$\lim_{u\in Q,\, u\to\theta} \frac{h(Fu)}{h(u)} = +\infty. \tag{2.3}$$

Obviously, if (H_3) holds, then (2.3) also holds. For a fixed $\delta > 0$, by (2.3) there exists R_1 with $R_0 > R_1 > 0$ such that

$$h(Fu) \geqslant \frac{1}{\lambda_0 r(K_0)}(1+\delta)h(u), \quad \forall u \in \bar{B}_{R_1} \cap Q. \tag{2.4}$$

Let $\psi \in Q \setminus \{\theta\}$. Now, we claim that

$$u \neq \lambda_0 Au + t\psi, \quad \forall u \in \partial B_{R_1} \cap Q,\ t \geqslant 0. \tag{2.5}$$

Indeed, if there exist $u_1 \in \partial B_{R_1} \cap Q$ and $t_0 \geqslant 0$ such that $u_1 = \lambda_0 Au_1 + t_0\psi$, then, by (2.4) we have

$$\begin{aligned} h(u_1) &\geqslant \lambda_0 h(Au_1) \geqslant \lambda_0 h(K_0Fu_1) = \lambda_0\big(K_0^* h\big)(Fu_1) \\ &= \lambda_0 r(K_0)h(Fu_1) \geqslant (1+\delta)h(u_1), \end{aligned}$$

and so $h(u_1) \leqslant 0$, which is a contradiction of

$$h(u_1) \geqslant \tau\|u_1\| = \tau R_1 > 0.$$

Thus, (2.5) holds. It follows from (2.2), (2.5) and Lemma 2.1 that $\lambda_0 A$ has at least one fixed point $u_0 \in Q \cap (B_{R_0} \setminus \bar{B}_{R_1})$. Hence, $(\lambda_0, u_0) \in \Sigma$. The proof is completed. □

Lemma 2.3. *Let $\Lambda = \{\lambda > 0 \mid$ there exists $u \in P \setminus \{\theta\}$ such that $(\lambda, u) \in \Sigma\}$. Assume* (H_2) *or* (H_3) *holds. Then Λ is a bounded set. Moreover, if $\lambda^* = \sup \Lambda$, then $(0, \lambda^*) \subset \Lambda$.*

Proof. Now, we divide the proof into two steps.

Step 1. We show that Λ is a bounded set.

For a given $q_1 > 0$, by (H_2), there exists $R > 0$ such that $h(Fu) \geqslant q_1 h(u)$ for all $u \in Q$ with $\|u\| \geqslant R$. Let q_2 be such that

$$0 < q_2 < \frac{h(F\theta)}{R\|h\|}.$$

Then, we have

$$h(Fu) \geqslant h(F\theta) \geqslant q_2\|h\|R \geqslant q_2 h(u), \quad \forall u \in \bar{B}_R \cap Q.$$

Thus, if $q = \min\{q_1, q_2, m\}$, then, by (H_3), we have

$$h(Fu) \geqslant qh(u), \quad \forall u \in Q. \tag{2.6}$$

For any $(\lambda, u) \in \Sigma$, by (2.6), we have

$$\|h\|\|u\| \geqslant h(u) = \lambda h(Au) \geqslant \lambda r(K_0)h(Fu) \geqslant \lambda r(K_0)qh(u) \geqslant \lambda r(K_0)q\tau\|u\|,$$

and so $\lambda \leqslant (r(K_0)q\tau)^{-1}\|h\|$, which means that Λ is a bounded set.

Step 2. We show that $(0, \lambda^*) \subset \Lambda$.

For any $\lambda \in (0, \lambda^*)$, we prove that $\lambda \in \Lambda$. By the definition of λ^*, there exists $\lambda_1 \in \Lambda$ such that $\lambda < \lambda_1 \leqslant \lambda^*$. Suppose that $(\lambda_1, u_1) \in \Sigma$. It is easy to see that u_1 is a sup-solution of λA. Let $u^{(0)} = u_1$, and $u^{(n)} = \lambda A u^{(n-1)}$ $(n = 1, 2, \ldots)$. Since λA is increasing and completely continuous, there exists u^λ such that

$$u^\lambda = \lambda A u^\lambda,$$
$$u^\lambda \leqslant \cdots \leqslant u^{(n)} \leqslant u^{(n-1)} \leqslant \cdots \leqslant u^{(1)} \leqslant u^{(0)} = u_1, \tag{2.7}$$

and

$$u^{(n)} \to u^\lambda \quad \text{as } n \to \infty. \tag{2.8}$$

Obviously, if (H_2) holds, then $u^\lambda > \theta$.

Now, we will show that $u^\lambda > \theta$ if (H_3) holds. First, we claim that

$$h(u^{(n)}) > 0, \quad n = 1, 2, \ldots. \tag{2.9}$$

In fact, if $h(u^{(n_0)}) = 0$ for some $n_0 \in N$, then, by (2.6) we have

$$0 = h(u^{(n_0)}) = \lambda h(Au^{(n_0-1)}) \geqslant \lambda r(K_0) h(Fu^{(n_0-1)}) \geqslant \lambda r(K_0) q h(u^{(n_0-1)}) \geqslant 0,$$

and so $h(u^{(n_0-1)}) = 0$. By inductive method, we have $h(u^{(0)}) = 0$, which is a contradiction to

$$h(u^{(0)}) = h(u_1) \geqslant \tau \|u_1\| > 0.$$

Hence, (2.9) holds. On the other hand,

$$h(u^{(n)}) = \lambda h(Au^{(n-1)}) \geqslant \lambda r(K_0) h(Fu^{(n-1)}).$$

Then, by (2.7) and (2.9), we have

$$\lim_{n\to+\infty} \frac{h(Fu^{(n-1)})}{h(u^{(n-1)})} \leqslant \lim_{n\to+\infty} \frac{h(Fu^{(n-1)})}{h(u^{(n)})} \leqslant \frac{1}{\lambda r(K_0)}.$$

If $u^\lambda = \theta$, together with (2.8), we have made a contradiction to (H_3). Hence, $u^\lambda > \theta$, and so $\lambda \in \Lambda$. The proof is completed. □

Lemma 2.4. *Let Λ, λ^* be defined as in Lemma* 2.3. *Assume* (H_2) *holds. Then* $\Lambda = (0, \lambda^*]$.

Proof. It follows from Lemma 2.3 that we only need to prove that $\lambda^* \in \Lambda$. Now, choose $\{\lambda_n\} \subset \Lambda$ such that $\lambda_n \to \lambda^*$ as $n \to +\infty$, and $\lambda_n \geqslant \lambda^*/2$ $(n = 1, 2, \ldots)$. Let $(\lambda_n, u_n) \in \Sigma$. Choose a positive number $M > 2(\lambda^* r(K_0))^{-1}$; then, by (H_2) there exists $R > 0$ such that $h(Fu) \geqslant Mh(u)$ for all $u \in Q$, $\|u\| \geqslant R$. Let $B = \{u_n\}$ and $\omega = \{\lambda_n - \lambda^*\}$. Now, we claim that B is a bounded set. In fact, if not, then there exists $u_{n_0} \in B$ such that $\|u_{n_0}\| \geqslant R$. Thus, we get $h(Fu_{n_0}) \geqslant Mh(u_{n_0})$. Since

$$h(u_{n_0}) = h(\lambda_{n_0} A u_{n_0}) \geqslant \lambda_{n_0} r(K_0) h(Fu_{n_0}) \geqslant \lambda_{n_0} r(K_0) M h(u_{n_0}),$$

and so $M \leqslant 2(\lambda^* r(K_0))^{-1}$, which is a contradiction to the assumption of M. Therefore, B is a bounded set.

Using the formula [9, p. 10]

$$\alpha(\bar{\Lambda}S) = \Big(\sup_{\lambda\in\bar{\Lambda}} |\lambda|\Big)\alpha(S),$$

where $\bar{\Lambda}$ is a bounded set of real numbers and $\bar{\Lambda}S = \{\lambda x\colon\ x \in S,\ \lambda \in \bar{\Lambda}\}$, we get

$$\begin{aligned}\alpha(B) &\leqslant \alpha\big(\omega A(B)\big) + \lambda^*\alpha\big(A(B)\big)\\ &\leqslant \frac{\lambda^*}{2}\alpha\big(A(B)\big) + \lambda^*\alpha\big(A(B)\big) = \frac{3\lambda^*}{2}\alpha\big(A(B)\big) = 0,\end{aligned}$$

which means that B is a compact set, and hence, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and $u^* \in Q$ such that $u_{n_k} \to u^*$ as $k \to +\infty$. From $u_{n_k} = \lambda_{n_k} A u_{n_k}$, letting $k \to +\infty$, we get $u^* = \lambda^* A u^*$. Thus, $\lambda^* \in \Lambda$. The proof is completed. □

Lemma 2.5. *Let λ^* be defined as in Lemma* 2.3. *Assume* (H_2) *holds. Then the operator equation* (1.2) *has at least two solutions for* $0 < \lambda < \lambda^*$.

Proof. It follows from Lemma 2.4 that $(\lambda^*, u^*) \in \Sigma$. For any $0 < \lambda < \lambda^*$, let

$$U^\lambda = \{x \in Q \mid \text{there exists } \eta > 0 \text{ such that } \lambda Ax \leqslant u^* - \eta A\theta\}.$$

Since

$$u^* = \lambda^* A u^* = (\lambda^* - \lambda)Au^* + \lambda A u^* \geqslant (\lambda^* - \lambda)A\theta + \lambda A u^*,$$

we get

$$\lambda A u^* \leqslant u^* - (\lambda^* - \lambda)A\theta. \tag{2.10}$$

Therefore, $u^* \in U^\lambda$, and so U^λ is a nonempty set.

For any $u \in U^\lambda$, there exists $\eta > 0$ such that

$$u^* \geqslant \eta A\theta + \lambda A u \geqslant \lambda K_0(Fu).$$

Then, by (2.6) we have

$$h(u^*) \geqslant \lambda r(K_0)h(Fu) \geqslant \lambda r(K_0)q\tau\|u\|,$$

and so $\|u\| \leqslant (\lambda q\tau r(K_0))^{-1}h(u^*)$, which means that U^λ is a bounded set.

Let $x_0 \in U^\lambda$ such that $\lambda A x_0 \leqslant u^* - \eta_0 A\theta$ for some $\eta_0 > 0$. For the given $\eta_0/(2\lambda) > 0$, by (H_2), there exists $\delta > 0$ such that

$$Fx - Fx_0 \leqslant \frac{\eta_0}{2\lambda}F\theta$$

for $x \in Q$ with $\|x - x_0\| < \delta$. Thus,

$$\begin{aligned}\lambda Ax &= \lambda(Ax - Ax_0) + \lambda A x_0 = \lambda K(Fx - Fx_0) + \lambda A x_0\\ &\leqslant \frac{\eta_0}{2}A\theta + u^* - \eta_0 A\theta = u^* - \frac{\eta_0}{2}A\theta\end{aligned}$$

for $x \in Q$ with $\|x - x_0\| < \delta$. This means that $x \in U^\lambda$. Hence, U^λ is a open set.

Now, we shall show that

$$\mu\lambda Au \neq u, \quad \forall\mu \in [0,1],\ u \in \partial U^{\lambda}, \tag{2.11}$$

where ∂U^{λ} denotes the boundary of U^{λ} in Q. In fact, if not, then there exist $\mu_0 \in [0,1]$ and $u_0 \in \partial U^{\lambda}$ such that $\mu_0\lambda Au_0 = u_0$. Since $u_0 \in \partial U^{\lambda}$, there exists $\{u_n\} \subset U^{\lambda}$ such that $u_n \to u_0$ as $n \to +\infty$. For any $n \in N$, we have

$$\lambda Au_n \leqslant u^* - \eta_n A\theta \leqslant u^*$$

for some $\eta_n > 0$. Letting $n \to +\infty$, we have $\lambda Au_0 \leqslant u^*$, and hence $u_0 = \mu_0\lambda Au_0 \leqslant u^*$. Then, by (2.10), we have

$$\lambda Au_0 \leqslant \lambda Au^* \leqslant u^* - (\lambda^* - \lambda)A\theta.$$

This means that $u_0 \in U^{\lambda}$, a contradiction to that $u_0 \in \partial U^{\lambda}$. Hence, (2.11) holds. By the properties of the fixed point index, we have

$$i(\lambda A, U^{\lambda}, Q) = i(\theta, U^{\lambda}, Q) = 1. \tag{2.12}$$

By (H_2), there exists $R > (\lambda\tau r(K_0)q)^{-1}h(u^*)$ such that

$$h(Fu) \geqslant \frac{2}{\lambda r(K_0)}h(u), \quad \forall u \in Q,\ \|u\| \geqslant R.$$

Let $\psi \in Q \setminus \{\theta\}$. Now, we will show that

$$u \neq \lambda Au + t\psi, \quad \forall u \in \partial B_R \cap Q,\ t \geqslant 0. \tag{2.13}$$

In fact, if not, there exist $u_0 \in \partial B_R \cap Q$ and $t_0 \geqslant 0$ such that $u_0 = \lambda Au_0 + t_0\psi$. Then we have

$$h(u_0) \geqslant h(\lambda Au_0) \geqslant \lambda r(K_0)h(Fu_0) \geqslant \lambda r(K_0)\frac{2}{\lambda r(K_0)}h(u_0) = 2h(u_0).$$

Thus, $h(u_0) = 0$, a contradiction of

$$h(u_0) \geqslant \tau\|u_0\| = \tau R > 0.$$

Thus, (2.13) holds. By the properties of the fixed point index, we have

$$i(\lambda A, B_R \cap Q, Q) = 0. \tag{2.14}$$

Then, by (2.12) and (2.14), we have

$$i(\lambda A, B_R \cap Q \setminus \bar{U}^{\lambda}, Q) = i(\lambda A, B_R \cap Q, Q) - i(\lambda A, U^{\lambda}, Q) = -1. \tag{2.15}$$

It follows from (2.12) and (2.15) that there exist $x_1 \in B_R \cap Q \setminus \bar{U}^{\lambda}$ and $x_2 \in U^{\lambda}$ such that $x_i = \lambda Ax_i$ $(i = 1, 2)$. Therefore, the operator equation (1.2) has at least two solutions for $0 < \lambda < \lambda^*$. The proof is completed. □

Theorem 2.1. *Assume* (H_1) *and* (H_2) *hold. Then there exists* $\lambda^* > 0$ *such that the operator equation* (1.2) *has at least two solutions for* $0 < \lambda < \lambda^*$*, at least one solution for* $\lambda = \lambda^*$ *and no solution for* $\lambda > \lambda^*$.

Proof. Clearly, Theorem 2.1 is an immediate consequence of Lemmas 2.2–2.5. □

Theorem 2.2. *Assume* (H_1) *and* (H_3) *hold. Then there exists* $\lambda^* > 0$ *such that the operator equation* (1.2) *has at least one solution for* $0 < \lambda < \lambda^*$ *and no solution for* $\lambda > \lambda^*$.

Proof. Theorem 2.2 is an immediate consequence of Lemmas 2.2–2.3. □

Remark 2.1. In [7], some results for the existence of multiple solutions of operator equation also had been obtained. Theorems 2.1 and 2.2 are different from the main results in [7]. Here, a special cone P_0 has been employed to discuss the existence and multiplicity of solutions of the operator equation.

3. Positive solutions of singular boundary value problems

In this section, we shall consider the existence of positive solutions of the singular boundary value problem (1.1).

Concerning the BVP (1.1), we make the following assumptions:

(H_4) g is increasing on R^+; $f > 0$ is continuous on $(0, 1)$ and can be singular at $t = 0, 1$;

$$\int_0^1 s(1-s) f(s)\, ds < +\infty.$$

(H_5) $g(0) > 0$, and

$$\lim_{u \to +\infty} \frac{g(u)}{u} = +\infty.$$

(H_6)
$$\lim_{u \to 0^+} \frac{g(u)}{u} = +\infty,$$

and there exists $d > 0$ such that $g(z) \geqslant dz$ for $z \in R^+$.

We have the following results.

Theorem 3.1. *Assume* (H_4) *and* (H_5) *hold. Then there exists* $\lambda^* > 0$ *such that the BVP* (1.1) *has at least two solutions for* $0 < \lambda < \lambda^*$*, at least one solution for* $\lambda = \lambda^*$ *and no solution for* $\lambda > \lambda^*$.

Theorem 3.2. *Assume* (H_4) *and* (H_6) *hold. Then there exists* $\lambda^* > 0$ *such that the BVP* (1.1) *has at least one solution for* $0 < \lambda < \lambda^*$ *and no solution for* $\lambda > \lambda^*$.

Let $E = C[I, R]$. For any $x \in E$, let $\|x\|_C = \max_{t \in I} |x(t)|$. It is easy to see that E is a Banach space with the norm $\|\cdot\|_C$. Let $P = \{x \in E \mid x(t) \geqslant 0,\ t \in I\}$ and $Q = \{x \in P \mid x(t) \geqslant \|x\|_C t(1-t),\ t \in I\}$. Clearly, P and Q are cones of E. Let $[\alpha, \beta] \subset (0, 1)$ be a fixed subinterval of I, and

$$f_0(t) = \begin{cases} f(t), & t \in [\alpha, \beta], \\ 0, & t \in [0, \alpha) \cup (\beta, 1]. \end{cases}$$

Let K, K_0 be two linear operators defined by

$$(Kx)(t) = \int_0^1 G(t,s) f(s) x(s)\, ds, \quad x \in P,$$

$$(K_0 x)(t) = \int_0^1 G(t,s) f_0(s) x(s)\, ds, \quad x \in P,$$

where

$$G(t,s) = \begin{cases} t(1-s), & t \leqslant s, \\ s(1-t), & t > s. \end{cases}$$

Let F, A be two nonlinear operators defined by

$$(Fx)(t) = g\big(x(t)\big), \quad x \in P$$

and $A = KF$. Clearly, $Ku \geqslant K_0 u$ for all $u \in P$.

Lemma 3.1 [8]. *Let Ω be a bounded open set in E and $\theta \in \Omega$. Suppose that $A : P \cap \bar{\Omega} \mapsto P$ is completely continuous, $A\theta = \theta$ and*

$$\inf_{x \in P \cap \partial\Omega} \|Ax\| > 0.$$

Then A has at least one eigenvector on $P \cap \partial\Omega$, which corresponds to a positive eigenvalue.

Lemma 3.2. *Assume* (H_4) *holds. Then K, K_0 and A are completely continuous operators from P into Q.*

Proof. We only prove that $K : P \mapsto Q$ is a completely continuous operator. In a similar way, we could prove that K_0 and A are completely continuous operators from P into Q.

For a fixed $x \in P$, let

$$y(t) = \int_0^1 G(t,s) f(s) x(s)\, ds$$

for $t \in I$. Then there exists $t_0 \in (0,1)$ such that $y(t_0) = \|y\|_C$. It is easy to see that

$$\frac{G(t,s)}{G(t_0,s)} = \left\{ \begin{array}{ll} \frac{t}{t_0}, & t,\ t_0 \leqslant s, \\ \frac{1-t}{1-t_0}, & t,\ t_0 \geqslant s, \\ \frac{s(1-t)}{t_0(1-s)}, & t \geqslant s,\ t_0 \leqslant s, \\ \frac{t(1-s)}{(1-t_0)s}, & t \leqslant s,\ t_0 \geqslant s, \end{array} \right\} \geqslant (1-t)t \quad \text{for } t, s \in (0,1), \tag{3.1}$$

and so

$$y(t) = \int_0^1 \frac{G(t,s)}{G(t_0,s)} G(t_0,s) f(s) x(s)\, ds \geqslant \|y\|_C t(1-t),$$

which implies that $K : P \mapsto Q$.

Now, we show that K is a completely continuous operator. Let $B \subset Q$ be a bounded set such that $\|x\| \leqslant L$ for all $x \in B$ and some $L > 0$. For any $\varepsilon > 0$, it follows from (H_4) that there exists $\delta_1 > 0$ such that

$$2g(L)\left(\int_0^{\delta_1} G(s,s) f(s)\, ds + \int_{1-\delta_1}^1 G(s,s) f(s)\, ds\right) < \frac{\varepsilon}{2}. \tag{3.2}$$

Let $a_0 = \max_{s \in [\delta_1, 1-\delta_1]} f(s)$. Since $G(t,s)$ is uniformly continuous on $I \times I$, then for the given $\varepsilon > 0$ there exists $\delta_1 > \delta > 0$ such that

$$\bigl|G(t_1,s) - G(t_2,s)\bigr| \leqslant \frac{\varepsilon}{2a_0 g(L)}, \quad t_1, t_2, s \in I,\ |t_1 - t_2| \leqslant \delta. \tag{3.3}$$

Then, by (3.2), (3.3) and using the fact that $G(t,s) \leqslant G(s,s)$ for any $(t,s) \in I \times I$, we have

$$\begin{aligned}
\bigl|(Kx)(t_1) - (Kx)(t_2)\bigr| &\leqslant \int_0^1 \bigl|G(t_1,s) - G(t_2,s)\bigr| f(s) g(L)\, ds \\
&\leqslant 2g(L)\left(\int_0^{\delta_1} G(s,s) f(s)\, ds + \int_{1-\delta_1}^1 G(s,s) f(s)\, ds\right) \\
&\quad + a_0 g(L) \int_{\delta_1}^{1-\delta_1} \bigl|G(t_1,s) - G(t_2,s)\bigr|\, ds \\
&< \varepsilon
\end{aligned}$$

for any $x \in B$, $t_1, t_2 \in I$, $|t_1 - t_2| \leqslant \delta$, which means that $K(B)$ is equicontinuous on I. Therefore, K is a completely continuous operator. The proof is completed. □

Lemma 3.3. *Assume* (H_4) *holds. Then* $x \in C^2(0,1) \cap C[0,1]$ *is a solution of the BVP* (1.1) *if and only if* $x \in C[0,1]$ *is a solution of the following integral equation*:

$$x(t) = \lambda \int_0^1 G(t,s) f(s) g\bigl(x(s)\bigr)\, ds, \quad t \in I. \tag{3.4}$$

Proof. This result can be easily obtained. For the sake of simplicity, we omit the process. □

Lemma 3.4. $r(K_0) > 0$.

Proof. Let

$$\psi(t) = \int_0^1 G(t,s) f_0(s)\,ds, \quad t \in I.$$

Then, we have

$$\psi(t) = \int_0^1 G(t,s) f_0(s)\,ds \leqslant \left(\int_\alpha^\beta f_0(s)\,ds\right) t(1-t), \quad t \in I. \tag{3.5}$$

It follows from Lemma 3.2 that $K_0\psi \in Q$, and so

$$(K_0\psi)(t) \geqslant \|K_0\psi\|_C t(1-t), \quad t \in I. \tag{3.6}$$

Let $b = \|K_0\psi\|_C [\int_\alpha^\beta f_0(s)\,ds]^{-1}$. By (3.5) and (3.6), we have

$$(K_0\psi)(t) \geqslant b\psi(t), \quad t \in I. \tag{3.7}$$

Set $S = \{\phi \in Q \mid \|\phi\| = 1\}$. For any $n \in N$, let K_n be a linear operator defined by

$$(K_n\phi)(t) = \int_0^1 G(t,s) f_0(s)\left(\phi(s) + \frac{1}{n}\psi(s)\right) ds, \quad \phi \in Q.$$

By (3.7), we get

$$K_n\phi \geqslant \frac{1}{n} K_0\psi \geqslant \frac{b}{n}\psi, \quad \forall \phi \in P.$$

Thus,

$$\inf_{\phi\in S} \|K_n\phi\|_C \geqslant \frac{b}{n}\|\psi\|_C > 0.$$

By Lemma 3.1, there exist $\lambda_n > 0$ and $\phi_n \in S$ such that

$$\lambda_n\phi_n = K_n\phi_n, \quad n = 1, 2, \ldots.$$

Let $\varepsilon_0 = \alpha(1-\beta)$ and $\vartheta_n = \lambda_n^{-1}$. Since $\phi_n \in Q$, we get

$$\begin{aligned}
\|\phi_n\|_C \geqslant \phi_n(t) &= \vartheta_n \int_0^1 G(t,s) f_0(s)\left(\phi_n(s) + \frac{1}{n}\psi(s)\right) ds \\
&\geqslant \vartheta_n \int_0^1 G(t,s) f_0(s)\phi_n(s)\,ds \\
&\geqslant \vartheta_n\varepsilon_0 \int_\alpha^\beta f_0(s) s(1-s)\,ds \|\phi_n\|_C
\end{aligned}$$

for $t \in [\alpha, \beta]$, and so

$$\vartheta_n \leqslant \left(\varepsilon_0 \int_\alpha^\beta f_0(s)s(1-s)\,ds \right)^{-1},$$

which means that $\{\vartheta_n\}$ is a bounded set. Since K_0 is a completely continuous operator, there exists a subsequence $\{n_i\}$ of $\{n\}$ and $\vartheta \geqslant 0$, $\phi \in S$ such that $\vartheta_{n_i} \to \vartheta$, $\phi_{n_i} \to \phi$ as $i \to +\infty$. Clearly, $\phi = \vartheta K_0 \phi$. Let $\lambda = \vartheta^{-1}$. It is easy to see that $\vartheta > 0$, and therefore, $\lambda > 0$ is a positive eigenvalue. Hence, $r(K_0) > 0$. The proof is completed. □

Proof of Theorem 3.1. By the Krein–Rutman theorem and Lemma 3.4, there exist $\phi \in P$ and $h \in P^*$ such that

$$r(K_0)\phi = K_0\phi, \qquad r(K_0)h = K_0^* h.$$

Obviously, $\phi \in Q$. We claim that h can be taken in the following form:

$$h(u) = \int_0^1 \phi(t) f_0(t) u(t)\,dt, \quad u \in E. \tag{3.8}$$

In fact, by the Fubini theorem, for $u \in E$, we have

$$\begin{aligned} r(K_0)h(u) &= \int_0^1 r(K_0)\phi(t) f_0(t) u(t)\,dt \\ &= \int_0^1 f_0(t)u(t)\,dt \int_0^1 G(t,s) f_0(s)\phi(s)\,ds \\ &= \int_0^1 \phi(s) f_0(s)\,ds \int_0^1 G(s,t) f_0(t) u(t)\,dt \\ &= h(K_0 u) = \big(K_0^* h\big)(u). \end{aligned} \tag{3.9}$$

Thus, (3.8) holds.

For any $u \in P$, we have

$$h(u) = \int_0^1 \phi(t) f_0(t) u(t)\,dt \geqslant \int_\alpha^\beta \phi(t) f_0(t) t(1-t)\,dt \|u\|_C.$$

Set $\tau = \int_\alpha^\beta \phi(t) f_0(t) t(1-t)\,dt$ and $P_0 = \{u \in P \mid h(u) \geqslant \tau \|u\|_C\}$. Obviously, P_0 is a cone of E and $Q \subset P_0$. It follows from Lemma 3.2 that $KP \subset Q$ and $K_0 P \subset Q$. Thus, (H_1) holds.

For any $M > 0$, by (H_5), there exists $R > 0$ such that $g(x) \geqslant Mx$ for $x \geqslant R$. Then, for any $u \in Q$ with $\|u\|_C \geqslant R(\alpha(1-\beta))^{-1}$, we have

$$\begin{aligned}\frac{h(F(u))}{h(u)} &\geqslant \frac{\int_\alpha^\beta \phi(t) f_0(t) g(\|u\|_C t(1-t))\,dt}{\int_\alpha^\beta \phi(t) f_0(t)\,dt\|u\|} \\ &\geqslant \frac{M\int_\alpha^\beta \phi(t) f_0(t) t(1-t)\,dt\|u\|_C}{\int_\alpha^\beta \phi(t) f_0(t)\,dt\|u\|_C} \geqslant M\alpha(1-\beta). \qquad (3.10)\end{aligned}$$

This means that

$$\lim_{u\in Q,\,\|u\|\to+\infty} \frac{h(Fu)}{h(u)} = +\infty.$$

Let $x_0 \in Q$ be fixed. By the uniformly continuity of g on $[0, \|x_0\|_C + 1]$, for any $\varepsilon > 0$, there exists $1 > \delta > 0$ such that

$$\left|g(x) - g(x')\right| < \varepsilon g(0)$$

for any $x, x' \in [0, \|x_0\|_C + 1]$ and $|x - x'| < \delta$. Then, we get

$$-\varepsilon g(0) < g\big(x(t)\big) - g\big(x_0(t)\big) < \varepsilon g(0)$$

for any $x \in Q$ and $\|x - x_0\|_C < \delta$. This implies that F is $F\theta$-continuous on Q. Hence, (H_2) holds. It follows from Theorem 2.1 that Theorem 3.1 holds. □

Proof of Theorem 3.2. It follows from the proof of Theorem 3.1 that we only need to check the condition (H_3) holds. For any $u \in Q$, we have

$$h(Fu) = \int_0^1 \phi(t) f_0(t) g\big(u(t)\big)\,dt \geqslant d\int_0^1 \phi(t) f_0(t) u(t)\,dt = h(u).$$

In a similar way as (3.10), we could prove that

$$\lim_{u\in Q,\,u\to\theta} \frac{h(Fu)}{h(u)} = +\infty.$$

Hence, (H_3) holds. It follows from Theorem 2.2 that Theorem 3.2 holds. □

Example. Consider the following singular boundary value problem:

$$\begin{cases} y'' + \lambda t^{-3/2}(1-t)^{-3/2} g(y) = 0, & 0 < t < 1, \\ y(0) = 0 = y(1), \end{cases}$$

where

$$g(y) = \begin{cases} y^{1/2}, & y \in [0, 1], \\ y, & y \in (1, +\infty). \end{cases} \qquad (3.11)$$

Conclusion. *There exists* $\lambda^* > 0$ *such that the BVP* (3.11) *has at least one solution for* $0 < \lambda < \lambda^*$, *has no solution for* $\lambda > \lambda^*$.

Proof. It is easy to check (H_4) and (H_6) hold. By Theorem 3.2, the conclusion holds. □

Remark 3.1. Theorems 3.1 and 3.2 are improvements of Theorem 3 in [2] and Theorem 1.1 in [3], respectively. In the Example, since $g(0) = 0$, the conclusion cannot be obtained by Theorem 1.1 in [3].

Remark 3.2. In a recent paper [6], an existence result of positive solutions of Sturm–Liouville type singular boundary value problem was obtained. Obviously, Theorems 2.1 and 2.2 can also be applied to that case.

4. Multiple positive radial solutions for an elliptic system on an annulus

Dunninger and Wang [5] studied the existence of positive radial solution of the elliptic system

$$\begin{cases} \Delta u + \lambda k_1(|x|) f(u, v) = 0, & \text{in } \Omega, \\ \Delta v + \lambda k_2(|x|) g(u, v) = 0, & \text{in } \Omega, \\ \alpha_1 u + \beta_1 \frac{\partial u}{\partial n} = 0, \quad \alpha_2 v + \beta_2 \frac{\partial v}{\partial n} = 0, & \text{on } |x| = R_1, \\ \gamma_1 u + \delta_1 \frac{\partial u}{\partial n} = 0, \quad \gamma_2 v + \delta_2 \frac{\partial v}{\partial n} = 0, & \text{on } |x| = R_2, \end{cases} \tag{4.1}$$

where $\alpha_i, \beta_i, \gamma_i, \delta_i \geqslant 0$ and $\rho_i = \gamma_i \beta_i + \alpha_i \gamma_i + \alpha_i \delta_i > 0$ for $i = 1, 2$.

It is easy to see that (4.1) is equivalent to the following system of integral equations (see [5]):

$$\begin{cases} u(t) = \lambda \int_0^1 k_1(t, s) h_1(s) f(u(s), v(s))\, ds, \\ v(t) = \lambda \int_0^1 k_2(t, s) h_2(s) g(u(s), v(s))\, ds, \end{cases} \tag{4.2}$$

where $h_i \in C[0, 1]$, and $k_i(t, s)$ $(i = 1, 2)$ is the Green's function

$$k_i(t, s) = \frac{1}{\rho_i} \begin{cases} (\beta_i + \alpha_i s)[\delta_i + \gamma_i(1 - t)], & s \leqslant t, \\ (\beta_i + \alpha_i t)[\delta_i + \gamma_i(1 - s)], & s > t. \end{cases}$$

Dunninger and Wang proved the following result.

Theorem 4.1 [5, Theorem 1.1]. *Assume*

(A$_1$) λ *is a positive parameter.*
(A$_2$) $k_1, k_2 : [R_1, R_2] \mapsto R^+$ *are continuous and do not vanish identically on any subinterval of* $[R_1, R_2]$.
(A$_3$) $f, g : R^+ \times R^+ \mapsto (0, +\infty)$ *are continuous.*
(A$_4$) $f(u_1, v_1) \leqslant f(u_2, v_2)$, $g(u_1, v_1) \leqslant g(u_2, v_2)$ *for* $0 \leqslant u_1 \leqslant u_2$, $0 \leqslant v_1 \leqslant v_2$.
(A$_5$) $$f_\infty = \lim_{u+v \to \infty} \frac{f(u, v)}{u + v} = +\infty, \qquad g_\infty = \lim_{u+v \to +\infty} \frac{g(u, v)}{u + v} = +\infty.$$

Then there exists a positive number λ^* *such that* (4.1) *has at least two positive solutions for* $0 < \lambda < \lambda^*$, *at least one positive solution for* $\lambda = \lambda^*$ *and no positive solution for* $\lambda > \lambda^*$.

As an application of Theorem 2.1, in this section, we will give a different proof of Theorem 1.1 in [5].

In what follows of this section, the norms in the Banach spaces $E = C[I, R^2]$ and $C[I, R]$ are denoted by $\|\cdot\|$ and $\|\cdot\|_C$, respectively, i.e., $\|\phi\|_C = \max_{t\in I} |\phi(t)|$ for any $\phi \in C[I, R]$ and

$$\|\phi\| = \|\phi^{(1)}\|_C + \|\phi^{(2)}\|_C, \quad \forall \phi = \begin{pmatrix} \phi^{(1)} \\ \phi^{(2)} \end{pmatrix} \in C[I, R^2].$$

Let $P = \{x \in E \mid x(t) \geqslant \theta,\ t \in I\}$, and $q(t) = \min\{q_1(t), q_2(t)\}$, where

$$q_i(t) = \big[(\alpha_i + \beta_i)(\gamma_i + \delta_i)\big]^{-1}(\beta_i + \alpha_i t)\big[\delta_i + \gamma_i(1-t)\big]$$

for $i = 1, 2$. Set

$$Q = \left\{ x = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \in P \;\middle|\; x_i(t) \geqslant \|x_i\|_C q_i(t) \text{ for } i = 1, 2,\ t \in I \right\}.$$

It is easy to see that Q is also a cone of E. Let us define the linear operator K_i by

$$(K_i x)(t) = \int_0^1 k_i(t,s) h_i(s) x(s)\, ds, \quad x \in C[I, R]$$

for $i = 1, 2$. Then define the operators K, F and A by

$$(Kx)(t) = \begin{pmatrix} (K_1 x_1)(t) \\ (K_2 x_2)(t) \end{pmatrix}, \quad \forall x = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \in P,$$
$$(Fx)(t) = \begin{pmatrix} f(x_1(t), x_2(t)) \\ g(x_1(t), x_2(t)) \end{pmatrix}, \quad \forall x = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \in P,$$

and $A = KF$.

Lemma 4.1. *The operators A and K are completely continuous operators from P into Q.*

Proof. The complete continuity of A and K are obvious.

Now, we prove that $K : P \mapsto Q$. For any $x = \binom{x_1(t)}{x_2(t)} \in P$, in a similar way as (3.1), we have

$$\frac{k_i(t,s)}{k_i(t_i,s)} \geqslant q_i(t), \quad i = 1, 2,\ t, s \in (0, 1),$$

where $t_i \in I$ such that $\|K_1 x_1\|_C = (K_1 x_1)(t_1)$ and $\|K_2 x_2\|_C = (K_2 x_2)(t_2)$. Then, we get

$$(K_1 x_1)(t) \geqslant \|K_1 x_1\|_C q_1(t), \qquad (K_2 x_2)(t) \geqslant \|K_2 x_2\|_C q_2(t).$$

This means that $K : P \mapsto Q$. In a similar way, we show that $A : P \mapsto Q$. The proof is completed. □

Lemma 4.2. $r(K) > 0$.

Proof. The proof is similar to that of Lemma 3.4, we only sketch it.

Let $S = \{x \in E \mid \|x\| = 1\}$, and $\psi = \binom{\psi^{(1)}}{\psi^{(2)}}$, where $\psi^{(i)} = K_i\omega$ for $i = 1, 2$, and $\omega(t) = 1$ for $t \in I$. Then, we have

$$K\psi = \begin{pmatrix} K_1\psi^{(1)} \\ K_2\psi^{(2)} \end{pmatrix} \geqslant \begin{pmatrix} \|K_1\psi^{(1)}\|_C q_1 \\ \|K_2\psi^{(2)}\|_C q_2 \end{pmatrix}. \tag{4.3}$$

On the other hand,

$$\psi = \begin{pmatrix} \psi^{(1)} \\ \psi^{(2)} \end{pmatrix} \leqslant \begin{pmatrix} \rho_1^{-1}(\alpha_1+\beta_1)(\gamma_1+\delta_1)q_1\int_0^1 h_1(s)\,ds \\ \rho_2^{-1}(\alpha_2+\beta_2)(\gamma_2+\delta_2)q_2\int_0^1 h_2(s)\,ds \end{pmatrix}. \tag{4.4}$$

Let

$$d_i = \rho_i\left[(\alpha_i+\beta_i)(\gamma_i+\delta_i)\int_0^1 h_i(s)\,ds\right]^{-1}, \quad i = 1, 2,$$

and

$$c = \min\bigl\{d_1\|K_1\psi^{(1)}\|_C,\, d_2\|K_2\psi^{(2)}\|_C\bigr\}.$$

By (4.3) and (4.4), we get $K\psi \geqslant c\psi$.

Then, by a similar argument as Lemma 3.4, we can show that $r(K) > 0$. The proof is completed. □

Proof of Theorem 4.1. It follows from Lemma 4.2 and the Krein–Rutman theorem that there exists $\varphi = \binom{\varphi^{(1)}}{\varphi^{(2)}} \in P$ and $h \in P^*$ such that

$$r(K)\varphi = K\varphi, \qquad r(K)h = K^*h.$$

By a similar argument as (3.9), we see that h can be taken in the form

$$h(u) = \int_0^1 \varphi^{(1)}(t)h_1(t)u_1(t)\,dt + \int_0^1 \varphi^{(2)}(t)h_2(t)u_2(t)\,dt, \quad \forall u = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \in E.$$

Let

$$\tau = \min\left\{\int_0^1 \varphi^{(1)}(t)h_1(t)q_1(t)\,dt,\, \int_0^1 \varphi^{(2)}(t)h_2(t)q_2(t)\,dt\right\}$$

and $P_0 = \{u \in P \mid h(u) \geqslant \tau\|u\|\}$. It is easy to see that P_0 is a cone of E and $Q \subset P_0$. Let $[\alpha, \beta]$ be a fixed subinterval of I and $\varepsilon_0 = \min_{t\in[\alpha,\beta]} q(t)$. For any $u = \binom{u_1(t)}{u_2(t)} \in Q$, we have

$$u_1(t) + u_2(t) \geqslant \bigl(\|u_1\|_C + \|u_2\|_C\bigr)q(t) \geqslant \varepsilon_0\|u\|, \quad \forall t \in [\alpha, \beta]. \tag{4.5}$$

By (A_5) and (4.5), in a similar way as (3.10), we can show that

$$\lim_{u\in Q,\, \|u\|\to+\infty} \frac{h(Fu)}{h(u)} = +\infty.$$

By Theorem 2.1, the conclusion of Theorem 1.1 in [5] holds. The proof is completed. □

Acknowledgment

The author wishes to express his sincere gratitude to the reviewers for their helpful suggestions in improving the paper.

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