Viscosity Solutions of Fully Nonlinear Second-Order Elliptic Partial Differential Equations

H. Ishii*

Department of Mathematics,
Chuo University, Bunkyo-ku, Tokyo 112 Japan

AND

P. L. Lions†

Ceremade, Université Paris-Dauphine,
Place de Lattre de Tassigny, 75775 Paris Cedex 16, France

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We investigate comparison and existence results for viscosity solutions of fully nonlinear, second-order, elliptic, possibly degenerate equations. These results complement those recently obtained by R. Jensen and H. Ishii. We consider various boundary conditions like for instance Dirichlet and Neumann conditions. We also apply these methods and results to quasilinear Monge–Ampère equations. Finally, we also address regularity questions.

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I. INTRODUCTION

This work is, in some sense, a sequel to some recent results of the first author (H. Ishii [26]). We are concerned with appropriate weak solutions of fully nonlinear, second-order, elliptic, possibly degenerate equations of the following form

\[ F(x, u, Du, D^2u) = 0 \quad (1.1) \]

in an open subset \( \Omega \) of \( \mathbb{R}^n \), where \( F \) is, say, a real-valued continuous function on \( \Gamma = \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{M}^n \) and \( \mathbb{M}^n \) will denote the space of \( n \times n \) real symmetric matrices. The unknown \( u \) will always be a scalar (real-valued) function on \( \Omega \), \( Du \) and \( D^2u \) denote respectively the gradient of \( u \) and the Hessian of \( u \). As it is customary to say, (1.1) is said to be (degenerate) elliptic if for all \((x, t, p, X) \in \Gamma, Y \in \mathbb{M}^n\) the following inequality holds

\[ F(x, t, p, X + Y) \leq F(x, t, p, X) \quad \text{if} \quad Y \succeq 0, \quad (1.2) \]

where we endow \( \mathbb{M}^n \) with the usual partial ordering. Note that this assumption covers in particular the completely degenerate case when \( F \) does not depend on \( X \), i.e., the particular case of first-order Hamilton–Jacobi equations of the following form

\[ F(x, u, Du) = 0. \quad (1.3) \]

Because of this possible degeneracy (and the highly nonlinear form of the equations considered here) classical solutions (smooth) cannot be expected to exist.

Therefore, we will work with the so-called viscosity solutions which, roughly speaking, are functions for which all inequalities expected from the classical maximum principle argument when comparing \( u \) with smooth test functions do hold. Precise definitions will be given below. Viscosity solutions were introduced by M. G. Crandall and P. L. Lions [16] and allowed a complete treatment of the first-order Hamilton–Jacobi Eqs. (1.3): general uniqueness results were first obtained in M. G. Crandall and P. L. Lions [16] and were subsequently improved, adapted, and reinterpreted in M. G. Crandall, L. C. Evans, and P. L. Lions [14], H. Ishii [27], [28], M. G. Crandall and P. L. Lions [17], [18], [19], M. G. Crandall, H. Ishii, and P. L. Lions [15].

Existence, based upon the vanishing viscosity method and approximation arguments, was studied in P. L. Lions [38], [39], G. Barles [4], [5], P. E. Souganidis [52], H. Ishii [28], [29], M. G. Crandall and P. L. Lions [17], [18], [19]. An important improvement and simplification of the existence program was achieved with the observation due to H. Ishii [30]...
that "good" uniqueness results yield existence via an adaptation of the classical Perron's method, a general statement which puts the emphasis upon uniqueness results. Let us mention at this stage that the whole first-order theory can be, with some care, carried out in infinite dimensions: see M. G. Crandall and P. L. Lions [19], [20], [21], [22] for further details. At this stage, it is worth mentioning that uniqueness depends crucially in these works upon the following obvious fact

$$D_x(\Phi(x - y)) = -D_y(\Phi(x - y))$$  \hspace{1cm} (1.4)

for any smooth (say $C^1$) function $\Phi$. This relation allows a certain cancellation in the uniqueness proof.

This very fact seemed to prevent any extension to second-order equations since we have now for second derivatives

$$D^2_x(\Phi(x - y)) = D^2_y(\Phi(x - y))$$  \hspace{1cm} (1.5)

and the desired cancellation disappears. However, uniqueness was to be expected from several arguments: one of which was the observation, made a posteriori after the introduction of viscosity solutions, that the notion of viscosity solutions is intimately connected with the accretivity of operators like those given in (1.1) in spaces like $C$ (or $L^\infty$), a fact which made the link between this theory and arguments, reminiscent of Minty's device, introduced by L. C. Evans [24], [25] to pass to the limit in equations like (1.1). The other argument in favour of general uniqueness results was the uniqueness results shown by P. L. Lions [41], [42] for Hamilton–Jacobi–Bellman equations (i.e., (1.1) with $F$ convex or concave in $D^2u$, $Du$, ...): the proof there was based upon stochastic control considerations, which translated in purely analytical arguments still require the use of comparison functions with some regularity—a fact which seems related to the convexity assumption. Note, however, that this method can be carried out in infinite dimensions as observed recently in P. L. Lions [45].

Major progress was made on the uniqueness question by R. Jensen [34]; as it was later reformulated and simplified in R. Jensen, P. L. Lions and P. E. Souganidis [35] (see also P. L. Lions and P. E. Souganidis [49]), the main idea in [31] is to regularize sub- or supersolutions of (1.1) by the so-called sup or inf convolutions essentially keeping the sub- or supersolution properties—a fact already noted by J. M. Lasry and P. L. Lions [37]—and then to use an appropriate version of the classical maximum principle—in fact, an adaptation of the maximum principles due to A. D. Alexandrov [1], I. Bakelman [2], C. Pucci [52], J. M. Bony [9], P. L. Lions [42]. Jensen's approach however was limited (in [34], [35], [49]) to cases when stringent restrictions on the $x$-dependence were made; this because
one was insisting on having sub- or supersolutions of the same equation after regularizations.

Recently, the first author (H. Ishii [26]) refined Jensen's argument by observing that regularization yields sub- and supersolutions of modified equations and that the Alexandrov type maximum principle gives nontrivial informations on some matrices (of second derivatives). Then, by a careful matrix analysis, uniqueness may be deduced, i.e., one can "control the modifications or perturbations of the original equation by the matrix bounds obtained through the maximum principle." This strategy was implemented effectively in two cases in [26], the newest of which is the class of the so-called Isaacs–Bellman equations.

One of our goals here is to improve, simplify (a bit), and extend this approach and also to apply it to other classes of equations. This is why we recall first briefly the main arguments of [34], [35], [26] in Section II and as we will see the whole uniqueness program will boil down eventually to matrix analysis. Once this is done, we will derive several uniqueness results for strictly elliptic equations (Section III), combinations of various cases and parabolic problems (Section IV), uniformly elliptic equations, quasilinear equations and Monge–Ampère equations (Section V). In all these sections, we will deal with Eq. (1.1) set in a bounded domain \( \Omega \) and Dirichlet-type boundary conditions: roughly speaking, solutions will have fixed values on the whole boundary \( \partial \Omega \). It is a routine work to adapt these results to the case of unbounded domains or of the whole space, provided of course appropriate restrictions at infinity and corresponding assumptions on \( F \) are made (as in [28], [18], [27] for instance ...).

But other boundary conditions are of interest, in particular for applications to control problems or differential games. We will show in Sections VI.1 and VI.2 how the viscosity formulation of Neumann type boundary conditions introduced in P. L. Lions [44]—see also B. Perthame and R. Sanders [50], H. Ishii [31]—may be combined with our approach in order to obtain general existence and uniqueness results in the case of Neumann type boundary conditions. Another important class of boundary conditions is the one which originates in state-constraints problems in Control (or Differential Games) theory: this boundary condition, as originally introduced by M. H. Soner [51], I. Capuzzo–Dolcetta and P. L. Lions [12] for first-order problems and used in J. M. Lasry and P. L. Lions [38] for "nondegenerate" second-order problems, can also be treated with our approach (see Section VI.3). Such "state-constraints" boundary conditions play also a crucial role in a more refined analysis of Dirichlet type conditions. Indeed, imposing values everywhere on the boundary \( \partial \Omega \) is in general hopeless for equations like (1.1) which may present important degeneracies. As it was realized in I. Capuzzo–Dolcetta and P. L. Lions [12], H. Ishii [31], G. Barles and B. Perthame [6], [7], at the boundary
points where the prescribed value is not achieved one has to impose those state-constraints boundary conditions (roughly speaking, those points play the same role as interior points for uniqueness purposes). This allows a more subtle interpretation of Dirichlet boundary conditions and this is precisely what we do in Section VI.4.

Finally, Section VII is devoted to the study of some regularity results. Let us point out that further regularity results for viscosity solutions of uniformly elliptic Eqs. (1.1) are to be found in L. Caffarelli \[10\], N. S. Trudinger \[55\], \[56\]. We would like to mention also that after all the results presented here were obtained, the second author was informed by Neil Trudinger that a variation of Jensen's original argument is possible in order to obtain general uniqueness results for uniformly elliptic equations: such results are thus proved by an entirely different method from ours.\(^1\)

Let us conclude this introduction with a few notations: for \(x, y \in \mathbb{R}^n, |x|, (x, y)\) denote, respectively, the usual norm and scalar product on \(\mathbb{R}^n\), \(B_r(x)\) denotes the open ball centered at \(x\) of radius \(r > 0\), \(B_r = B_r(0)\), \(\mathbb{M}(m, n)\) denotes the space of real \(m \times n\) matrices. For \(A \in \mathbb{M}(m, n), \ tA, \ U(A), \ Tr(A)\) stand respectively for the transpose, the norm, and the trace of \(A\) (if \(m = n\)): we take for instance the following norm \(\|A\| = (\sum_{i,j} a_{ij}^2)^{1/2}\) if \(A = (a_{ij})\). Let us also mention that each time this is not ambiguous \(B_r\) will denote indifferently an open ball of radius \(r\) in any euclidean space or even in \(\mathbb{M}(m, n)\) or \(\mathbb{M}^n\). Recall also that in everything that follows \(\Omega\) is assumed to be bounded unless explicit mention of the contrary is made. Finally, we will denote by \(I\) and \(0\) the identity matrix and the zero matrix, respectively.

\[\text{II. THE GENERAL STRATEGY FOR UNIQUENESS AND COMPARISON RESULTS}\]

We first recall the definition of viscosity solutions of (1.1).

**Definition.** Let \(u\) be an upper-semicontinuous function (resp. lower semi-continuous) on \(\Omega\); \(u\) is said to be a viscosity subsolution of (1.1) (resp. supersolution) if for all \(\phi \in C^2(\Omega)\) the following inequality holds at each local maximum (resp. minimum) point \(x_0 \in \Omega\) of \(u - \phi\)

\[
F(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) \leq 0
\]

(resp. \(F(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) \geq 0\)).

\[\text{\(^1\) After this work was completed, we learned from R. Jensen that he obtained structure conditions somewhat similar to the one presented in Section IV.}\]
Then, \( u \in C(\Omega) \) is said to be a viscosity solution of (1.1) if \( u \) is a viscosity subsolution and a viscosity supersolution of (1.1).

**Remarks.**

1. It is possible to replace local by global, or local strict, or global strict.

2. It is possible to give an intrinsic formulation of the notion in terms of generalized derivatives (see [40]) but we will not do so here.

3. One may define (as in [32]) viscosity solutions which do not satisfy (semi-) continuity properties by requiring in the subsolution case that the usc envelope of \( u \), namely
   \[
   u^*(x) = \limsup_{y \to x, \, y \in \Omega} u(y),
   \]
   is finite and a viscosity subsolution (similarly one uses the lsc envelope \( u_* = -(-u)^* = \liminf_{y \in \Omega, \, y \to x} u(y) \) for the supersolution part) but we will not need to do so here.

Next, we want to describe the main ingredients in our strategy to prove uniqueness and comparison results. More precisely, we will try to compare an usc bounded viscosity subsolution \( u \) of (1.1) and a lsc bounded viscosity supersolution \( v \) of (1.1). We will denote by
   \[
   M = \sup_{x \in \partial \Omega} \{u^*(x) - v_*(x)\}^+
   \]  
(2.3) 
and we will in fact always use the following observation
   \[
   \lim_{n} \{u(x_n) - v(y_n)\} \leq M
   \]  
(2.4) 
whenever \( x_n, \, y_n \in \Omega, \, \text{dist}(x_n, \partial \Omega) \to 0, \, x_n - y_n \to 0 \) (and thus \( d(y_n, \partial \Omega) \to 0 \)). The typical assertion we want to prove is
   \[
   u(x) - v(x) \leq M \quad \text{in} \quad \Omega.
   \]  
(2.5) 

Before going into the description of the first steps of the proof of (2.5), let us immediately mention the implication of such a result on existence.

**Proposition II.1.** Assume that the comparison assertion (2.5) holds for any pair \((u, v)\) with the properties indicated above and assume that there exist
   \( f, \, g \in C(\overline{\Omega}) \) respectively viscosity subsolution and supersolution of (1.1) such that
   \( f = g = \varphi \) on \( \partial \Omega \). Then, there exists a unique viscosity solution \( u \in C(\overline{\Omega}) \) of (1.1) such that \( u = \varphi \) on \( \partial \Omega \) (and \( f \leq u \leq g \) on \( \Omega \)).

This is an immediate application of Perron's method as in Ishii [30], [26]: indeed, one considers the function \( u \) defined as the supremum of, say, all usc subsolutions of (1.1) equal to \( \varphi \) on \( \partial \Omega \). Observe that \( f \leq u \leq g \) in
The main idea of Perron's construction (maximality) translates into the statement that $u_\star$ which is lsc in $\bar{\Omega}$ (and continuous at $\partial \Omega$) is a viscosity supersolution of (1.1) while the stability of viscosity subsolutions through sup operations yields the fact that $u^\star$ which is usc on $\bar{\Omega}$ (and continuous at $\partial \Omega$, taking $\varphi$ as boundary values) is a viscosity subsolution of (1.1). Therefore, by the uniqueness-comparison assumption, $u^\star \leq u_\star$ in $\Omega$ while obviously by definition $u^\star \geq u_\star$ in $\partial \Omega$. Thus, $u$ is continuous and is the unique viscosity solution of (1.1)!

The first step towards the proof of (2.5) is the construction of appropriate regularizations of $u$ and $v$. The one we take here is the one used in [34], which is a bit different from the one originally considered by R. Jensen [53] and much simpler. We set for $\varepsilon \in (0, 1]$

$$u^\varepsilon(x) = \sup_{y \in \Omega} \left\{ u(y) - \frac{1}{\varepsilon} |x - y|^2 \right\}, \quad \forall x \in \Omega \quad (2.6)$$

$$v_\varepsilon(x) = \inf_{z \in \Omega} \left\{ v(z) + \frac{1}{\varepsilon} |x - z|^2 \right\}, \quad \forall x \in \bar{\Omega}, \quad (2.7)$$

(i.e., the "usual" sup convolution or inf convolution). It is straightforward to check that $u^\varepsilon, v_\varepsilon$ are bounded, Lipschitz continuous on $\bar{\Omega}$, the $y$'s and the $z$'s in (2.6)–(2.7) may be restricted by

$$|x - y| \leq C_0 \sqrt{\varepsilon}, \quad |x - z| \leq C_0 \sqrt{\varepsilon}, \quad (2.8)$$

where $C_0 = [(2 \sup_{\Omega} |u|) \vee (2 \sup_{\Omega} |v|)]^{1/2}$, and that $u^\varepsilon, v_\varepsilon$ are respectively semi-convex, semi-concave on $\Omega$ that is, more precisely,

$$u^\varepsilon + \frac{1}{\varepsilon} |x|^2 \text{ is convex on } \Omega \quad (2.9)$$

$$v_\varepsilon - \frac{1}{\varepsilon} |x|^2 \text{ is concave on } \Omega. \quad (2.10)$$

These easy claims (and more refined bounds) may be found in J. M. Lasry and P. L. Lions [36].

Next, the argument made in [34], [28] adapts and yields easily the following

**PROPOSITION II.2.** Assume (to simplify) that $F$ satisfies

$$F(x, t, p, A) \geq F(x, s, p, A), \quad \forall x \in \Omega, \forall t \geq s, \forall p \in \mathbb{R}^n, \forall A \in \mathbb{M}^n. \quad (2.11)$$

Then, $u^\varepsilon$ is a viscosity subsolution of

$$F_\varepsilon(x, u^\varepsilon, Du^\varepsilon, D^2u^\varepsilon) = 0 \quad \text{in } \Omega_\varepsilon, \quad (2.12)$$
where $F_\epsilon(x, t, p, A) = \min \{ F(y, t, p, A)/|y-x| \leq C_0 \epsilon^{1/2} \}$ and $\Omega_\epsilon = \{ x \in \Omega, \ \text{dist}(x, \partial \Omega) \leq C_0 \epsilon^{1/2} \}$. Similarly, $\nu_\epsilon$ is a viscosity supersolution of

$$F_\epsilon(x, \nu_\epsilon, D\nu_\epsilon, D^2\nu_\epsilon) = 0 \quad \text{in } \Omega_\epsilon, \quad (2.13)$$

where $F_\epsilon(x, t, p, A) = \max \{ F(y, t, p, A)/|x-y| \leq C_0 \epsilon^{1/2} \}$.

The second step in the proofs of (2.5) we will make in the next section is given by the following result which is an immediate adaptation of Proposition 5.1 in H. Ishii [26].

**Proposition II.3.** Let $\varphi \in C^2(\Omega \times \Omega)$, set $w(x, y) = u^\epsilon(x) - v_\epsilon(y)$ for $x, y \in \Omega_\epsilon$ and assume that $w - \varphi$ achieves its maximum over $\Omega_\epsilon \times \Omega_\epsilon$ at a point $(\bar{x}, \bar{y}) \in \Omega_\epsilon \times \Omega_\epsilon$. Then, if (2.11) holds there exist matrices $X, Y \in \mathbb{M}^n$ such that

$$F_\epsilon(\bar{x}, u^\epsilon(\bar{x}), D_x \varphi(\bar{x}, \bar{y}), X) \leq 0 \quad (2.14)$$

$$F_\epsilon(\bar{y}, v_\epsilon(\bar{y}), -D_y \varphi(\bar{x}, \bar{y}), -Y) \geq 0 \quad (2.15)$$

$$-\frac{2}{\epsilon} I \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq D^2 \varphi(\bar{x}, \bar{y}). \quad (2.16)$$

Note that (2.16) implies in particular that $X + Y \leq 0$ as soon as we choose $\varphi(x, y) = \Phi(x - y)$: indeed, in this case $D^2 \varphi$ becomes the matrix $\begin{pmatrix} D^2 \Phi(x - y) & -D^2 \Phi(x - y) \\ -D^2 \Phi(x - y) & D^2 \Phi(x - y) \end{pmatrix}$ and any point of the form $(\xi, \eta)$ for $\xi, \eta \in \mathbb{R}^n$ is in its kernel, so that by (2.16)

$$(X \xi, \xi) + (Y \xi, \xi) \leq 0$$

proving thus our claim. This simple consequence of (2.16) is a form of the maximum principle shown in R. Jensen [34]: indeed, $\bar{x}, \bar{y}$ can be made as close as we wish to a maximum point of $u - v$ by convenient choices of $\varphi$...

Note also that the "hopeless equality" (1.5) is now reflected by the equality of the diagonal terms of the above matrix. And, roughly speaking, the difficulties associated with (1.5) will be circumvented by exploiting fully the complete matrix instead of looking only at its diagonal part.

We will not re-prove Proposition II.3 here and we refer instead to H. Ishii [26]: we will only mention that its proof in [26] is a direct consequence of the Alexandrov type maximum principle shown in R. Jensen [34] that we described in the Introduction.

We now deduce from Proposition II.3 above one particular uniqueness result, namely the one obtained in R. Jensen, P. L. Lions, and P. E. Souganidis [35]—a variant of it being also independently given in H. Ishii [26]. To this end, we will use the following assumptions

$$\forall R < \infty, \ \exists \gamma_R > 0, \quad F(x, t, p, A) \geq F(x, s, p, A) + \gamma_R(t-s) \quad (2.17)$$
for all $x \in \Omega$, $R \geq t \geq s \geq -R$, $p \in \mathbb{R}^n$, $A \in \mathbb{M}^n$.

$$|F(x, t, p, A) - F(y, t, p, A)| \leq \omega_R(|x - y|(1 + |p|))$$  \hspace{1cm} (2.18)

if $x, y \in \Omega$, $|t| \leq R$, $p \in \mathbb{R}^n$, $A \in \mathbb{M}^n$, for all $R < \infty$; or

$$|F(x, t, p, A) - F(y, t, p, A)| \leq \omega_R(|x - y|)$$  \hspace{1cm} (2.19)

if $x, y \in \Omega$, $|t| \leq R$, $|p| \leq R$, $A \in \mathbb{M}^n$, for all $R < \infty$ where $\omega_R(\sigma) \to 0$ as $\sigma \to 0_+$.

**Theorem II.1.** Assume (2.17). If, in addition, (2.18) holds or if (2.19) holds and $u$ or $v$ is locally Lipschitz in $\Omega$, then the assertion (2.5) holds.

**Sketch of proof.** Using first the fact that $X + Y \leq 0$ if we choose $\varphi(x, y) = \Phi(x - y)$, we deduce with such a choice from the ellipticity condition (1.2)

$$F^e(\bar{y}, v_\varepsilon(\bar{y}), -D_y \varphi(x, \bar{y}), X) \geq 0.$$  \hspace{1cm} (2.20)

Therefore, by (2.17) and (2.18), there exists $\gamma > 0$ such that

$$\gamma(u^e(\tilde{x}) - v_\varepsilon(\tilde{y}))^+ \leq \omega\left\{(|\tilde{x} - \tilde{y}| + 2C_0 \varepsilon^{1/2})(1 + |D\Phi(\tilde{x} - \tilde{y})|)\right\},$$

where $\omega(\sigma) \to 0$ as $\sigma \to 0_+$ ($\omega$ may be taken continuous). Next, we let $\varepsilon$ go to $0_+$, we choose $\varphi(x, y) = (1/2\delta) |x - y|^2$, and (extracting subsequences if necessary) we may assume that $(\tilde{x}, \tilde{y})$ converges to a maximum point $(x_0, y_0) \in \Omega \times \Omega$ of $u(x) - v(y) - (1/2\delta) |x - y|^2$ over $\Omega \times \Omega$: indeed, the case of boundary points is easily taken care of using (2.4). Thus, we have

$$\gamma(u(x_0) - v(y_0))^+ \leq \omega\left\{|x_0 - y_0|^2 \left(1 + \frac{1}{\delta} |x_0 - y_0| \right)\right\},$$

and we conclude easily observing that $|x_0 - y_0|^2/\delta \to 0$ as $\delta \to 0_+$.

As it is standard in the theory of viscosity solutions, if $u$ or $v$ is Lipschitz, one can replace (2.18) in the proof by the weaker condition (2.19) observing that $D\Phi(\tilde{x} - \tilde{y})$ is bounded independently of $\varepsilon$ and $\delta$ (at least if all maxima remain in a compact subset of $\Omega$).

We now conclude this section by giving another example of an application of Proposition II.3 above: we will explain briefly how Proposition II.3 yields the uniqueness for Isaacs–Bellman equations, i.e., we take

$$F(x, t, p, A) = \inf_{\alpha \in \mathcal{A}} \sup_{\beta \in \mathcal{B}} \left\{-\text{Tr}\left\{\Sigma_{\alpha\beta}(x) \cdot \Sigma_{\alpha\beta}(x) \cdot A\right\}ight\}$$

$$+ (b_{\alpha\beta}(x), p) + c_{\alpha\beta}(x)t - f_{\alpha\beta}(x),$$

(2.20)
where $\mathcal{A}, \mathcal{B}$ are two given sets, $\Sigma_{x\beta}$ is (for all $x \in \overline{\Omega}$, $\alpha \in \mathcal{A}$, $\beta \in \mathcal{B}$) an element of $\mathbb{M}(n, m)$, $\Sigma_{x\beta}$, $b_{x\beta}$, $c_{x\beta}$, $f_{x\beta}$ are bounded uniformly continuous on $\overline{\Omega}$ uniformly in $\alpha \in \mathcal{A}$, $\beta \in \mathcal{B}$. We will use the following conditions

\begin{align}
&\exists c_0 > 0, \quad c_{x\beta}(x) \geq c_0 \quad \text{for all } \alpha \in \mathcal{A}, \beta \in \mathcal{B}, x \in \overline{\Omega} \quad (2.21) \\
&\|\Sigma_{x\beta}(x) - \Sigma_{x\beta}(y)\| \leq C |x - y|^\theta, \quad \forall x, y \in \overline{\Omega}, \forall \alpha \in \mathcal{A}, \forall \beta \in \mathcal{B} \quad (2.22) \\
&\left| b_{x\beta}(x) - b_{x\beta}(y) \right| \leq C |x - y|, \quad \forall x, y \in \overline{\Omega}, \forall \alpha \in \mathcal{A}, \forall \beta \in \mathcal{B} \quad (2.23)
\end{align}

for some $C \geq 0$.

The following result is essentially due to H. Ishii [26].

**Theorem II.2.** Assume that $F$ is given by (2.20) and that (2.21) holds. Then, if (2.23) and (2.22) hold with $\theta = 1$ or if $u$ or $v$ is locally Lipschitz in $\Omega$ and (2.22) holds with $\theta > \frac{1}{2}$, the comparison assertion (2.5) holds.

**Sketch of proof.** The main idea of the proof is the following consequence of (2.16). Choose $\omega(x, y) = (1/2\delta) |x - y|^2$ so that

$$D^2\omega(\tilde{x}, \tilde{y}) = \frac{1}{\delta} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix},$$

then, observing that if $\Sigma_1, \Sigma_2 \in \mathbb{M}(n, m)$ the following matrix

$$\begin{pmatrix} \Sigma_1 \cdot T\Sigma_1 & \Sigma_1 \cdot T\Sigma_2 \\ \Sigma_2 \cdot T\Sigma_1 & \Sigma_2 \cdot T\Sigma_2 \end{pmatrix}$$

is nonnegative, we may multiply (2.16) by this matrix and take the trace. In this way, we obtain

$$\text{Tr}(\Sigma_1 \cdot T\Sigma_1 \cdot X) + \text{Tr}(\Sigma_2 \cdot T\Sigma_2 \cdot Y) \leq \frac{1}{\delta} \text{Tr}\{((\Sigma_1 - \Sigma_2) \cdot T(\Sigma_1 - \Sigma_2))\}. \quad (2.24)$$

Then, the rest of the proof is quite similar to the proof of Theorem II.1: with the same notations, one obtains using (2.21), (2.22), (2.23), and (2.24)

$$c_0(u(x_0) - v(y_0))^+ \leq \omega(\sqrt{\delta}) + \frac{C}{\delta} |x_0 - y_0|^2$$

and one concludes easily since $(1/\delta) |x_0 - y_0|^2 \to 0$ as $\delta \to 0_+$. The case when $u$ or $v$ is Lipschitz is a standard adaptation.
III. Strictly Elliptic Equations

III.1. Main Uniqueness Result

We will consider throughout this section a particular class of Eqs. (1.1) that we call strictly elliptic equations. We will assume

\[ F(x, t, p, A) \leq F(x, t, p, B) - v_R \operatorname{Tr}(A - B) \quad \text{for some } v_R > 0 \quad (3.1) \]

if \( A \geq B, \ x \in \bar{\Omega}, \ |t| \leq R, \ p \in \mathbb{R}^n, \) for all \( R < \infty, \) and we will also use the weaker version of (3.1) where \( p \) varies just in \( B_R; \) we will denote by (3.1w) this weaker condition. It is quite clear that (3.1) is a much stronger notion of ellipticity than (1.2).

We will need to restrict the \( x \)-dependence of \( F \) and we will have to assume

\[ |F(x, t, p, A) - F(y, t, p, A)| \leq C_R + \omega_R(|x - y|) \ |x - y|^\tau \ |p|^{2+\tau} + \mu_R(|x - y|) \ |A| \quad (3.2) \]

for \( x, y \in \bar{\Omega}, \ |t| \leq R, \ p \in \mathbb{R}^n, \ A \in \mathbb{M}^n \) and \( R > 0, \) where \( \tau \in [0, 1], \ C_R \) is a positive constant and \( \omega_R, \mu_R \) are nonnegative functions on \([0, \infty)\) satisfying, respectively, \( \omega_R(\sigma) \to 0 \) as \( \sigma \to 0_+ \) and \( \int_{0}^{\infty} (\mu_R(\sigma)/\sigma) \ d\sigma < \infty. \)

We will also use another condition

\[ |F(x, t, p, A) - F(y, t, p, A)| \leq \omega_R(|x - y|^\theta (1 + \|A\|)) \quad (3.3) \]

for some \( \theta > \frac{1}{2}, \) if \( x, y \in \bar{\Omega}, \ |t| \leq R, \ |p| \leq R, \ A \in \mathbb{M}^n, \) where \( \omega_R(\sigma) \to 0 \) as \( \sigma \to 0_+ \) and \( \omega_R(r)(1 + r) \) is bounded for \( r \geq 0, \) for all \( R < \infty. \)

As usual, typical examples of \( \omega_R \) are \( \omega_R(\sigma) = K_R \sigma \) for some \( K_R \geq 0 \) in which case if we take (for instance) \( C_R = \tau - 0 \) in (3.2), (3.2) really means that the \( x \)-derivative of \( F \) grows at most quadratically in \( p \) and linearly in \( A \)—the so-called natural conditions ...

We may now state our main result.

**Theorem III.1.** (1) Assume (2.17), (3.1), (3.2), and (3.3), then the comparison assertion (2.5) holds.

(2) Assume (2.17), (3.1w), and (3.3). Then, if \( u \) or \( v \) is locally Lipschitz in \( \Omega, \) the comparison assertion (2.5) holds.

**Remarks.** (1) We do not know if conditions (3.2) or (3.3) are sharp; they are possibly not in the case of uniformly elliptic equations with Lipschitz solutions.
(2) Let us introduce a stronger form of (3.3):
\[
|F(x, t, p, A) - F(y, t, p, A)| \leq \omega_R(|x - y|(1 + |p| + \|A\|)) \tag{3.3s}
\]
for \(x, y \in \Omega\), \(|t| \leq R\), \(p \in \mathbb{R}^n\), \(A \in \mathcal{M}^n\) and \(R > 0\), where \(\omega_R(\sigma) \to 0\) as \(\sigma \to 0_+\).

As the proof of part (2) of the above theorem will show, assertion (2.5) holds if (2.17), (3.1), and (3.3s) are satisfied.

The proof of Theorem III.1 is presented in the next two subsections: we begin with the Lipschitz case in Section III.3, while the proof of (1) will be given in Section III.2.

III.2. Proof of Part (2) of Theorem III.1

We begin by proving part (2) of Theorem III.1. The first step consists in passing to the limit as \(\varepsilon\) goes to 0, in Proposition II.3. To this end, we consider \(\psi(x, y) = (1/2\delta)|x - y|^2\) and maximize over \(\Omega \times \Omega\) the function
\[
u(x) - w \frac{1}{2\delta} |x - y|^2 = u(x) - v(y) - \varphi(x, y).
\]

To simplify the presentation, we argue by contradiction and assume that \(\sup_{\Omega \times \Omega} (u - v) > M\). From this, we deduce easily that for \(\delta\) small enough, the above function achieves its maximum over \(\Omega \times \Omega\) and that its maxima lie in a fixed compact subset of \(\Omega \times \Omega\). We then fix \(\delta\) small enough and we consider for \(\varepsilon \in (0, 1]\) the function \(w_{\varepsilon}(x, y) = u^\varepsilon(x) - v^\varepsilon(y) - \varphi(x, y)\), and we maximize it over \(\Omega \times \Omega\). From the definitions (2.6)-(2.7) of \(u^\varepsilon, v^\varepsilon\), we see that
\[
\sup_{\Omega \times \Omega} \{u^\varepsilon(x) - v^\varepsilon(y) - \varphi(x, y)\} \geq \sup_{\Omega \times \Omega} \{u(x) - v(y) - \varphi(x, y)\}. \tag{3.4}
\]

On the other hand if \((\bar{x}_\varepsilon, \bar{y}_\varepsilon)\) is a maximum point, there exist \((\tilde{x}_\varepsilon, \tilde{y}_\varepsilon) \in \Omega \times \Omega\) such that
\[
u^\varepsilon(\tilde{x}_\varepsilon) = u(\tilde{x}_\varepsilon) - \frac{1}{\varepsilon} |\tilde{x}_\varepsilon - \tilde{x}_\varepsilon|^2, \quad v^\varepsilon(\tilde{y}_\varepsilon) = v(\tilde{y}_\varepsilon) + \frac{1}{\varepsilon} |\tilde{y}_\varepsilon - \tilde{y}_\varepsilon|^2 \tag{3.5}
\]
and \(|\tilde{x}_\varepsilon - \tilde{x}_\varepsilon| \leq C_0 \sqrt{\varepsilon}, |\tilde{y}_\varepsilon - \tilde{y}_\varepsilon| \leq C_0 \sqrt{\varepsilon}\). Hence, we deduce from (3.4) and (3.5)
\[
\lim_{\varepsilon} \{u(\tilde{x}_\varepsilon) - v(\tilde{y}_\varepsilon) - \varphi(\tilde{x}_\varepsilon, \tilde{y}_\varepsilon)\} = \lim_{\varepsilon} \{u^\varepsilon(\tilde{x}_\varepsilon) - u^\varepsilon(\tilde{y}_\varepsilon) - \varphi(\tilde{x}_\varepsilon, \tilde{y}_\varepsilon)\} \geq \sup_{\Omega \times \Omega} \{u(x) - v(y) - \varphi(x, y)\}
\]
therefore (extracting subsequences if necessary), \((\tilde{x}_\varepsilon, \tilde{y}_\varepsilon)\) converges to a maximum over \(\Omega \times \Omega\) of \(u(x) - v(y) - \phi(x, y)\) which lie in a compact subset of \(\Omega \times \Omega\). In conclusion, we deduce that, for \(\varepsilon\) small enough, \(w\) admits a maximum point over \(\bar{\Omega}_\varepsilon \times \bar{\Omega}_\varepsilon\) that we denote by \((\tilde{x}_\varepsilon, \tilde{y}_\varepsilon)\) which belongs to \(\Omega_\varepsilon \times \Omega_\varepsilon\). Furthermore, \((\tilde{x}_\varepsilon, \tilde{y}_\varepsilon)\) converges to a maximum \((\tilde{x}, \tilde{y})\) over \(\Omega \times \Omega\) of \(u(x) - v(y) - \phi(x, y)\) and \(u'(\tilde{x}_\varepsilon), v'(\tilde{y}_\varepsilon)\) converge respectively to \(u(\tilde{x}), v(\tilde{y})\). Finally, observe that since \(u\) or \(v\) is locally Lipschitz on \(\Omega\) then \(D_x \phi(\tilde{x}_\varepsilon, \tilde{y}_\varepsilon) = -D_y \phi(\tilde{x}_\varepsilon, \tilde{y}_\varepsilon) = (1/\delta)(\tilde{x}_\varepsilon - \tilde{y}_\varepsilon)\) is bounded independently of \(\varepsilon\) and \(\delta\).

We may now apply Proposition II.3 and we deduce the existence of \(X_\varepsilon, Y_\varepsilon \in \mathbb{M}^n\) such that

\[
F_\varepsilon(\tilde{x}_\varepsilon, u'(\tilde{x}_\varepsilon), \frac{1}{\delta}(\tilde{x}_\varepsilon - \tilde{y}_\varepsilon), X_\varepsilon) \leq 0
\]

(3.6)

\[
F'(\tilde{y}_\varepsilon, v'(\tilde{y}_\varepsilon), \frac{1}{\delta}(\tilde{x}_\varepsilon - \tilde{y}_\varepsilon), -Y_\varepsilon) \geq 0
\]

(3.7)

\[-\frac{2}{\varepsilon} I \preceq \begin{pmatrix} X_\varepsilon & 0 \\ 0 & Y_\varepsilon \end{pmatrix} \preceq \frac{1}{\delta} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.
\]

(3.8)

Using (3.1w), we deduce from (3.6) that for some \(\nu > 0\)

\[
\nu \text{ Tr}(X_\varepsilon^+ - X_\varepsilon) \leq -F_\varepsilon(\tilde{x}_\varepsilon, u'(\tilde{x}_\varepsilon), \frac{1}{\delta}(\tilde{x}_\varepsilon - \tilde{y}_\varepsilon), X_\varepsilon^+) \leq C_\delta
\]

(3.9)

for some positive constant \(C_\delta\) independent of \(\varepsilon\). The last inequality is due to (3.8) which implies obviously \(X_\varepsilon \leq (1/\delta)I\) and thus

\[
0 \leq X_\varepsilon^+ \leq \frac{1}{\delta} I,
\]

i.e., \(X_\varepsilon^+\) is bounded independently of \(\varepsilon\). But this combined with (3.9) immediately implies that \(X_\varepsilon\) is bounded independently of \(\varepsilon\). Similarly, one shows that \(Y_\varepsilon\) is bounded independently of \(\varepsilon\). Extracting subsequences if necessary, we may assume without loss of generality that \(X_\varepsilon \to X \in \mathbb{M}^n\), \(Y_\varepsilon \to Y \in \mathbb{M}^n\). And passing to the limit in (3.6)–(3.8) we deduce

\[
F\left(\tilde{x}, u(\tilde{x}), \frac{1}{\delta}(\tilde{x} - \tilde{y}), X\right) \leq 0
\]

(3.10)

\[
F\left(\tilde{y}, v(\tilde{y}), \frac{1}{\delta}(\tilde{x} - \tilde{y}), -Y\right) \geq 0
\]

(3.11)

\[
\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq \frac{1}{\delta} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.
\]

(3.12)
Recall also from the above considerations that there exists a positive constant $C$ independent of $\delta$ such that

$$\frac{1}{\delta} \left| \bar{x} - \bar{y} \right| \leq C. \quad (3.13)$$

We want now to conclude the proof of (2.5) using these inequalities and the assumptions made upon $F$. To this end, we recall that (3.12) implies $X + Y \leq 0$ and using (3.11w) we deduce from subtracting (3.11) from (3.10) and using (2.17)

$$\gamma(u(\bar{x}) - v(\bar{y}))^+ + v \left| \text{Tr}(X + Y) \right| \leq \omega(|\bar{x} - \bar{y}|^\theta(1 + \|X\|)) \quad (3.14)$$

for some $\gamma, v > 0$ (independent of $\delta$) and where $\omega(\sigma) \to 0$ as $\sigma \to 0_+$. We will need the following “matrix analysis” result.

**Lemma III.1.** Let $X, Y \in \mathbb{M}^n$ satisfy (3.12). Then, there exists a positive constant $C_1$ (depending only on $n$) such that we have

$$\|X\|, \|Y\| \leq C_1 \left\{ \|\delta^{-1/2}\|\text{Tr}(X + Y)\|^{1/2} + \|\text{Tr}(X + Y)\| \right\}. \quad (3.15)$$

We first conclude the uniqueness proof and then we show Lemma III.1. Now, if we plug the estimates (3.13), (3.15) in (3.14), we deduce

$$\gamma(u(\bar{x}) - v(\bar{y}))^+ \leq \omega(C\delta^{-\theta/2} \|\text{Tr}(X + Y)\|^{1/2}$$

$$+ C\delta^\theta \|\text{Tr}(X + Y)\| - v \|\text{Tr}(X + Y)\| \leq \sup_{t \geq 0} \left[ \omega(C\delta^{-\theta/2} t^{1/2} + C\delta t) - vt \right]$$

and we conclude observing that the right-hand side goes to 0 as $\delta$ goes to $0_+$ since $\theta > \frac{1}{2}$ and $\omega(\sigma) \to 0$ as $\sigma \to 0_+$.

**Proof of Lemma III.1.** Recalling that $X + Y \leq 0$, we see that

$$\|X + Y\| \leq C \|\text{Tr}(X + Y)\|, \quad (3.16)$$

where $C$ denotes various positive constants depending only on $n$. Therefore, in order to prove (3.15), we just have to bound $\|X - Y\|$.

To this end, we multiply (3.12) to the right by ($\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}$) and then to the left by the same matrix. In this way, we find

$$\begin{pmatrix} X + Y & X - Y \\ Y - Y & X + Y \end{pmatrix} \leq \frac{4}{\delta} \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}. \quad (3.17)$$
Next, let \( t \in \mathbb{R}, \, \xi \in \mathbb{R}^n \) with \( |\xi| = 1 \), calculating the quadratic forms associated with the matrices entering (3.17) on \( (\xi) \) we obtain
\[
(1 + t^2)((X + Y)\xi, \xi) + 2t((X - Y)\xi, \xi) \leq \frac{4}{\delta} \quad \text{for all} \quad t \in \mathbb{R}
\]
hence for all \( |\xi| = 1 \)
\[
((X - Y)\xi, \xi)^2 \leq \left( \frac{4}{\delta} - ((X + Y)\xi, \xi) \right) \cdot (-((X + Y)\xi, \xi))
\]
and we conclude easily recalling (3.16).

III.3. Proof of Part (1) of Theorem III.1

Without loss of generality, we may assume that \( M = 0 \) and that arguing by contradiction
\[
d = \sup_{\Omega} (u - v) > 0. \tag{3.18}
\]
The main idea of the proof below is to replace the Lipschitz continuity of \( u \) or \( v \) by some ad hoc estimate. The precise form of this estimate is
\[
u(x) - v(y) \leq d + C |x - y| \quad \text{for} \quad x, y \in \Omega \tag{3.19}
\]
for some positive constant \( C \).

We first admit this estimate and we prove (2.5). We follow the arguments of Section III.2 and we find \((\bar{x}, \bar{y}) \in \Omega \times \Omega \) a maximum point of \( u(x) - v(y) - (1/2\delta) |x - y|^2 \) (for \( \delta > 0 \) small enough) where by Proposition II.3 (3.10)–(3.12) hold. Therefore, we just have to explain why (3.19) implies (3.13) and then the rest of the argument presented in Section III.2 applies and we conclude. Indeed, we just observe that on one hand
\[
u(\bar{x}) - v(\bar{y}) - \frac{1}{2\delta} |\bar{x} - \bar{y}|^2 = \max_{(x, y) \in \Omega \times \Omega} u(x) - v(y) - \frac{1}{2\delta} |x - y|^2
\]
\[
\geq \max_{x \in \Omega} u(x) - v(x) = d
\]
while on the other hand (3.19) implies
\[
u(\bar{x}) - v(\bar{y}) - \frac{1}{2\delta} |\bar{x} - \bar{y}|^2 \leq d + C |\bar{x} - \bar{y}| - \frac{1}{2\delta} |\bar{x} - \bar{y}|^2.
\]
Therefore,

\[ C |\bar{x} - \bar{y}| \geq \frac{1}{2\delta} |\bar{x} - \bar{y}|^2 \]

and (3.13) is proven.

In order to complete the proof of Theorem III.1, we just have to show (3.19), the proof of it is given by the

**PROPOSITION III.1.** We assume (2.17), (3.1), (3.2), \( M = 0 \) and \( \text{sup}_{\Omega} (u - v) > 0 \). Then, (3.19) holds.

**Proof.** We adapt an argument due to [10]. To simplify the presentation we observe that if \( R_0 = \text{sup}_{\Omega}(|u| + |v|) \), then by simple scalings we may always assume that \( v_{R_0} = 1 \) in (3.1) and we denote \( \omega = \omega_{R_0}, \mu = \mu_{R_0}, \) and \( C_0 = C_{R_0} \). We may assume that \( \mu \) is continuous and nondecreasing on \([0, \infty)\) and \( \mu(r) \geq r \) for \( r \geq 0 \). We set

\[
l(r) = \int_0^r ds \int_0^s \frac{\mu(\sigma)}{\sigma} d\sigma \quad \text{for} \quad r \geq 0,
\]

\[
\Phi(x) = d + MK|x| - Ml(K|x|) \quad \text{and} \quad \varphi(x, y) = \Phi(x - y)
\]

for \( x, y \in \mathbb{R}^n \), where \( M \) and \( K \) are constants to be chosen later. We choose \( r_0 > 0 \) so that \( l(r_0) = \frac{1}{2} \) and observe that if \( K|x| \leq r_0 \), then \( \Phi(x) \geq d + (M/2)K|x| \) since \( l(r) \leq rl'(r) \). We then set \( \Delta_K = \{(x, y) \in \Omega \times \Omega/|x - y| < r_0/K\} \) and fix \( M \) so that \( \text{sup}_{x, y \in \Omega} (u(x) - v(y)) \leq (M/2)r_0 \). Next, we remark that there exists \( \alpha > 0 \) such that if \( |x - y| < \alpha \) and \( x \) or \( y \in \partial \Omega \), then \( u(x) - v(y) \leq d/2 \), and therefore

\[
u(x) - v(y) \leq d + C|x - y| \quad \text{for} \quad (x, y) \in \partial(\Omega \times \Omega) \quad (3.20)
\]

for some constant \( C > 0 \). We thus deduce that if \( K \) is restricted so that \( MK \geq 2C \), then

\[
u(x) - v(y) - \varphi(x, y) \leq 0 \quad \text{for} \quad (x, y) \in \partial \Delta_K. \quad (3.21)
\]

We claim that taking \( K \) large enough, (3.21) implies

\[
u(x) - v(y) - \varphi(x, y) \leq 0 \quad \text{for} \quad (x, y) \in \Delta_K. \quad (3.22)
\]

If this is the case, (3.19) is proven. To show (3.22) we argue by contradiction, i.e., we assume

\[
\text{sup}_{(x, y) \in \Delta_K} (\nu(x) - v(y) - \varphi(x, y)) > 0. \quad (3.23)
\]
Next, we apply the proof of Section III.2, to find \((\bar{x}, \bar{y}) \in A_k\) and matrices \(X, Y \in \mathbb{M}^n\) such that

\[
F(\bar{x}, u(\bar{x}), D\Phi(\bar{x} - \bar{y}), X) \leq 0 \leq F(\bar{y}, v(\bar{y}), D\Phi(\bar{x} - \bar{y}), -Y),
\]

(3.24)

\[
\begin{pmatrix}
X & 0 \\
0 & Y
\end{pmatrix} \leq \begin{pmatrix}
B & -B' \\
-B & B
\end{pmatrix}, \quad B = D^2\Phi(\bar{x} - \bar{y}),
\]

(3.25)

\[
u(\bar{x}) - v(\bar{y}) - \Phi(\bar{x} - \bar{y}) = \sup_{(x, y) \in A_k} (u(x) - v(y) - \Phi(x - y)).
\]

(3.26)

Observe also that (3.23) and (3.26) show that \(\bar{x} \neq \bar{y}\) (so that \(D\Phi\) and \(D^2\Phi\) make sense at \(\bar{x} - \bar{y}\)). Lemma III.1 and its proof then yield

\[
X + Y \leq 0, \quad X + Y \leq 4B,
\]

(3.27)

\[
\|X\|,\|Y\| \leq C\{|\text{Tr}(X + Y)| + \|B\|^{1/2} |\text{Tr}(X + Y)|^{1/2}\}
\]

(3.28)

for some constant \(C > 0\) depending only on \(n\).

We next claim that if \(P \in \mathbb{M}^n\) and \(0 \leq P \leq I\), then

\[
\text{Tr}(X + Y) \leq 4 \text{Tr} PB.
\]

(3.29)

Indeed, since \(X + Y \leq 0\) and \(P \leq I\), we have

\[
\text{Tr}(X + Y) \leq \text{Tr} P(X + Y),
\]

while, since \(X + Y \leq 4B\) and \(P \geq 0\), we have

\[
\text{Tr} P(X + Y) \leq 4 \text{Tr} PB.
\]

We then use (3.29) with \(P = (1/|\bar{x} - \bar{y}|^2)(\bar{x} - \bar{y}) \otimes (\bar{x} - \bar{y})\). Computing \(\text{Tr} PB\), we find easily

\[
\text{Tr} PB = -MK^2l^n(K|\bar{x} - \bar{y}|) = -MK \mu(K|\bar{x} - \bar{y}|)/|\bar{x} - \bar{y}|.
\]

(3.30)

Therefore, by (3.24), (2.17), (3.1), (3.2), and (3.28), we deduce

\[
0 \geq |\text{Tr}(X + Y)| - C_0 - \omega(|\bar{x} - \bar{y}|)|\bar{x} - \bar{y}|^\tau |D\Phi(\bar{x} - \bar{y})|^{2+\tau}
- C\mu(|\bar{x} - \bar{y}|)(|D^2\Phi(\bar{x} - \bar{y})|^{1/2}|\text{Tr}(X + Y)|^{1/2} + |\text{Tr}(X + Y)|).
\]

From this, using (3.30) and

\[
|D\Phi(\bar{x} - \bar{y})| \leq MK(1 - l'(r_0)),
\]

\[
|D^2\Phi(\bar{x} - \bar{y})| \leq C_1 MK(1 + \mu(r_0))/|\bar{x} - \bar{y}|,
\]
where \( C_1 \) is a constant independent of \( K \), we get

\[
\frac{1}{2} |\text{Tr}(X + Y)| + 2MK^2 \frac{\mu(K\bar{x} - \bar{y})}{K|\bar{x} - \bar{y}|} \\
\leq C_2\mu(|\bar{x} - \bar{y}|) |\text{Tr}(X + Y)| + MK^2 \frac{\mu(|\bar{x} - \bar{y}|)}{K|\bar{x} - \bar{y}|} \\
+ C_0 + C_2K^2\omega(|\bar{x} - \bar{y}|)(K|\bar{x} - \bar{y}|)^\gamma,
\]

where \( C_2 \) is a constant independent of \( K \). This yields a contradiction for \( K \) large enough, which proves (3.22) for large \( K \) and hence (3.19).

IV. EXTENSIONS AND ADAPTATIONS

IV.1 General Conditions

We have seen in the preceding sections three different situations where we can prove (2.5). We want to make in this section various remarks indicating possible extensions of the preceding results. Let us warn the reader that we will not give precise results here.

The first observation is that we may combine what we did above to treat nonlinearities \( F \) given by

\[
F = F_1 + F_2 + F_3,
\]

(4.1)

where \( F_1 \) satisfies (2.18) (or (2.19) if we deal with a locally Lipschitz sub-solution or supersolution), \( F_2 \) is given by (2.20) and (2.21)–(2.22) hold with \( \theta = 1 \) (or \( \theta > \frac{1}{2} \) if \( u \) or \( v \) is locally Lipschitz) and \( F_3 \) is of the type considered in Section III. Let us remark further that \( F_3 \) may act only on some second-order derivatives, for instance

\[
F_3 = F_2(x, t, p, A_k),
\]

(4.2)

where \( A_k = P_kAP_k, P_k = (t_0 t_0) \) and \( I_k \) is the \( k \times k \) identity matrix for some \( 1 \leq k \leq n \). In this case, (3.1) is required with \( \text{Tr}(A - B) \) replaced by \( \text{Tr}(A_k - B_k) \).

Another possible extension is to replace (2.20) by

\[
F_2 = \inf_{\alpha \in \mathcal{A}} \sup_{\beta \in \mathcal{B}} \{ -\text{Tr}\{\Sigma_{x\beta}(x, p) \cdot T\Sigma_{x\beta}(x, p) \cdot A\} + H_{x\beta}(x, t, p)\},
\]

(4.3)

where \( H_{x\beta} \in C(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n) \) satisfy (uniformly in \( \alpha, \beta \)) (2.17) and

\[
|H_{x\beta}(x, t, p) - H_{x\beta}(y, t, p)| \leq \omega_R(|x - y|(1 + |p|))
\]

(4.4)
for all $x, y \in \bar{Q}$, $|t| \leq R$, $p \in \mathbb{R}^n$, where $\omega_R(\sigma) \to 0$ as $\sigma \to 0$, for all $R < \infty$—as usual, if $u$ or $v$ is locally Lipschitz, we may further restrict $p$ to lie in $B_R$ in condition (4.4). We also have to assume that $\Sigma_{x\beta}$ satisfy (2.21) uniformly in $p \in \mathbb{R}^n$ (or $p \in B_R$ if $u$ or $v$ is locally Lipschitz).

A related remark is that we may take inf sup over collections of nonlinearities $F_{x\beta}$ each of which satisfy one of the assumptions given above (including (4.1) with $F_3$ depending only on some derivatives, which may even depend on the parameters $\alpha, \beta$ ...).

Our final remark consists in observing that if there exists $F_\epsilon$ satisfying the above conditions such that $F_\epsilon$ converges uniformly on $\bar{Q} \times [-R, R] \times \mathbb{R}^n \times \mathcal{M}^n$ (or $\bar{Q} \times [-R, R] \times \bar{B}_R \times \mathcal{M}^n$ if $u$ or $v$ is locally Lipschitz) to $F$ then the comparison assertion (2.5) holds true.

However, the precise class of nonlinearities $F$ we can achieve with all these remarks is not clear at all and we will come back on general structure conditions in Section IV.3 below.

IV.2. Parabolic Problems

We want to explain in this section how to adapt the preceding arguments in order to treat parabolic type problems, i.e., equations of the following form

$$\frac{\partial u}{\partial t} + F(x, t, u, D_x u, D_x^2 u) = 0 \quad \text{in } Q,$$

where $Q = \Omega \times (0, T)$ for some $T > 0$, $F \in C(\bar{Q} \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{M}^n)$. Observe that, as usual, replacing $x$ by $(x, t)$ immediately allows us to write (4.5) as a special case of (1.1) while the special structure of (4.5) will enable us to achieve a little more generality in our uniqueness results.

Again, we will not state precise results. Instead, we will only indicate what are the necessary adaptations in order to prove uniqueness. Let us first mention that all the above uniqueness results are still true provided we make the same assumptions on $F$ uniformly in $t \in [0, T]$. We may even relax (2.17) and assume only

$$F(x, t, p, A) - F(x, s, p, A) \geq \gamma_R(t-s) \quad \text{if } (x, t) \in \bar{Q},$$

$R \geq t \geq s > -R$, $p \in \mathbb{R}^n$, $A \in \mathcal{M}^n$ for some $\gamma_R > -\infty$, for all $R < \infty$.

The strategy of the proof is basically the same as in Section II: we just have to replace Proposition II.3 by the following result (keeping the same notations as in Proposition II.3 ...).
**Proposition IV.1.** Let \( \varphi \in C^2(\Omega \times \Omega \times [0, T]) \), set
\[
 w(x, y, t) = u^\varepsilon(x, t) - v_\varepsilon(y, t) \quad \text{for } x, y \in \Omega_\varepsilon, t \in [0, T] \tag{4.7}
\]
and assume that \( w - \varphi \) achieves its maximum over \( \tilde{\Omega}_\varepsilon \times \Omega_\varepsilon \times [0, T] \) at a point \((\tilde{x}, \tilde{y}, \tilde{t}) \in \Omega_\varepsilon \times \Omega_\varepsilon \times (0, T)\). Then, if \( (2.11) \) holds for all \( t \in [0, T] \), there exist matrices \( X, Y \in \mathbb{M}^n \) such that
\[
 \frac{\partial}{\partial t} \varphi(\tilde{x}, \tilde{y}, \tilde{t}) + F_u(\tilde{x}, \tilde{t}, u^\varepsilon(\tilde{x}, \tilde{t}), D_x \varphi(\tilde{x}, \tilde{y}, \tilde{t}), X) \\
 - F_v(\tilde{y}, \tilde{t}, v_\varepsilon(\tilde{y}, \tilde{t}), -D_y \varphi(\tilde{x}, \tilde{y}, \tilde{t}), -Y) \leq 0 \tag{4.8}
\]
\[
 -\frac{2}{\varepsilon} I \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq D^2_{\varphi}(\tilde{x}, \tilde{y}, \tilde{t}). \tag{4.9}
\]

**Remarks.**
1. The super and subscripts \( \varepsilon \) refer to the same formulas as in Section II that is regularizations in \( x \) only.
2. Assuming \( (2.11) \) is not a restriction since, as usual, the case when \( F \) satisfies only \( (4.6) \) can be reduced to that case by a simple change of unknowns.

The proof of Proposition IV.1 is very much the same as the proof of Proposition II.3: one just introduces for \( \varepsilon' \in (0, 1) \)
\[
 u^{\varepsilon, \varepsilon'}(x, t) = \sup_{y \in \Omega, s \in [0, T]} \left\{ u(y, s) - \frac{1}{\varepsilon} |x - y|^2 - \frac{1}{\varepsilon'} (t-s)^2 \right\} \tag{4.10}
\]
and
\[
 v_{\varepsilon, \varepsilon'}(x, t) = \inf_{y \in \Omega, s \in [0, T]} \left\{ v(y, s) + \frac{1}{\varepsilon} |x - y|^2 + \frac{1}{\varepsilon'} (t-s)^2 \right\}. \tag{4.11}
\]
Then, one shows the exact analogue of Proposition II.3 for such double-parameters regularizations and one finally lets \( \varepsilon' \) go to 0, to obtain Proposition IV.1.

**IV.3. Another Look at the Uniqueness Proofs and Structure Conditions**

In this section, we want to shed a slightly different light at the uniqueness strategy presented in Section II. These observations will in fact lead to structure conditions on \( F \) which yield uniqueness results.

We begin by recalling the notions of sub- and superdifferentials of order two as defined in P.L. Lions [37]: let \( u \) be an upper semicontinuous
function (resp. lower semicontinuous) on a bounded open set $\Omega$ and let $x_0 \in \Omega$, we denote by

$$D_+^2 u(x_0) = \{(p, A) \in \mathbb{R}^n \times \mathbb{M}^n / u(x) \leq u(x_0) + (p, x - x_0) + \frac{1}{2}(Ax - x_0 - x_0) + o(|x - x_0|^2) \text{ for } x \in \Omega \}$$

(resp. $D_-^2 u(x_0) = \{(p, A) \in \mathbb{R}^n \times \mathbb{M}^n / u(x) \geq u(x_0) + (p, x - x_0) + \frac{1}{2}(Ax - x_0 - x_0) + o(|x - x_0|^2) \text{ for } x \in \Omega \}$)

and we will say that $D_+^2 u(x_0)$ (resp. $D_-^2 u(x_0)$) is the superdifferential of order two of $u$ at $x_0$ (resp. subdifferential). Next, we denote by

$$\bar{D}_+^2 u(x_0) = \{(p, A) \in \mathbb{R}^n \times \mathbb{M}^n / \exists x_n \in \Omega, x_n \to x_0, u(x_n) \to u(x_0),$$

$$\exists(p_n, A_n) \in D_+^2 u(x_n), p_n \to p, A_n \to A \}$$

(resp. $\bar{D}_-^2 u(x_0) = \{(p, A) \in \mathbb{R}^n \times \mathbb{M}^n / \exists x_n \in \Omega, x_n \to x_0, u(x_n) \to u(x_0),$

$$\exists(p_n, A_n) \in D_-^2 u(x_n), p_n \to p, A_n \to A \} \).$$

Then, as in [40], $u$ is a viscosity subsolution (resp. supersolution) of (1.1) if and only if

$$F(x, (u(x), p, A)) \leq 0, \forall (p, A) \in \bar{D}_+^2 u(x), \forall x \in \Omega \quad (4.12)$$

(resp. $F(x, (u(x), p, A)) \geq 0, \forall (p, A) \in \bar{D}_-^2 u(x), \forall x \in \Omega \). \quad (4.13)$$

Recall also that we always assume (1.2).

It is possible to look at the uniqueness strategy as explained in Section II with a slightly different viewpoint and this will lead to some structure condition implying the uniqueness. Let $u, v$ be respectively upper semicontinuous, lower semicontinuous and bounded on $\Omega$. We denote by

$$M = \sup_{x \in \Omega} u(x) - v(x), \quad (4.14)$$

and in the course of proving uniqueness or comparison, we may always assume that

$$\sup \{ \lim_{n} u(x_n) - v(y_n)/(x_n)_n \subset \Omega, (y_n)_n \subset \Omega, x_n - y_n \to 0, \text{dist}(x_n, \partial \Omega) \to 0, \text{dist}(y_n, \partial \Omega) \to 0 \} < M. \quad (4.15)$$

Then, Proposition II.3 can be translated in a general statement about continuous (or even semicontinuous) functions, namely,
**Lemma IV.1.** With the above notations and assumptions, for $\varepsilon > 0$, $\delta > 0$ small enough, $u'(x) - v'(y) - (1/2\delta) |x - y|^2$ achieves its maximum over $\Omega \times \Omega$ at $(\tilde{x}, \tilde{y})$ and there exist $\tilde{x}, \tilde{y} \in \Omega$, $X, Y \in \mathbb{M}^n$ such that

$$u(x) - v(y) - \frac{1}{2(\varepsilon + \delta)} |x - y|^2$$

achieves its maximum over $\Omega \times \Omega$ at $\tilde{x}, \tilde{y}$,

$$\left(\frac{\tilde{x} - \tilde{y}}{\varepsilon + \delta}, X\right) \in D^+_2 u(\tilde{x}), \quad \left(\frac{\tilde{x} - \tilde{y}}{\varepsilon + \delta}, -Y\right) \in D^-_2 v(\tilde{y}),$$

$$X \geq \frac{2}{\varepsilon} I, \quad Y \geq \frac{2}{\varepsilon} I,$$

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq \frac{1}{\delta} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

**Remarks.** (1) It would be extremely interesting to have a direct proof of this result which does not use measure theory.

(2) Note of course that if $u, v$ are twice differentiable at $\tilde{x}, \tilde{y}$ then (4.17) and (4.19) are obvious.

Using this result, we deduce immediately the following uniqueness result.

**Theorem IV.1.** Assume (2.7) and that $u, v$ are respectively a viscosity subsolution, supersolution of (1.1). Let $F$ satisfy the following structure condition for $\delta, \varepsilon$ small enough

$$|x - y|^2 I + \frac{1}{\delta} |x - y|^2$$

for all $x, y \in \Omega$, $|x - y| \leq \gamma$, $|t| \leq R$, $X, Y \in \mathbb{M}^n$ satisfying (4.18)-(4.19),

$$F(\frac{x - y}{\delta + \varepsilon}, X) - F(\frac{x - y}{\delta + \varepsilon}, -Y) \geq -\omega_R (|x - y| + \frac{|x - y|^2}{\delta})$$

for some $\gamma > 0$, where $\omega_R(\sigma) \to 0$ if $\sigma \to 0_+$ $(\forall R < \infty)$. Then, the comparison assertion (2.5) holds.

**Remarks.** (1) It is not difficult to check that the proofs made for the uniqueness results presented in the preceding sections consist in fact in checking the structure condition (4.20).

(2) The main difficulty in checking this condition consists clearly in the generality of matrices $X, Y$ satisfying (4.18) and (4.19). This is why it is worth noting that if $X, Y$ satisfy (4.19) then first of all $X, Y < (1/\delta)I$ and furthermore $Y \leq -X(I - \delta X)^{-1}$. Hence, it is enough to check (4.20) with $Y = -X(I - \delta X)^{-1}$ (use indeed the ellipticity condition (1.2)).
To prove the claim about \( Y \) one just has to prove it when \( X \) is diagonal, i.e., \( X = (x_i \delta \iota) \) with \( x_i < 1/\delta \). Then, one observes that for all \( \xi, \eta \in \mathbb{R}^n \)

\[
0 \geq ((Y + X(I - \delta X)^{-1}) \eta, \eta) + \sum_i x_i \xi_i^2 - \sum_i \frac{x_i}{1 - \delta x_i} \eta_i^2 - \frac{1}{\delta} \sum_i (\xi_i - \eta_i)^2,
\]

i.e.,

\[
0 \geq ((Y + X(I - \delta X)^{-1}) \eta, \eta) - \sum_i \left( \frac{1}{\delta} - x_i \right) \xi_i^2 + \frac{1}{\delta(1 - \delta x_i)} \eta_i^2 + \frac{2}{\delta} \xi_i \eta_i \geq ((Y + X(I - \delta X)^{-1}) \eta, \eta) - \sum_i \frac{1 - \delta x_i}{\delta} \left( \xi_i + \frac{1}{1 - \delta x_i} \eta_i \right)^2,
\]

and this clearly yields our claim. It is possible to give another proof as follows:

\[
(X \xi, \xi) - (Y \eta, \eta) \leq \frac{1}{\delta} |\xi - \eta|^2.
\]

Take \( \xi = (I - \delta X)^{-1} \) here. Then

\[
(I - \delta X)^{-1} X(I - \delta X)^{-1} + Y \leq \frac{1}{\delta} ((I - \delta X)^{-1} - I)^2
\]

\[
= \frac{1}{\delta} ((I - \delta X)^{-1}(I - I + \delta X))^2
\]

\[
= (I - \delta X)^{-1} X^2(I - \delta X)^{-1}.
\]

Hence,

\[
Y \leq - (I - \delta X)^{-1} X(\delta X - I)(\delta X - I)^{-1} = -(I - \delta X)^{-1} X.
\]

Let us remark finally that if \( Y = -X(I - \delta X)^{-1} \) and \( X < (1/\delta)I \) then \( Y < (1/\delta)I \) and \( X = -Y(I - \delta Y)^{-1} \) (i.e., the structure condition obtained from (4.20) by considering only \( Y = -X(I - \delta X)^{-1} \) is "symmetric in \( X, Y \)).

V. OTHER CLASSES OF EQUATIONS

In this section, we want to explain how to obtain uniqueness results for special classes of equations when (2.17) does not hold (but we still have (2.11)). To this end, it is worth noting that if (2.11) holds, then (2.5) is still
valid in the context of all the results above but where we replace (3.17) by
the following assumption

\[ u \text{ is a strict subsolution of (1.1)} \]

or

\[ v \text{ is a strict supersolution of (1.1)}. \]

Here, by strict we mean that, for instance in the case of subsolutions, there
exists \( g \in C(\overline{\Omega}) \) such that \( u \) is viscosity subsolution of

\[ F(x, u, Du, D^2u) = g \quad \text{in } \Omega, \quad \text{and } g < 0 \quad \text{in } \Omega. \]

Then, a typical strategy in order to show (2.5) without assuming (5.1) is
to make some small perturbation of \( u \) (or \( u \)) which will yield a new sub-
solution (or supersolution) satisfying (5.1).

V.1. Strictly Elliptic Equations without Zeroth Order Terms

In this subsection, we will always assume that \( F \) satisfies (3.1), (3.2),
(3.3), and (2.11) (we may replace (3.1)–(3.2) by (3.1w) and (3.3) if \( u \) or \( v \)
is locally Lipschitz in \( \Omega \)). As we explained above, we are going to perturb
\( u \) or \( v \). To simplify the presentation, we will consider here only a perturba-
tion of the subsolution \( u \).

We first claim that (5.2) is valid if we assume that there exist \( \varepsilon, \gamma, > 0, 
\phi, \in C^2(\overline{\Omega}) \) such that for all \( R < \infty \) (or at least \( R = \|u\|_{\infty} \)) and for each
compact subset \( K \) we have

\[ F(x, \alpha_m t + \phi_m(x), \alpha_m p + D\phi_m(x), \alpha_m A + D^2\phi_m(x)) \leq -\varepsilon_m < 0, \]

whenever \( F(x, t, p, A) \leq 0 \) and \( x \in K, |t| \leq R, p \in \mathbb{R}^n, A \in \mathbb{M}^n \)

\[ \text{for some } \varepsilon_m > 0, \text{ and furthermore } \alpha_m \to 1, \phi_m \to 0 \text{ in } C^2. \]

As usual, if \( u \) is Lipschitz on \( \Omega \), we may restrict \( p \) in (5.3) to lie in \( B_{R_0} \) where
\( R_0 = \|Du\|_{\infty} \). Indeed, if we consider \( u_m = \alpha_m u + \phi_m \), we check immediately
using (5.3) that \( u_m \) is a strict subsolution of (1.1). Therefore, (5.1) holds
and we conclude in view of the preliminary observation we made above.

Next, we want to give a few examples where (5.3) holds. The first exam-
ple consists in assuming that for all \( R < \infty \) (again \( R = \|u\|_{\infty} \) is enough)
there exists \( \nu_R > 0, C_R > 0 \) such that

\[ F(x, t, p + q, A + B) \leq F(x, t, p, A) - \nu_R \text{ Tr } B + C_R |q| \]

if \( x \in \overline{\Omega}, |t| \leq R, p, q \in \mathbb{R}^n, A, B \in \mathbb{R}^n, B \geq 0 \)

\[ \text{(if } u \text{ is Lipschitz on } \Omega, \text{ it is enough to take } |p| \leq \|Du\|_{\infty} \text{).} \]


We claim that (5.3) then holds (and in fact \( \varepsilon_n \) does not depend on \( K \)) choosing \( \alpha_m = 1, \quad \varphi_m = (1/m)\{\exp[\lambda(|x|^2 - M^2)/2] - 1\} \) where \( \lambda > 0 \) will be determined later on, and \( M > 0 \) is such that \( \Omega \subset B_M \). Observe that \( \varphi_m \leq 0 \) in \( \Omega \), \( D\varphi_m(x) = (\lambda x/m) \exp[\lambda(|x|^2 - M^2)/2] \), \( D^2\varphi_m(x) = (\lambda I/m) \exp[\lambda(|x|^2 + M^2)/2] + (\lambda^2/m)x \otimes x \exp[\lambda(|x|^2 - M^2)/2] \geq 0 \) so that in view of (5.4)

\[
F(x, t + \varphi_m(x), p + D\varphi_m(x), A + D^2\varphi_m(x)) \\
\leq F(x, t, p, D\varphi_m(x), A + D^2\varphi_m(x)) \\
\leq F(x, t, p, A) - \left\{ v_R \left( n \frac{\lambda}{m} + \frac{\lambda^2 |x|^2}{m} \right) - C R \frac{\lambda |x|}{m} \right\} \exp[\lambda(|x|^2 - M^2)/2]
\]

and we conclude easily choosing \( \lambda \) large enough so that

\[
1 + \lambda |x|^2 \geq 2 \frac{C}{v_R} |x|.
\]

The second example we wish to mention is the case when \( F \) is convex in \((t, p, A)\). Then, if we assume there exists \( \varphi \in C^2(\Omega) \) such that

\[
F(x, \varphi(x), D\varphi(x), D^2\varphi(x)) < 0 \quad \text{in } \Omega,
\]

we claim that (5.3) holds. Indeed, we just have to choose \( \alpha_m = 1 - 1/m, \quad \varphi_m = (1/m)\varphi \) and we observe that

\[
F(x, \alpha_m t + \varphi_m, \alpha_m p + D\varphi_m, \alpha_m A + D^2\varphi_m) \\
\leq \alpha_m F(x, t, p, A) + (1 - \alpha_m) F(x, \varphi(x), D\varphi(x), D^2\varphi(x)),
\]

and we conclude easily in view of (5.5).

**V.2. Quasilinear Equations**

In this subsection, we will study quasilinear degenerate elliptic equations (not in divergence form) of the following type

\[
-\text{Tr} \{ a(Du) \cdot D^2u \} = 0 \quad \text{in } \Omega,
\]

where \( a \in C(\mathbb{R}^n; M^n) \) and there exists \( \sigma \in C(\mathbb{R}^n; M(n, m)) \) (for some \( m \geq 1 \)) such that

\[
a(\xi) = \sigma(\xi) \cdot T\sigma(\xi), \quad \forall \xi \in \mathbb{R}^n.
\]

We could treat as well more general equations of the form

\[
-\text{Tr} \{ a(Du) \cdot D^2u \} + b(x, u, Du) = 0 \quad \text{in } \Omega.
\]
We consider as in the preceding sections an USC viscosity solution $u$ of (5.6) and a lsc viscosity supersolution $v$ of (5.6) and we want to show (2.5). Taking $\sigma = 0$, we see immediately that some nondegeneracy assumption is necessary in order to obtain (2.5). We will use the following assumption

$$\exists \zeta \in C^2(\overline{\Omega}), \quad \inf_{\xi \in R^n} \text{Tr}(a(\xi) \cdot D^2z(x)) > 0 \quad \text{in } \Omega \quad (5.9)$$

and in the case when $u$ or $v$ is locally Lipschitz, it will be enough to assume

$$\exists \zeta \in C^2(\overline{\Omega}), \quad \inf_{\xi \in R^n} \text{Tr}(a(\xi) \cdot D^2z) > 0 \quad \text{in } \Omega, \text{ for all } R < \infty. \quad (5.10)$$

**EXAMPLE.** We can consider the so-called Levi's equation (studied by A. Debiard and B. Gaveau [21], E. Bedford, and B. Gaveau [7])

$$-\left(1 + \left(\frac{\partial \varphi}{\partial x_3}\right)^2 \sum_{i=1}^{2} \frac{\partial^2 \varphi}{\partial x_i^2} - \left(\sum_{i=1}^{2} \left(\frac{\partial \varphi}{\partial x_i}\right)^2\right) \frac{\partial^2 \varphi}{\partial x_3^2}\right) + 2 \left(\frac{\partial \varphi}{\partial x_3} \frac{\partial \varphi}{\partial x_1} - \frac{\partial \varphi}{\partial x_3} \frac{\partial \varphi}{\partial x_2}\right) = 0 \quad \text{in } \Omega. \quad (5.11)$$

Observing that $a(\xi)$, which is given here by

$$a(\xi) = \begin{pmatrix} 1 + \xi_1^2 & 0 & \xi_2 - \xi_1 \xi_3 \\ 0 & 1 + \xi_2^2 & -\xi_3 - \xi_2 \xi_3 \\ \xi_2 - \xi_1 \xi_3 & -\xi_1 - \xi_2 \xi_3 & \xi_1 + \xi_2 \xi_3 \end{pmatrix} \quad (5.12)$$

is nonnegative and smooth in $\xi$, we see that (5.7) holds for some $\sigma$ which is even Lipschitz since the second derivatives of $a$ are bounded on $R^n$. We also notice that (5.9) holds in this case even if $a(\xi)$ is never definite positive: indeed, since $\text{Tr} a(\xi) = 2 + |\xi|^2 + \xi_3^2 \geq 2$, we immediately see that $\zeta = |x|^2$ satisfies (5.9).

We may now state our main comparison result.

**THEOREM V.1.** We assume that either (5.9) holds and $\sigma$ is Lipschitz or (5.10) holds, $\sigma$ is locally Lipschitz and $u$ or $v$ is locally Lipschitz in $\Omega$, then (2.5) holds.

**Proof.** We consider, say in the first case, $u_m = u + (1/m)z$ which is clearly a viscosity subsolution of

$$-\text{Tr} \left\{ a \left( Du_m - \frac{1}{m} Dz(x) \right) \cdot D^2u_m \right\} = -\frac{1}{m} \inf_{\xi} \text{Tr}(a(\xi) \cdot D^2z) < 0 \quad \text{in } \Omega$$
and we want to show

\[
\sup_{\Omega} (u_m - v) \leq M + \frac{1}{m} \|z\|_\infty. \tag{5.13}
\]

As usual we argue by contradiction and by the arguments of Section II (see the proof of Theorem II.2), we obtain for some positive constants \(C, v\) independent of \(m, \delta > 0\)

\[
\frac{1}{m} v \leq \frac{C}{\delta} \frac{1}{m^2}
\]

and we easily reach a contradiction taking \(m\) large. ■

**Remarks.** (1) It is easily seen that the Lipschitz continuity of \(\sigma\) may be replaced by Hölder continuity with an exponent \(\theta > \frac{1}{2}\) as in Sections II, III.

(2) It is worth remarking that (5.6) is equivalent to

\[
- \text{Tr}(a'(Du) \cdot D^2u) = 0 \quad \text{in } \Omega
\]

provided \(a'(Du) = p(Du) a(Du)\) and \(p \in C(\mathbb{R}^n, (0, \infty))\). And checking (5.9) (or (5.1)) or the Lipschitz continuity of \(\sigma\) may be easier for \(a'\) than for \(a\) when choosing appropriately the weight \(p\).

(3) In particular, the above result applies to Levi's Eq. (5.11) since (5.9) holds and \(\sigma\) is Lipschitz in this example.

Combining this result with the general existence strategy based upon Perron's method, we deduce the following existence result

**Corollary V.1.** (1) Assume that (5.9) holds, \(\sigma\) is Lipschitz and there exist \(\varphi \in C(\partial \Omega), y, \bar{u} \in C(\overline{\Omega})\) such that \(y\) (resp. \(\bar{u}\)) is a viscosity subsolution of (5.6) (resp. supersolution) and \(y = \bar{u} = \varphi\) on \(\partial \Omega\). Then, there exists a unique viscosity solution \(u \in C(\Omega)\) of (5.6) such that \(u = \varphi\) on \(\partial \Omega\).

(2) Assume that (5.10) holds, \(\sigma\) is locally Lipschitz and there exist \(\varphi \in C(\partial \Omega), y, \bar{u}\) Lipschitz on \(\overline{\Omega}\) such that \(y\) (resp. \(\bar{u}\)) is a viscosity subsolution of (5.6) (resp. supersolution) and \(y = \bar{u} = \varphi\) on \(\partial \Omega\). Then, there exists a unique Lipschitz viscosity solution of (5.6) such that \(u = \varphi\) on \(\partial \Omega\).

**Remark.** If we assume for instance

\[
\Psi \in C^2(\overline{\Omega}), \quad \Psi = 0 \quad \text{on } \partial \Omega, \quad \inf_{\xi \in \mathbb{R}^n} \text{Tr}(a(\xi) \cdot D^2\Psi(x)) > 0 \quad \text{in } \overline{\Omega}
\]

(5.14)

(as it is the case when \(\Omega\) is strongly convex and \(\text{Tr}(a(\xi)) \geq v > 0\) on \(\mathbb{R}^n\)) then we claim that for each \(\varphi \in C(\partial \Omega)\), there exist \(y, \bar{u}\) with the above
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properties. Furthermore, if \( \varphi \) is smooth (say \( C^{1,1} \)) then \( y, \tilde{u} \) are Lipschitz on \( \tilde{\Omega} \). Indeed, when \( \varphi \) is smooth we just consider for \( \lambda \) large enough

\[
\tilde{u} = \Phi + \lambda \Psi, \quad y = \Phi - \lambda \Psi,
\]

where \( \Phi \) is any smooth extension of \( \varphi \) to \( \tilde{\Omega} \) (\( \lambda \geq C_0 \|D^2 \Phi\|_{\infty} \)). While, when \( \varphi \in C(\partial \Omega) \), we approximate \( \varphi \) by a sequence \( \varphi_n \) of smooth functions converging uniformly to \( \varphi \) and we observe that by the preceding arguments there exists a solution \( u_n \) (Lipschitz) of (5.6) such that \( u_n = \varphi_n \) on \( \partial \Omega \). Now, by Theorem V.1, \( \sup_{\tilde{\Omega}} |u_n - u| = \sup_{\partial \Omega} |u_n - u| = \sup_{\partial \Omega} |\varphi_n - \varphi_n| \); therefore, \( u_n \) converges uniformly to the unique viscosity solution \( u \) of (5.6) such that \( u = \varphi \) on \( \partial \Omega \). In other words, we deduce from Corollary V.1 the

**COROLLARY V.2.** We assume (5.14) and that \( \sigma \) is locally Lipschitz. If \( \varphi \) is smooth on \( \partial \Omega \) (and so is \( \partial \Omega \)) then there exists a unique Lipschitz viscosity solution \( u \) of (5.6) such that \( u = \varphi \) on \( \partial \Omega \). Furthermore, if \( \varphi \in C(\partial \Omega) \) and \( \sigma \) is Lipschitz, there exists a unique viscosity solution \( u \in C(\Omega) \) of (5.6) such that \( u = \varphi \) on \( \partial \Omega \).

We conclude this section by explaining the proof of part (2) of Corollary V.1: truncating \( a \) and \( \sigma \) by the following operation

\[
a_R(\xi) = a(\Pi_R \xi), \quad \sigma_R(\xi) = \sigma(\Pi_R \xi),
\]

where \( \Pi_R \) denotes the projection on the ball \( B_R \), we have for \( R \) large enough the existence of a viscosity solution \( u_R \in C(\Omega) \) of

\[
-\text{Tr}\{a_R(Du_R) \cdot D^2 u_R\} = 0 \quad \text{in } \Omega
\]
such that \( u \leq u_R \leq \tilde{u} \) in \( \tilde{\Omega} \). One then shows that \( u_R \) is Lipschitz with a Lipschitz bound independent of \( R \) and we conclude letting \( R \) go to \( \infty \). To show the Lipschitz continuity of \( u_R \), we merely adapt the standard method of translations: let \( h \in \mathbb{R}^n \), \( u_R(\cdot + h) \) is a viscosity solution of the same equation as that for \( u_R \) but now set in \( \Omega - h \); hence, by Theorem V.1,

\[
\sup_{\Omega \cap (\tilde{\Omega} - h)} |u_R - u_R(\cdot + h)| = \sup_{\partial(\Omega \cap (\Omega - h))} |u_R - u_R(\cdot + h)| \\
\leq \sup_{\partial(\Omega \cap (\Omega - h))} \max\{|\tilde{u}(x) - u(x + h)|, |u(x) - \tilde{u}(x + h)|\} \\
\leq C_0 |h|
\]
since \( y, \tilde{u} \) are Lipschitz and \( y = \tilde{u} \) on \( \partial \Omega \).
V.3 Monge–Ampère Equations

In this section, we want to apply the theory of viscosity solutions to real Monge–Ampère equations of the following type

$$\det(D^2u) = f(x, u, Du) \quad \text{in} \quad \Omega, \ u \text{ convex on} \ \partial \Omega,$$

where $f \in C(\Omega \times \mathbb{R} \times \mathbb{R}^n; [0, \infty))$, $\Omega$ is a bounded, convex open subset of $\mathbb{R}^n$. Even if a notion of weak solutions of (5.15) already exists (see Pogorelov [49], I. Bakelman [3], S. Y. Cheng and S. T. Yau [13], P. L. Lions [45], and the references therein ...), we believe viscosity solutions also provide a convenient sense of solutions which adapts immediately to other classes of “curvatures” equations. Another advantage is the possibility of treating as well equations like

$$\det(D^2u) = f(x, u, Du, D^2u) \quad \text{in} \quad \Omega, \ u \text{ convex in} \ \partial \Omega,$$

where $f$ is elliptic, i.e., satisfies (1.2); this extension is straightforward with our method but we will skip it to simplify the presentation.

First of all, we have to define viscosity solutions of (5.15). This is an immediate adaptation of the notion introduced in Section II. Let $u$ (resp. $v$) be a bounded usc (resp. lsc) convex function on $\Omega$, we will say that $u$ (resp. $v$) is a viscosity subsolution (resp. supersolution) of (5.15) if for all functions $q \in C^2(\Omega)$ the following holds: at each local maximum (resp. minimum) point of $u - q$, then

$$\det(D^2u(x_0)) \geq f(x_0, u(x_0), D^2u(x_0))$$

(resp. $\det(D^2v(x_0)) \leq f(x_0, u(x_0), D^2v(x_0))$) if $D^2\phi(x_0) \geq 0$). (5.18)

The restriction in (5.18) is natural since if $u$ were smooth then at a maximum point of $u - \phi$ we would have

$$0 \leq D^2u(x_0) \leq D^2\phi(x_0) \quad \text{and thus} \quad \det(D^2\phi(x_0)) \geq \det(D^2u(x_0))$$

while at a minimum point we would have

$$D^2\phi(x_0) \leq D^2u(x_0)$$

and if $D^2\phi(x_0)$ is nonnegative, we cannot conclude in general that (5.18) holds (at least in even dimensions). Another way of understanding the viscosity formulation of (5.15) is to recall the following way of writing Monge–Ampère equations based upon the matrix identity for $A \in \mathbb{M}^n$

$$\inf \left\{ \frac{\text{Tr}(AB)}{B} \in \mathbb{M}^n, B \succeq 0, \det B = \frac{1}{n!} \right\} = (\det A)^{1/n} \quad \text{if} \quad A \succeq 0$$

$$= -\infty \quad \text{if} \quad A \ngeq 0$$

(5.19)
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hence, (5.15) is equivalent to
\[
\sup \left\{ -\text{Tr}(B \cdot D^2u)/B \in \mathbb{M}^n, B \geq 0, \det B = \frac{1}{n^n} \right\} + f(x, u, Du)^{1/n} = 0 \text{ in } \Omega, u \text{ convex in } \bar{\Omega}.
\]

(5.20)

Our main comparison result is the

**THEOREM V.2.** Assume that \( f \) satisfies

\[
f(x, t, p) \text{ is nondecreasing with respect to } t \text{ for all } x \in \bar{\Omega}, p \in \mathbb{R}^n
\]

(5.21)

\[
|f^{1/n}(x, t, p) - f^{1/n}(x, t, q)| \leq C_R |p - q|, \quad \forall x \in \bar{\Omega}, \forall |t| \leq R,
\]

(5.22)

\[
\forall p, q \in \overline{B}_R, \text{ for some } C_R \geq 0, \text{ for all } R < \infty.
\]

Then, (2.5) holds.

**Remarks.** (1) The fact that no restriction on the \( x \)-dependence is necessary is a simple consequence of the local Lipschitz continuity of convex functions.

(2) As explained in Section II, the Perron’s method readily yields the existence of solutions with prescribed boundary values provided we have a sub- and a supersolution having the right boundary values.

(3) (5.22) is only needed to make some perturbation argument as in the preceding subsection; in particular, if \( f(x, t, p) \geq f(x, s, p) + \gamma_R(r - s) \) for all \( x \in \bar{\Omega}, R \geq t \geq s \geq -R, |p| \leq R \) for some increasing \( \gamma_R \) (for all \( R < \infty \)) then (5.22) is not needed.

We will not prove Theorem V.2 since its proof is very much similar to the one made in Section II. Let us only observe that, in this particular example, the regularization \( v_\varepsilon \) of the supersolution \( v \) given by

\[
v_\varepsilon(x) = \inf_{y \in \bar{\Omega}} \left\{ v(y) + \frac{1}{\varepsilon} |x - y|^2 \right\}
\]

(5.23)

yields as usual a semi-concave, Lipschitz function on \( \bar{\Omega} \) but, in addition, since \( v \) is convex, \( v_\varepsilon \) is also convex say in \( \bar{\Omega}_\varepsilon \) (see for instance J. M. Lasry and P. L. Lions [34] for a related observation), therefore, in conclusion, \( v_\varepsilon \) is \( C^{1,1} \) on \( \bar{\Omega}_\varepsilon \). This allows, in fact, a sensible simplification of the arguments used and recalled in Section II.

Let us also mention that the perturbation argument involved in the proof of Theorem V.2 consists in replacing the subsolution \( u \) by \( u_m = \)
\[ u + \left( \frac{1}{m} \right) \{ \exp(\beta |x|^2/2) - M \}, \]
using the fact that \( v \) is Lipschitz on every compact subset of \( \Omega \) and that

\[
\det(D^2u_m)^{1/n} \geq f^{1/n}(x, u, Du) + \frac{1}{m} \beta e^{\beta |x|^2/2} (1 + |x|^2)^{1/n}
\]

on every compact subset of \( \Omega \) if \( \beta \) is large enough and then \( M \) and \( m \) chosen large enough.

**VI. Other Boundary Conditions**

In this section, we will show how to deal with uniqueness questions under some boundary conditions, like the Neumann type boundary conditions and the state-constraints condition, other than the Dirichlet condition discussed above.

Such boundary problems were treated first by P. L. Lions [43] and then by several authors [31, 51, 7, 12, 48] in the case of first order equations extending the notion of viscosity solutions in order to take boundary conditions into account. The point here is to adapt and modify the techniques exploited in the case of first order equations so that we can treat second order equations.

We always assume here that \( F \) is defined and continuous on \( \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{M}^n \).

**VI.1. Neumann Type Boundary Conditions**

We always assume in this section that \( \partial \Omega \) is of class \( C^2 \). Let \( \nu(x) \) designate the outer unit normal of \( \Omega \) at \( x \in \partial \Omega \). According to [43] the Neumann problem

\[
F(x, u, Du, D^2u) = 0 \quad \text{in} \ \Omega,
\]

\[
\frac{\partial u}{\partial \nu} = 0 \quad \text{on} \ \partial \Omega
\]

is formulated in the viscosity sense as follows. An usc function \( u \) on \( \bar{\Omega} \) is said to be a viscosity subsolution of (6.1) if whenever \( \phi \in C^2(\bar{\Omega}) \) and \( u - \phi \) attains its maximum at \( x_0 \in \bar{\Omega} \), then

\[
F(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) \leq 0 \quad \text{if} \quad x_0 \in \Omega,
\]

(6.2)
and
\[ \frac{\partial \varphi}{\partial v}(x_0) \leq 0 \quad \text{or} \quad F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leq 0 \quad \text{if} \quad x \in \partial \Omega. \] (6.3)

A lsc function \( u \) on \( \bar{\Omega} \) is said to be a viscosity supersolution of (6.1) if whenever \( \varphi \in C^2(\bar{\Omega}) \) and \( u - \varphi \) attains a minimum at \( x_0 \in \bar{\Omega} \), then
\[ F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq 0 \quad \text{if} \quad x_0 \in \Omega, \] (6.4)
and
\[ \frac{\partial \varphi}{\partial v}(x_0) \geq 0 \quad \text{or} \quad F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq 0 \quad \text{if} \quad x_0 \in \partial \Omega. \] (6.5)

More generally, the nonlinear boundary problem
\[ \begin{align*}
F(x, u, Du, D^2u) &= 0 \quad \text{in} \quad \Omega, \\
B(x, u, Du) &= 0 \quad \text{on} \quad \partial \Omega,
\end{align*} \] (6.6)
where \( B \in C(\partial \Omega \times \mathbb{R} \times \mathbb{R}^n) \), is formulated in the viscosity sense similarly to (6.1) but with the inequality
\[ B(x_0, u(x_0), D\varphi(x_0)) \leq 0 \quad (\text{resp.} \geq 0) \]
in place of the first inequality in (6.3) (resp. (6.5)).

First, we consider the case when sub- and supersolutions are not necessarily Lipschitz continuous. We need the following assumption:
\[ |F(x, t, p, A) - F(x, t, q, B)| \leq \omega_\varepsilon(|p - q| + \|A - B\|) \] (6.7)
if \( x \in U_\varepsilon, \ |t| \leq \varepsilon, \ p, q \in \mathbb{R}^n, \) and \( A, B \in \mathbb{M}^n \), for \( \varepsilon \in (0, \infty) \), where \( U_\varepsilon \) is a neighborhood of \( \partial \Omega \) and \( \omega_\varepsilon(\sigma) \to 0 \) as \( \sigma \to 0^+ \).

**Theorem VI.1.** Assume (6.7). Assume that either (2.17) and (2.18) (as in Theorem II.1) or (2.20), (2.21), (2.22) with \( \theta = 1 \) and (2.23) (as in Theorem II.2) hold. Let \( u, v \) be, respectively, viscosity sub- and supersolutions of (6.1). Then
\[ u \leq v \quad \text{on} \quad \bar{\Omega}. \] (6.8)

**Proof.** The main idea to prove this is to find "strict" sub- and supersolutions of (6.1), respectively, close to \( u \) and \( v \). To find such sub- and supersolutions, we fix \( \varepsilon \) so that \( R > \sup_{\Omega} |u| + \sup_{\Omega} |v|, \) write \( U \) and \( \omega,
respectively, for $U_R$ and $\omega_R$, where $U_R$ and $\omega_R$ are from (6.7), and choose $\zeta \in C^2(\Omega)$ so that
\[ \sup \zeta < U \quad \text{and} \quad (v, D\zeta) \geq 1 \quad \text{on } \partial \Omega. \quad (6.9) \]

For $\alpha > 0$ we set $\tilde{u} = u - \alpha \zeta$ on $\bar{\Omega}$. We see from (6.7) that if $\alpha$ is sufficiently small, then $\tilde{u}$ is a viscosity subsolution of
\[ F(x, u, Du, D^2u) = \gamma x \|\zeta\|_\infty + \omega(\alpha \|D\zeta\|_\infty + \alpha \|D^2\zeta\|_\infty) \quad \text{in } \Omega \]
\[ \frac{\partial u}{\partial v} = -\alpha \quad \text{on } \partial \Omega, \quad (6.10) \]
where $\gamma = \gamma_R$ (resp. $\gamma = c_0$) when (2.17) and (2.18) (resp. (2.20)-(2.23)) are assumed. Moreover, setting
\[ \tilde{u} = u - \alpha \zeta - \alpha \|\zeta\|_\infty - \frac{1}{\gamma} \omega(\alpha \|D\zeta\|_\infty + \alpha \|D^2\zeta\|_\infty) - \frac{\alpha}{\gamma} \quad \text{on } \bar{\Omega}, \]
we see that $\tilde{u}$ is a viscosity subsolution of
\[ F(x, u, Du, D^2u) = \alpha \quad \text{in } \Omega \]
\[ \frac{\partial u}{\partial v} = -\alpha \quad \text{on } \partial \Omega, \quad (6.11) \]
if $\alpha > 0$ is small enough. Similarly, the function
\[ \tilde{v} = v + \alpha \zeta + \alpha \|\zeta\|_\infty + \frac{1}{\gamma} \omega(\alpha \|D\zeta\|_\infty + \alpha \|D^2\zeta\|_\infty) + \frac{\alpha}{\gamma} \quad \text{on } \bar{\Omega} \]
is a viscosity supersolution of
\[ F(x, v, Du, D^2v) = \alpha \quad \text{in } \Omega, \]
\[ \frac{\partial v}{\partial v} = \alpha \quad \text{on } \partial \Omega, \]
if $\alpha > 0$ is small enough.

To conclude the proof, we have to prove that $\tilde{u} \leq \tilde{v}$ on $\Omega$ for small $\alpha > 0$. To this end, we let $u$ and $v$ be, respectively, a viscosity subsolution of (6.10) and a viscosity supersolution of (6.11) and show that $u \leq v$ on $\partial \Omega$. We need the following observations on inf- and supconvolutions. First of all, we note that for bounded functions $u, v$ on $\Omega$ the formula
\[ u^\varepsilon(x) = \sup_{y \in \Omega} \left\{ u(y) - \frac{1}{\varepsilon} |x - y|^2 \right\}, \quad x \in \mathbb{R}^n \quad (6.12) \]
and

\[ v_\varepsilon(x) = \inf_{y \in \Omega} \left\{ \frac{1}{\varepsilon} |x - y|^2 / y \in \overline{\Omega}, \ |y - x| \leq (C_0 + C) \varepsilon^{1/2} \right\} , \quad x \in \mathbb{R}^n \]  

(6.13)

define functions on \( \mathbb{R}^n \) which are bounded and Lipschitz continuous on bounded subsets of \( \mathbb{R}^n \). Moreover, \( u^\varepsilon \) and \( v_\varepsilon \) are semiconvex and semiconcave on \( \mathbb{R}^n \), respectively. Also, we have

\[ u^\varepsilon(x) = \sup \left\{ u(y) - \frac{1}{\varepsilon} |x - y|^2 / y \in \overline{\Omega}, \ |y - x| \leq (C_0 + C) \varepsilon^{1/2} \right\} \]

and

\[ v_\varepsilon(x) = \inf \left\{ v(y) + \frac{1}{\varepsilon} |x - y|^2 / y \in \overline{\Omega}, \ |y - x| \leq (C_0 + C) \varepsilon^{1/2} \right\} , \]

if \( C > 0 \) and \( \text{dist}(x, \overline{\Omega}) \leq C \varepsilon^{1/2} \), where \( C_0 = \left[ (2 \sup_{\Omega} |u|) \vee (2 \sup_{\Omega} |v|) \right]^{1/2} \).

It is easy to see that if a function \( \phi \) on \( \mathbb{R}^n \times \mathbb{R}^n \) satisfies

\[ \phi(x, y) \geq C_1 \quad \text{and} \quad \phi(x, x) \leq C_2 \]

for \( x, y \in \mathbb{R}^n \) and \( C_1, C_2 \in \mathbb{R} \) and if \( u^\varepsilon(x) - v_\varepsilon(y) - \phi(x, y) \) attains a maximum at \( (x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}^n \) then

\[ \text{dist}(x_0, \overline{\Omega}) \vee \text{dist}(y_0, \overline{\Omega}) \leq C_3 \varepsilon^{1/2} , \]  

(6.14)

where \( C_3 = \left[ (C_2 - C_1 + 2 (\sup_{\Omega} |u| + \sup_{\overline{\Omega}} |v|) \right]^{1/2} \). The argument in \[32\], [24], is now adapted easily to yield

**Proposition VI.1.** Assume that functions \( F(x, \cdot, p, A) \) and \( B(x, \cdot, p) \) are nondecreasing on \( \mathbb{R} \) for \( x \in \overline{\Omega}, \ p \in \mathbb{R}^n \) and \( A \in \mathbb{M}^n \). Fix \( C > 0 \) and define

\[ \Omega^c = \{ x \in \mathbb{R}^n / \text{dist}(x, \Omega) < C \varepsilon^{1/2} \} , \]

\[ F_c(x, t, p, A) = \min \{ F(y, t, p, A) \wedge B(y, t, p) / y \in \overline{\Omega}, \ |y - x| \leq (C_0 + C) \varepsilon^{1/2} \} \]

(6.15)

and

\[ F^c(x, t, p, A) = \max \{ F(y, t, p, A) \vee B(y, t, p) / y \in \overline{\Omega}, \ |y - x| \leq (C_0 + C) \varepsilon^{1/2} \} \]

(6.16)

for \( (x, t, p, A) \in \Omega^c \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{M}^n \), where the conventions that \( F(y, t, p, A) \vee B(y, t, p) = F(y, t, p, A) \) and \( F(y, t, p, A) \wedge B(y, t, p) = F(y, t, p, A) \) for \( y \in \Omega \) are understood. Let \( u \) (resp. \( v \)) be a viscosity subsolution (resp.
supersolution) of (6.6). Then $u_\varepsilon$ (resp. $v_\varepsilon$) is a viscosity subsolution (resp. supersolution) of

$$ F_\varepsilon(x, u, Du, D^2u) = 0 \quad \text{in } \Omega^\varepsilon $$

(resp.

$$ F_\varepsilon(x, v, Dv, D^2v) = 0 \quad \text{in } \Omega^\varepsilon. $$

Finally, we observe that $F_\varepsilon$ (resp. $F^\varepsilon$) is lsc (resp. usc) and so that the assertion of Proposition II.3 with $\Omega^\varepsilon$ in place of $\Omega_\varepsilon$ holds under the above assumptions and notations.

Proof continued. We fix $C > 0$ so that

$$ C > \left[2 \left( \sup_{\Omega} |u| + \sup_{\Omega} |v| \right) \right]^{1/2}, $$

choose $\varphi(x, y) = (1/\delta) |x - y|^2$, $\delta > 0$, as in Section II and proceed as in Section II with $\Omega^\varepsilon$ in place of $\Omega_\varepsilon$.

Now, note that $u'(x) - v_\varepsilon(y) - \varphi(x, y)$ does not attain a maximum over $\overline{\Omega}^\varepsilon \times \overline{\Omega}^\varepsilon$ on $\partial(\Omega^\varepsilon \times \Omega^\varepsilon)$. Next, let $(\tilde{x}, \tilde{y}) \in \Omega^\varepsilon \times \Omega^\varepsilon$ be a point where $u'(x) - v_\varepsilon(y) - \varphi(x, y)$ achieves its maximum over $\overline{\Omega}^\varepsilon \times \overline{\Omega}^\varepsilon$, and let $X, Y \in M^n$ satisfy (2.14)-(2.16) where $F_\varepsilon$ and $F^\varepsilon$ are defined, respectively, by (6.15) with $B(x, t, p) = (v(x), p) + \alpha$ and (6.16) with $B(x, t, p) = (v(x), p) - \alpha$. Assume, for example, that

$$ F_\varepsilon(\tilde{x}, u'(\tilde{x}), D_x \varphi(\tilde{x}, \tilde{y}), X) = (v(\tilde{x}), D_x \varphi(\tilde{x}, \tilde{y})) + \alpha \quad (6.17) $$

for some $\tilde{x} \in \partial \Omega$ with $|\tilde{x} - \tilde{x}| \leq (C_0 + C) \varepsilon^{1/2}$. By the regularity of $\partial \Omega$ we see ([44], [31]) that

$$ (v(x), x - y) + C_1 |x - y|^2 \geq 0, \quad \forall x \in \partial \Omega, \forall y \in \overline{\Omega} \quad (6.18) $$

for some constant $C_1 \geq 0$. Therefore we have

$$ (v(\tilde{x}), D_x \varphi(\tilde{x}, \tilde{y})) \geq O \left( \frac{|\tilde{x} - \tilde{y}|^2}{\delta} + \varepsilon^{1/2} \right). $$

This means that we do not have (6.17) if $\varepsilon^{1/2}/\delta$ and $|\tilde{x} - \tilde{y}|^2/\delta$ are small enough. With these observations at hand one easily goes through the rest of the proof as in the proof of Theorems II.1 and II.2.

Assume hereafter that $\partial \Omega$ is of class $C^3$. Let $\gamma \in C^2(\mathbb{R}^n, \mathbb{R}^n)$ satisfy

$$ (\gamma(x), v(x)) > 0, \quad \forall x \in \partial \Omega, $$

$$ \forall x \in \partial \Omega, $$
and consider the boundary problem

\[ F(x, u, Du, D^2u) = 0 \quad \text{in} \, \Omega, \]
\[ \frac{\partial u}{\partial n} = 0 \quad \text{on} \, \partial \Omega. \]  

(6.19)

The above argument is easily modified to produce the following.

**Theorem VI.2.** Let \( u \) and \( v \) be, respectively, viscosity sub- and supersolutions of (6.19). Then, under the assumptions on \( F \) in Theorem VI.1, (6.8) holds.

The only modification needed when proving this theorem is the choice of \( \zeta \) and \( \varphi \). We now choose \( \zeta \in C^2(\Omega) \) so that the first condition in (6.9) and \( (\gamma, D\zeta) \geq 1 \) on \( \partial \Omega \) are satisfied, and choose

\[ \varphi(x, y) = \frac{1}{2\delta} (A(x)(x - y), x - y), \quad \delta > 0, \]  

(6.20)

where \( A = (a_{ij}) \in C^2(\mathbb{R}^n, \mathbb{M}^n) \) is selected to satisfy

\[ A(x) \geq c_0 I \quad \forall x \in \mathbb{R}^n \quad \text{and} \quad A(x) \gamma(x) = \nu(x), \quad \forall x \in \partial \Omega \]  

(6.21)

for some constant \( c_0 > 0 \) (see [44]). We may assume that \( A(x) = I \) on \( \Omega \setminus U_R \), where \( U_R \) is a neighbourhood of \( \partial \Omega \) from (6.7). The first term of the right-hand side of the equation corresponding to (6.17) in our problem is

\[ (\gamma(\hat{x}), D_x \varphi(\hat{x}, \hat{y})), \]

where \( \hat{x} \in \partial \Omega \) and \( |\hat{x} - \hat{y}| \leq C \varepsilon^{1/2} \) for some constant \( C > 0 \), which is estimated from below again by

\[ O\left( \frac{|\hat{x} - \hat{y}|^2}{\delta} + \frac{\varepsilon^{1/2}}{\delta} \right) \]

because of our choice of \( \varphi \). The discussions in Section II are also affected a bit by the change of \( \varphi \). Indeed, \( D^2 \varphi(x, y) \) now has the form

\[ \frac{1}{\delta} \left( \begin{array}{cc} A + B + TB + C & -A - B \\ -A - TB & A \end{array} \right), \]

where \( A = A(x), \ B = B(x, y), \) and \( C = C(x, y) \) are matrices satisfying

\[ B(x, y) = O(|x - y|), \quad C(x, y) = O(|x - y|^2). \]
On the other hand, we have

\[
\begin{pmatrix}
A + B + TB & -A - B \\
-A - TB & A
\end{pmatrix}
\begin{pmatrix}
\xi \\
\xi
\end{pmatrix} = 0, \quad \forall \xi \in \mathbb{R}^n,
\]

and

\[
\text{Tr}
\begin{pmatrix}
\Sigma_1^T \Sigma_1 & \Sigma_1^T \Sigma_2 \\
\Sigma_2^T \Sigma_1 & \Sigma_2^T \Sigma_2
\end{pmatrix}
\begin{pmatrix}
A + B + TB & -A - B \\
-A - TB & A
\end{pmatrix}
\]

\[
= \text{Tr}(\Sigma_1 - \Sigma_2)^T(\Sigma_1 - \Sigma_2)A + \text{Tr}(\Sigma_1 - \Sigma_2)^T \Sigma_1 B
\]

\[
+ \text{Tr} \Sigma_1^T(\Sigma_1 - \Sigma_2)^TB, \quad \forall \Sigma_1, \Sigma_2 \in \mathbb{M}(m, n).
\]

Therefore, if \( X, Y \in \mathbb{M}^n \) satisfy

\[
\begin{pmatrix}
X & 0 \\
0 & Y
\end{pmatrix} \leq D^2 \varphi(x, y),
\]

then we have

\[
X + Y \leq O \left( \frac{|x - y|^2}{\delta} \right)
\]

and also

\[
\text{Tr}(\Sigma_1^T \Sigma_1 X) + \text{Tr}(\Sigma_2^T \Sigma_2 Y)
\]

\[
\leq \text{Tr}(\Sigma_1 - \Sigma_2)^T(\Sigma_1 - \Sigma_2)A + \text{Tr}(\Sigma_1 - \Sigma_2)^T \Sigma_1 B
\]

\[
+ \text{Tr} \Sigma_1^T(\Sigma_1 - \Sigma_2)^TB + O \left( \frac{|x - y|^2}{\delta} \right), \quad \forall \Sigma_1, \Sigma_2 \in \mathbb{M}(m, n).
\]

With these observations the proof is quite similar to that of Theorem VI.1.

VI.2. Nonlinear Boundary Conditions

In this section we assume that \( \partial \Omega \) is of class \( C^3 \) and consider the problem

\[
F(x, u, Du, D^2u) = 0 \quad \text{in } \Omega,
\]

\[
\frac{\partial u}{\partial \gamma} + f(x, u) = 0 \quad \text{on } \partial \Omega,
\]

where \( f(x, t) \in C(\partial \Omega \times \mathbb{R}) \) is nondecreasing in \( t \) and, as above, \( \gamma \in C^2(\mathbb{R}^n, \mathbb{R}^n) \) satisfies \( (\nu, \gamma) > 0 \) on \( \partial \Omega \).
THEOREM VI.3. Let $u$ and $v$ be respectively, viscosity sub- and supersolutions of (6.22). Then, under the assumptions on $F$ in Theorem VI.1, (6.8) holds.

In order to prove this theorem, we need to change a bit the structure of the proof employed above. We here localize the treatment at a point where $u-v$ achieves its maximum.

Sketch of proof. We may assume that $u$ and $v$ are, respectively, a viscosity subsolution of

$$F(x, u, Du, D^2u) = -\alpha$$

in $\Omega$,

$$\frac{\partial u}{\partial \gamma} + f(x, u) = -\alpha$$
on $\partial \Omega$,

and a viscosity supersolution of

$$F(x, v, Dv, D^2v) = \alpha$$

in $\Omega$,

$$\frac{\partial v}{\partial \gamma} + f(x, v) = \alpha$$
on $\partial \Omega$,

for some constant $\alpha > 0$. As usual, we assume that $\max_{\partial \Omega}(u-v) > 0$. By Theorems II.1 and II.2 we see that $\max_{\partial \Omega}(u-v) = \max_{\Omega}(u-v)$. We fix a point $z \in \partial \Omega$ so that $(u-v)(z) = \max_{\partial \Omega}(u-v)$, and consider the case where $f(z, u(z)) = 0$. If this is not the case, then we replace $u$ and $v$, respectively, by $u + \xi$ and $v + \xi$, where $\xi \in C^2(\mathbb{R}^n)$ is chosen to satisfy $(\partial \xi/\partial \gamma)(z) = f(z, u(z))$, and just follow the discussion below. We choose

$$\varphi(x, y) = \frac{1}{2\delta} (A(x)(x-y), x-y) + |x-z|^4, \quad \delta > 0,$$

where $A$ is as in (6.20). Then, the function $u(x) - v(x) - \varphi(x, x)$ attains its strict maximum at $z$. Therefore, letting $(x_{\delta, \varepsilon}, y_{\delta, \varepsilon})$ be a point where $u^\varepsilon(x) - v^\varepsilon (y) - \varphi(x, y)$ achieves its maximum over $\bar{\Omega}^\varepsilon \times \bar{\Omega}^\varepsilon$ and selecting sequences $\varepsilon_j \to 0_+$ and $\delta_j \to 0_+$ so that the double limits $\lim_{\delta_j} \lim_{\varepsilon_j} x_{\delta_j, \varepsilon_j}$ and $\lim_{\varepsilon_j} \lim_{\delta_j} y_{\delta_j, \varepsilon_j}$ exist, we see that these two limits are equal to $z$. The rest of the proof is now standard with this remark.

Remark. The existence of continuous viscosity solution of (6.22) is easily shown by Perron's method under the assumptions on $F$ in Theorem VI.1 and the assumption of the existence of a viscosity subsolution $u$ and a viscosity supersolution $v$ of (6.22).
VI.3 Remarks on Lipschitz Continuous Solutions

Here we only assume, regarding the regularity of $\partial \Omega$ and $\gamma$, that $\partial \Omega \in C^1$ and $\gamma \in C(\mathbb{R}^n, \mathbb{R}^n)$. Under these weaker assumptions, we still have a function $\zeta \in C^2(\overline{\Omega})$ for any neighbourhood of $\partial \Omega$ such that $\text{supp} \ z \subset U$ and $\partial z/\partial \gamma \geq 1$ on $\partial \Omega$. On the other hand, we can choose only a continuous $\mathbb{M}^n$-valued function $A = (a_{ij})$ so that (6.21) is satisfied and, in general, it cannot be a $C^2$ function anymore. Also, we now have a weaker estimate

$$(v(x), x - y) + |x - y| \omega(|x - y|) \geq 0 \quad \forall x \in \partial \Omega, \forall y \in \overline{\Omega} \quad (6.23)$$

instead of (6.18), where $\omega(r) \to 0$ as $r \to 0_+$. One of our assumptions on $F$ is:

$$(6.24)$$

The following corresponds to Theorems II.1 and II.2 in the Lipschitz case.

**Theorem VI.4.** Assume (2.17), (6.24), and either (2.19) or (2.20–2.22) with $\theta > \frac{1}{2}$. Then, if $u$ and $v$ are, respectively, Lipschitz continuous, viscosity sub- and supersolutions of (6.22), (6.8) holds.

The argument of the proof of Theorem VI.3 with a new choice of $\varphi$, i.e.,

$$\varphi(x, y) = \frac{1}{2\delta} (A(z)(x - y), x - y) + |x - z|^4, \quad \delta > 0,$$

applies to prove this theorem. The only new observation needed is this easy estimate: if $\hat{x} \in \partial \Omega$, $|x - \hat{x}| \leq C\varepsilon^{1/2}$ and $\text{dist}(y, \Omega) \leq C\varepsilon^{1/2}$ for some constant $C > 0$, then

$$(\gamma(\hat{x}), D_x \varphi(x, y)) \geq -\frac{|x - y|}{\delta} \{\omega(|x - y|) + \omega(|x - z|)\} - \frac{\omega(\varepsilon)}{\delta},$$

where $\omega(\sigma) \to 0$ as $\sigma \to 0_+$ and $\omega$ is chosen independently of $\hat{x}, x,$ and $y$.

One of the main assertions in Section III is also valid. Indeed, we have:

**Theorem VI.5.** Let $u$ and $v$ be, respectively, Lipschitz continuous, viscosity sub- and supersolutions of (6.22). Assume that (2.17), (3.1w), (3.3), and (6.24) are satisfied. Then (6.8) holds.

An obvious mixture of techniques above and those in Section III yields this result, and we leave the details of the proof to the reader.
VI.4. Back to the Dirichlet Problem

In this section we come back to the Dirichlet problem

\[ F(x, u, Du, D^2u) = 0 \quad \text{in } \Omega, \]
\[ u = h(x) \quad \text{on } \partial \Omega, \]

where \( h \in C(\partial \Omega) \), and now look at this problem exactly in the same way as in Section VI.1. Indeed, (6.25) is the special case of (6.6) obtained by setting \( B(x, t, p) = t - h(x) \), and thus we have the notion of viscosity sub- and supersolutions to (6.25) as defined in Section VI.1. This viewpoint is very important when we deal with degenerate operators \( F \) as it is already realized in the case of first order equations (see [6, 12, 31]).

Throughout this section we assume that each \( z \in \partial \Omega \), there is \( \eta \in \mathbb{R}^n \), and \( a > 0 \) such that

\[ B_{\eta a}(x + \eta t) \subset \Omega, \quad \forall x \in \overline{\Omega} \cap B_{\eta}(z), \quad 0 < \forall t \leq 1. \]  

Theorem VI.5. Let \( u \) and \( v \) be, respectively, viscosity sub- and supersolutions of (6.25).

1. If \( u \) and \( v \) are continuous at points of \( \partial \Omega \) and the assumptions on \( F \) in Theorem VI.1 are satisfied, then (6.8) holds.

2. If \( u \) and \( v \) are Lipschitz continuous on \( \Omega \) and if (2.17), (3.1k), (3.3), and (6.24) are satisfied, then (6.8) holds.

Sketch of proof. We adapt an idea of H. Soner [51]. We begin with part (1). First, we observe that for any \( 0 < s < 1 \), (6.26) holds with \( s \eta \) and \( sa \), respectively, in place of \( \eta \) and \( a \). That is, we can choose \( \eta \in \mathbb{R}^n \) in (6.26) so that \( |\eta| > 0 \) is as small as desired. We argue by contradiction and assume \( \sup_{\partial \Omega}(u - v) > 0 \). We observe by Theorems II.1 and II.2 that \( \sup(u - v) = (u - v)(z) \) for some \( z \in \partial \Omega \). Our arguments are divided into two cases; one is the case where \( u(z) > h(z) \) and the other is the case where \( v(z) < h(z) \). We first consider the case where \( u(z) > h(z) \). We let

\[ \phi(x, y) = \frac{1}{2\delta^2} |x - y + \delta \eta|^2 + |x - z|^4, \quad 0 < \delta < 1, \]

where \( \eta \) is from (6.26), and \( (x_{\delta, \epsilon}, y_{\delta, \epsilon}) \) be a maximum point of \( u^\delta(x) - v^\epsilon(x) - \phi(x, y) \) over \( \mathbb{R}^n \times \mathbb{R}^n \). Recall that \( \text{dist}(x_{\delta, \epsilon}, \Omega) \leq C \delta^{1/2} \) and \( \text{dist}(y_{\delta, \epsilon}, \Omega) \leq C \delta^{1/2} \) for some \( C > 0 \) independent of \( \delta, \epsilon > 0 \). Taking subsequences if necessary, the points \( (x_{\delta, \epsilon}, y_{\delta, \epsilon}) \) converge to a maximum point \( (x_{\delta}, y_{\delta}) \in \overline{\Omega} \times \overline{\Omega} \) of \( w(x, y) = u(x) - v(y) - \phi(x, y) \) as \( \epsilon \to 0_+ \), and the
points $x_\delta, y_\delta$ converge, as $\delta \to 0_+$, to $z$, the unique maximum of $u(x) - v(x) - |x - z|^4$. In these convergences we have

$$
\lim_{\delta} u(x_\delta) = u(z), \quad \text{and} \quad \lim_{\delta} v(y_\delta) = v(z).
$$

Therefore, if we choose $\delta$ first and then $\varepsilon$ small enough and if $|x - x_{\delta, \varepsilon}| \leq Ce^{1/2}$, then $u(x_{\delta, \varepsilon}) > h(x)$. Next, we observe that $w(x_{\delta, \varepsilon}, y_{\delta}) \geq w(z, z + \delta \eta)$ since $z + \delta \eta \in \Omega$, and hence

$$
\frac{1}{\delta^2} |x_\delta - y_\delta + \delta \eta|^2 = o(1) \quad \text{as} \quad \delta \to 0_+.
$$

This together with (6.26) implies that $y_\delta \in \Omega$ if $\delta$ is small enough. Moreover, if we choose $\delta$ and then $\varepsilon$ small enough and if $|y - y_{\delta, \varepsilon}| \leq Ce^{1/2}$, then $y \in \Omega$. Thus, we conclude from Proposition VI.1 that if $\delta, \varepsilon$ are small enough then there are $X, Y \in M^n$ such that

$$
F(\bar{x}, \nu(\bar{x}), D\varphi(\bar{x}, \bar{y}), X) \leq 0,
$$

$$
F(\bar{y}, \nu(\bar{y}), -D\varphi(\bar{x}, \bar{y}), -Y) \geq 0,
$$

$$
\begin{pmatrix} X \\ 0 \\ 0 \\ Y \end{pmatrix} \leq D^2\varphi(\bar{x}, \bar{y})
$$

for some $\bar{x}, \bar{y} \in \mathbb{R}^n$, where $\bar{x} = x_{\delta, \varepsilon}, \quad \bar{y} = y_{\delta, \varepsilon}, \quad |\bar{x} - \bar{y}| \leq Ce^{1/2}$ and $|\bar{y} - \bar{y}| \leq Ce^{1/2}$. Now, using (6.17), we argue as in the proof of Theorems II.1 and II.2 and deduce after sending $\varepsilon \to 0_+$ that

$$
(u(x_\delta) - v(y_\delta))^{+} \leq \omega \left( |x_\delta - y|^2 + \frac{|x_\delta - y_\delta + \delta \eta|^2}{\delta^2} |x_\delta - y_\delta| + \frac{|x_\delta - y_\delta|^2}{\delta^2} \right).
$$

Finally, letting $\delta \to 0_+$ and $\eta \to 0$, we obtain a contradiction. The other case is treated similarly with

$$
\varphi(x, y) = \frac{1}{2\delta^2} |x - y - \delta \eta|^2 + |y - z|^4, \quad 0 < \delta < 1,
$$

instead of the above choice of $\varphi$.

We now turn to part (2). The proof goes similarly as the proof of part (1). We assume $\sup(u - v) = (u - v)(z) > 0$ for $z \in \partial \Omega$ and only consider the case $u(z) > h(z)$. We fix $x \in (1, 2\theta)$, where $\theta$ is from (3.3), and set

$$
\varphi(x, y) = \frac{1}{2\delta^2} |x - y + \delta \eta|^2 + |x - z|^4, \quad 0 < \delta < 1.
$$
Using the Lipschitz continuity of \( v \), we observe that for any maximum point \((x_\delta, y_\delta)\) of \( w(x, y) = u(x) - v(y) - \phi(x, y) \), we have

\[
x_\delta - y_\delta + \delta \eta = O(\delta^2)
\]
as \( \delta \to 0_+ \),
since, if \( \delta \) is small enough, then \( x_\delta + \delta \eta \in \Omega \) and hence \( w(x_\delta, y_\delta) \geq w(x_\delta, x_\delta + \delta \eta) \). With this estimate we deduce as in the proof of part (1) that

\[
F(\dot{x}, u'(\ddot{x}), D_x \phi(\ddot{x}, \ddot{y}), X) \leq 0,
\]

\[
F(\ddot{y}, c_x(\dot{y}), -D_y \phi(\ddot{x}, \ddot{y}), -Y) \geq 0,
\]

\[
\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq D^2 \phi(\ddot{x}, \ddot{y})
\]

for some \( X, Y \in \mathbb{M}^n \) and \( \ddot{x}, \ddot{y} \in \mathbb{R}^n \), where \((\ddot{x}, \ddot{y})\) is a maximum point of \( u'(x) - v'(y) - \phi(x, y) \) and \( \dddot{x}, \dddot{y} \) satisfy \(|\dddot{x} - \ddot{x}| \leq C\delta^{1/2} \) and \(|\dddot{y} - \ddot{y}| \leq C\delta^{1/2} \) for some constant \( C > 0 \). As in the proof of Theorem III.1 we may send \( \varepsilon \to 0_+ \) to get

\[
F(\ddot{x}, u(\ddot{x}), D_x \phi(\ddot{x}, \ddot{y}), X) \leq 0,
\]

\[
F(\ddot{y}, v(\ddot{y}), -D_y \phi(\ddot{x}, \ddot{y}), -Y) \geq 0,
\]

\[
\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq D^2 \phi(\ddot{x}, \ddot{y})
\]

for some \( X, Y \in \mathbb{M}^n \) and \( \ddot{x}, \ddot{y} \in \mathbb{R}^n \). Moreover, \((\ddot{x}, \ddot{y})\) is a maximum point of \( u(x) - v(y) - \phi(x, y) \). We see that

\[
D^2 \phi(\ddot{x}, \ddot{y}) = \frac{1}{\delta^2} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \left( \begin{pmatrix} O(|\dddot{x} - z|^2) \\ 0 \end{pmatrix} \right),
\]

where \( O(|\dddot{x} - z|^2) \) is an element of \( \mathbb{M}^n \). Thus, from Lemma III.1, we find that

\[
\|X\| \leq O(\delta^{-\gamma/2}(|\text{Tr}(X + Y)|^{1/2} + |\ddot{x} - z|) + |\text{Tr}(X + Y)|),
\]

while, as in the proof of Theorem III.1, we deduce

\[
(u(\ddot{x}) - v(\ddot{y}))^+ + v|\text{Tr}(X + Y)| \leq \omega(|\dddot{x} - \ddot{y}|^\theta(1 + \|X\|) + O(|\dddot{x} - z|^2)
\]

for some \( \nu > 0 \) and function \( \omega \) satisfying \( \omega(\sigma) \to 0 \) as \( \sigma \to 0_+ \). Thus, combining these two inequalities and sending \( \delta \to 0_+ \), we conclude that \( (u(\ddot{z}) - v(\ddot{z}))^+ \leq 0 \), a contradiction. 

It is possible to obtain the uniqueness (and comparison ...) of viscosity solutions by a different method adapted from I. Capuzzo-Colletta and
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P. L. Lions [12]. We then replace the restrictive assumption (6.7) by one of the following assumptions: either \( \Omega \) is strictly starshaped, i.e.,

\[ \exists \lambda_0 > 1, \quad \exists \gamma > 0, \quad \forall \lambda \in (1, \lambda_0), \quad \text{dist}(x, \partial \Omega) \geq \gamma (\lambda - 1) \quad \text{if} \quad x \in \lambda \Omega, \]

\( F \) satisfies in a neighborhood \( U_R \) of \( \partial \Omega \)

\[ F(x, t, p, A) \text{ is uniformly continuous in } p, \text{ uniformly for } x \in U_R, \quad |t| \leq R, \quad A \in \mathbb{M}^n \]  

(6.7')

or \( F \) satisfies (6.7') and

\[ F(x, t, p, A) \text{ is convex in } (p, A) \text{ for each } x \in U_R, \quad |t| + |p| \leq R \]

(and \( R < \infty \) is arbitrary) and \( \Omega \) is a smooth domain. Exactly as in [12], more general assumptions are possible and when we deal with Lipschitz continuous solutions on \( \Omega \) the convexity of \( F \) in \( A \) for \( x \in U_R, \quad |t| + |p| \leq R \) is enough.

In an (hopeless) attempt to restrict the length of this paper, we will not give a precise theorem, we will only explain how to adapt the proofs in [12] for the case where \( \Omega \) is strictly starshaped and we are in the situation of part (1) of Theorem VI.5. Furthermore, to unify a bit the presentation we will replace the structure conditions on \( F \) assumed in Theorem VI.1 by the general structure condition (4.20) and we want to conclude that (6.8) holds.

To prove this claim, we consider for \( \nu > 0, \delta > 0 \), the function \( \tilde{u} \) defined on \( (1 + \nu \delta) \Omega \) by

\[ \tilde{u}(x) = (1 + \nu \delta)^2 \frac{x}{1 + \nu \delta}. \]

It is very easy to check that \( \tilde{u} \) is a viscosity subsolution of

\[ F\left( \frac{x}{1 + \nu \delta}, \frac{1}{1 + \nu \delta} \tilde{u}, \frac{1}{1 + \nu \delta} D\tilde{u}, D^2 \tilde{u} \right) = 0 \quad \text{in} \quad (1 + \nu \delta) \Omega \]

and that the appropriate "Dirichlet" condition holds on \( \partial((1 + \nu \delta) \Omega) \).

Then, we apply the strategy of proof described in Section IV.3 and Lemma IV.1. And we find for \( \delta, \epsilon \) small enough, using the starshapedness of \( \Omega \), that there exist \( \bar{x} \in (1 + \nu \delta) \Omega, \quad \bar{y} \in \partial \Omega, \quad X, Y \in \mathbb{M}^n \) satisfying (4.18), (4.19) such that

\[ F\left( \frac{\bar{x}}{1 + \nu \delta}, \frac{1}{1 + \nu \delta} \tilde{u}(\bar{x}), \frac{1}{1 + \nu \delta} \frac{\bar{x} - \bar{y}}{\delta + \epsilon}, X \right) \leq 0 \]

\[ F\left( \frac{\bar{y}}{\delta + \epsilon}, v(\bar{y}), \frac{\bar{x} - \bar{y}}{\delta + \epsilon}, -Y \right) \geq 0. \]
We then introduce \( \dot{x} = \ddot{x}/(1 + \nu \delta) \) and we observe that

\[
\left| \frac{\dot{x} - \ddot{y}}{\delta + \varepsilon} - \frac{\ddot{x} - \ddot{y}}{\delta + \varepsilon} \right| \leq C_0 \nu, \quad \left| \frac{\dot{x} - \ddot{y}}{\delta + \varepsilon} - \frac{1}{1 + \nu \delta} \frac{\ddot{x} - \ddot{y}}{\delta + \varepsilon} \right| \leq C_0 \nu
\]

for some \( C_0 \) independent of \( \nu, \delta, \varepsilon \). We may now conclude easily using (6.7') and the structure condition (6.20), letting \( \varepsilon \to 0, \delta \to 0 \), and then \( \nu \to 0 \).

**Remark.** If we replace in the above argument \( \ddot{u} \) by

\[
F(x, t, p, A) \text{ is uniformly continuous in } A, \text{ uniformly for } x \in U_R, |t| \leq R, p \in \mathbb{R}^n.
\]

then (6.7') may be replaced by

\[
(1 + \nu \delta) u \left( \frac{x}{1 + \nu \delta} \right)
\]

VI.5 State-constraints Boundary Conditions

According to H. Soner [51], viscosity sub- and supersolutions of the state-constraints problem for (1.1) are defined as follows. An usc function \( u \) on \( \bar{Q} \) is said to be viscosity subsolution of this problem if it is a viscosity subsolution of (1.1). A lsc function \( u \) on \( \bar{Q} \) is said to be a viscosity supersolution of the problem if whenever \( \phi \in C^2(\bar{Q}) \) and \( u - \phi \) achieves its maximum at \( x_0 \in \bar{Q} \), then

\[
F(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) \geq 0.
\]

It is obvious that viscosity sub- and super solutions of the state-constraints problem are, respectively, those of (6.25) with \( h(x) = +\infty \). Looking at the state-constraints problem this way, we conclude that the same assertion for the state-constraints as Theorem VI.5 holds.

Let us finally mention that the reader interested in the state-constraints problem and more precisely in some existence and uniqueness results should consult [52, 12] for the case of first order Hamilton–Jacobi equations and [38] for some second order uniformly elliptic equations.

VII. Regularity Results

In this section we present some regularity results for viscosity solutions of (1.1).
VII.1. Hölder Continuity of Solutions

In this section we are concerned with strictly elliptic equations.

**Theorem VII.1.** Assume (3.1), (3.2), and that

\[ F(x, \cdot, p, A) \text{ is nondecreasing on } \mathbb{R} \text{ for all } x, p, A. \]  

Let \( u \in C(\Omega) \) be a viscosity solution of (1.1), and assume

\[ |u(x) - u(y)| \leq C |x - y|^{\alpha}, \quad \forall (x, y) \in \partial(\Omega \times \Omega), \]  

for some constants \( 0 < \alpha \leq 1 \) and \( C > 0 \). Then

\[ |u(x) - u(y)| \leq C' |x - y|^{\alpha}, \quad \forall x, y \in \Omega, \]  

for some constant \( C' > 0 \).

**Remark.** In the case when \( 0 < \alpha < 1 \), the assumptions of the above theorem may be relaxed a bit.

**Sketch of Proof.** The proof of part (1) of Theorem III.1 yields this assertion for \( \alpha = 1 \) with minor changes.

We prove the assertion in case where \( 0 < \alpha < 1 \). For simplicity, fixing \( R > 0 \) so that \( R > 2 \sup_{\Omega} |u| \), we write \( \omega \) both for \( \omega_{R} \) and \( \mu_{R} \). We may assume \( \nu_{R} = 1 \). For \( r, A > 0 \) we set

\[ Q(x) = M |x|^{\alpha}, \quad A_{r} = \{(x, y) \in \Omega \times \Omega | |x - y| < r\}. \]

We first observe that if \( M \geq C \) and \( Mr^{\alpha} \geq R \), then \( u(x) - u(y) \leq \Phi(x - y) \) for \( (x, y) \in \partial A_{r} \). To complete the proof, we assume that \( M \geq C, Mr^{\alpha} \geq R \) and \( u(x) - u(y) > \Phi(x - y) \) for some \( (x, y) \in A_{r} \), and follow the proof the part (1) of Theorem III.1 with the above \( \Phi \) and \( v = u \). We then find that

\[ |\text{Tr}(X + Y)| \leq C_{0} \left\{ 1 + \omega(\|\bar{x} - \bar{y}\|)(\alpha K)^{2 + t} (\bar{x} - \bar{y})^{2(t + 2) - 2} \right. \]

\[ + \omega(\|\bar{x} - \bar{y}\|)(\|B\| + |\text{Tr}(X + Y)|) \} \]

for some \((\bar{x}, \bar{y}) \in A_{r}\), with \( \bar{x} \neq \bar{y} \), matrices \( X, Y \in \mathbb{M}^{n} \) and constant \( C_{0} > 0 \), where \( B = D^{2}\Phi(\bar{x} - \bar{y}) \). Therefore, using (3.29) and

\[ \text{Tr} \frac{X \otimes X}{|x|^{2}} D^{2}\Phi(x) = -\alpha(1 - \alpha) M |x|^{\alpha - 2}, \]
we obtain

\[ 0 \geq \left( \frac{1}{2} - C_0 \omega(r) \right) |\text{Tr}(X + Y)| \]
\[ + |\hat{x} - \hat{y}|^{-2} \left\{ 2(1 - \alpha)t - C_1 \left[ \frac{t^{2/\alpha}}{(\alpha M)^{2/\alpha}} + \omega(r) t^{2 + \alpha} + \omega(r)t \right] \right\}, \]

where \( t = \alpha M |\hat{x} - \hat{y}|^\alpha \leq Mr^\alpha \) and \( C_1 \) is a constant independent of \( M \) and \( r \). Now, we choose \( M = R/r^\alpha \) and observe that the expression inside of the above braces is positive for \( 0 \leq t \leq R \) and \( M \geq C \) if \( r \) is small enough. Thus we obtain a contradiction, proving (7.3).

The above methods are easily modified to yield interior regularity results.

**Theorem VII.2.** Assume (7.1), (3.2), and that for any \( R > 0 \) there are constants \( v_R > 0, C_R > 0, \) and \( 0 < \lambda < 2 \) such that

\[ F(x, t, p + q, A + B) \leq F(x, t, p, A) - v_R \text{ Tr } B + C_R (|p|^2 + 1) \]

for all \( x \in \Omega, t \in \mathbb{R}, p, q \in \mathbb{R}^n, \) and \( A, B \in \mathbb{M}^n \) with \( |t| \leq R, |q| \leq R, \) and \( B > 0 \). Let \( u \in C(\Omega) \) be a viscosity solution of (1.1). Then for any \( \delta > 0 \) there is a constant \( C > 0 \) such that

\[ |u(x) - u(y)| \leq C |x - y|^\lambda, \quad \forall x, y \in \Omega_\delta. \]

**Remark.** If one replaces (3.2) by the weaker assumption indicated in the remark after Theorem VII.1 in the assumptions of the above theorem, then the following conclusion holds: for any \( \delta > 0 \) and \( 0 < \alpha < 1 \) there is a constant \( C > 0 \) such that

\[ |u(x) - u(y)| \leq C |x - y|^{\alpha}, \quad \forall x, y \in \Omega_\delta. \]

**Sketch of proof.** Let \( \omega, \mu, l, \Phi, \) and \( r_0 \) be as in the proof of part (1) of Theorem VII.1 but with \( \alpha = 0 \). In the definition of \( \Phi \) we now choose \( M \) so that \( Mr_0 \geq 4 \sup u \). We set \( \delta = r_0/K, \) and fix \( z \in \Omega_\delta \). Also we set \( \phi(x, y) = \Phi(x - y) + L |x - z|^2, \) where \( L = 2 \sup \Omega \| |u|/\delta^2, \) and

\[ A_\delta = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n/|x - y| < \delta, |x - z| < \delta \}. \]

By the choice of \( M \) and \( L, \) we see that if \( (x, y) \in \partial A_\delta, \) then \( \phi(x, y) \geq 2 \sup A_\delta |u|. \) It is clear that if \( (x, y) \in A \) then \( x, y \in \Omega. \) We will prove that \( u(x) - u(y) \leq \phi(x, y) \) for \( (x, y) \in A_\delta \) provided \( K \) is large enough. Once we prove this, plugging \( x = z \) into this, we conclude (7.5). To this end, we assume the contrary and proceed as in the proof of part (1) of Theorem III.1. The only difference in the calculations from the proof of
part (1) of Theorem III.1 comes, of course, from the fact that \( \varphi(x, y) \) is not a function of \( x - y \), and this is easily dealt with by taking the inequality

\[
\begin{pmatrix}
X & 0 \\
0 & Y
\end{pmatrix} \leq D^2 \varphi(x, y),
\]

where \( X, Y \in M^n \) and \( x, y \in \mathbb{R}^n \), as

\[
\begin{pmatrix}
X - 2LI & 0 \\
0 & Y
\end{pmatrix} \leq \begin{pmatrix}
D^2 \Phi(x - y) & -D^2 \Phi(x - y) \\
-D^2 \Phi(x - y) & D^2 \Phi(x - y)
\end{pmatrix}.
\]

We thus obtain

\[
|\text{Tr}(X + Y)| \leq C_0 \{1 + (L|\vec{x} - z|)^2 + \omega(|\vec{x} - \vec{y}|) |\vec{x} - \vec{y}|^\theta |D\Phi(\vec{x} - \vec{y})|^2 + \theta
+ \mu(|\vec{x} - \vec{y}|)(|D\Phi(\vec{x} - \vec{y})|)^{1/2}
+ |\text{Tr}(X + Y - 2LI)|^{1/2}) |\text{Tr}(X + Y - 2LI)|^{1/2}\}
\]

for some \( (\vec{x}, \vec{y}) \in \Delta_2 \), with \( \vec{x} \neq \vec{y} \), matrices \( X, Y \in M^n \) and constant \( C_0 > 0 \). Since \( L = O(K^2) \) and \( L|\vec{x} - z| = O(K) \) as \( K \to \infty \), we get a contradiction from the above inequality as in the proof of part (1) of Theorem III.1 if \( K \) is sufficiently large, which completes the proof.

VIII.2. Semiconcavity of Solutions

We show here that a combination of the techniques in [25] and a variant of Proposition II.3 provides a way to prove semiconcavity of solutions of Bellman equations. These semi-concavity results were first proved by probabilistic arguments by N. V. Krylov [35], P. L. Lions [46]. The function \( F \) now takes the form

\[
F(x, t, p, A) = \sup_{a \in \mathcal{A}} \{-\text{Tr} \Sigma_a(x)^T \Sigma_a(x) A - (b_a(x), p) + c_a(x)t - f_a(x)\},
\]

where \( \mathcal{A} \) is a given set and \( \Sigma_a(x) \in M(n, m) \) and \( b_a(x) \in \mathbb{R}^n \) for all \( a \in \mathcal{A} \) and \( x \in \mathbb{R}^n \). We use the following assumptions:

\[
\exists c_0 > 0, \quad c_a(x) \geq c_0 \quad \text{for} \quad a \in \mathcal{A} \quad \text{and} \quad x \in \mathbb{R}^n, \quad (7.8)
\]

\[
\Sigma_a, b_a, c_a, f_a \in W^{2, \infty}(\mathbb{R}^n) \quad \text{for} \quad a \in \mathcal{A} \quad (7.9)
\]

and moreover,

\[
\sup_{a \in \mathcal{A}} \|\varphi_a\|_{W^{2, \infty}(\mathbb{R}^n)} < \infty \quad \text{for} \quad \varphi_a = \Sigma_a, b_a, c_a, f_a.
\]
VISCOSITY SOLUTIONS

THEOREM VII.3. Let $\Omega$ be the whole space $\mathbb{R}^n$, and $u \in C(\mathbb{R}^n) \cap W^{1,\infty}(\mathbb{R}^n)$ be a viscosity solution of (1.1). Under the above assumptions, if $c_0$ is sufficiently large, then $u$ is semiconcave, i.e.,

$$u(x + h) + u(x - h) - 2u(x) \leq C |h|^2 \quad \forall x, h \in \mathbb{R}^n$$

for some constant $C > 0$.

Sketch of proof: We are going to prove

$$u(x) + u(y) - 2u(z) \leq C(|x - z|^4 + |y - z|^4 + |x + y - 2z|^2)^{1/2},$$

$$\forall x, y, z \in \mathbb{R}^n,$$

for some constant $C > 0$; this obviously yields (7.10) by plugging $x = z + h$ and $y = z - h$. It is easy to see that the above inequality is equivalent to the following one:

$$u(x) + u(y) - 2u(z) \leq C \left\{ \delta + \frac{1}{\delta} \left( |x - z|^4 + |y - z|^4 + |x + y - 2z|^2 \right) \right\},$$

$$\forall \delta > 0, \forall x, y, z \in \mathbb{R}^n,$$

(7.11)

for some constant $C > 0$. To prove (7.11), we fix any $\delta > 0$, $\gamma > 0$ and $M > 0$, and set

$$\psi(x, y, z) = |x - z|^4 + |y - z|^4 + |x + y - 2z|^2,$$

and

$$\phi(x, y, z) = M \left( \delta + \frac{1}{\delta} \psi(x, y, z) \right) + \gamma |x|^2.$$

It is clear that the function $u(x) + u(y) - 2u(z) - \phi(x, y, z)$ on $\mathbb{R}^{3n}$ achieves a maximum. We assume as usual that this maximum value is positive, and will get a contradiction for $c_0, M$ large enough and $\gamma$ small enough, proving (7.11). Now, we fix $\epsilon > 0$ and observe that $u'(x) + u'(y) - 2u_\epsilon(z) - \phi(x, y, z)$ still has a positive maximum. Let $(\tilde{x}, \tilde{y}, \tilde{z})$ be one of its maximum points. It is obvious that $\gamma |\tilde{x}|^2$ is bounded as $\gamma \to 0_+$. The proof of Proposition II.3 (see [24]) is now applied to show that there are matrices $X, Y, Z \in \mathbb{M}^n$ such that

$$F_\epsilon(\tilde{x}, u'(\tilde{x}), D_x \phi(x, \tilde{y}, \tilde{z}), X) \leq 0,$$

(7.12)

$$F_\epsilon(\tilde{y}, u'(\tilde{y}), D_y \phi(\tilde{x}, \tilde{y}, \tilde{z}), Y) \leq 0,$$

(7.13)

$$F_\epsilon(\tilde{z}, u'(\tilde{z}), -\frac{1}{2} D_z \phi(\tilde{x}, \tilde{y}, \tilde{z}), -\frac{1}{2} Z) \geq 0,$$

(7.14)
The rest of the proof is similar to that of Theorem 11.2. We calculate
\[ D^2 \psi(x, y, z) \]
\[ = 4 \begin{pmatrix} |x-z|^2 I & 0 & -|x-z|^2 I \\ 0 & |y-z|^2 I & -|y-z|^2 I \\ -|x-z|^2 I & -|y-z|^2 I & (|x-z|^2 + |y-z|^2)I \end{pmatrix} \]
\[ + 8 \begin{pmatrix} (x-z) \otimes (x-z) & 0 & -(x-z) \otimes (x-z) \\ 0 & (y-z) \otimes (y-z) & -(y-z) \otimes (y-z) \\ -(x-z) \otimes (x-z) & -(y-z) \otimes (y-z) & (x-z) \otimes (x-z) + (y-z) \otimes (y-z) \end{pmatrix} \]
\[ + \begin{pmatrix} I & I & -2I \\ I & I & -2I \\ -2I & -2I & 4I \end{pmatrix}, \]
and then that for \( \Sigma_1, \Sigma_2, \Sigma_3 \in M(m, n), \)
\[ \text{Tr} \left( \begin{pmatrix} \Sigma_1^T \Sigma_1 & \Sigma_1^T \Sigma_2 & \Sigma_1^T \Sigma_3 \\ \Sigma_2^T \Sigma_1 & \Sigma_2^T \Sigma_2 & \Sigma_2^T \Sigma_3 \\ \Sigma_3^T \Sigma_1 & \Sigma_3^T \Sigma_2 & \Sigma_3^T \Sigma_3 \end{pmatrix} D^2 \psi(x, y, z) \right) \]
\[ = 4 \text{Tr} \{|x-z|^2 (\Sigma_1 - \Sigma_3)^T (\Sigma_1 - \Sigma_3) + |y-z|^2 (\Sigma_2 - \Sigma_3)^T (\Sigma_2 - \Sigma_3)\} \]
\[ + 8 \text{Tr} \{(x-z) \otimes (x-z) (\Sigma_1 - \Sigma_3)^T (\Sigma_1 - \Sigma_3)\} \]
\[ + (y-z) \otimes (y-z) (\Sigma_2 - \Sigma_3)^T (\Sigma_2 - \Sigma_3) \}
\[ + \text{Tr}(\Sigma_1 + \Sigma_2 - 2\Sigma_3)^T (\Sigma_1 + \Sigma_2 - 2\Sigma_3). \]

Multiplying (7.15) by the nonnegative matrix
\[ \begin{pmatrix} \Sigma_1^T \Sigma_1 & \Sigma_1^T \Sigma_2 & \Sigma_1^T \Sigma_3 \\ \Sigma_2^T \Sigma_1 & \Sigma_2^T \Sigma_2 & \Sigma_2^T \Sigma_3 \\ \Sigma_3^T \Sigma_1 & \Sigma_3^T \Sigma_2 & \Sigma_3^T \Sigma_3 \end{pmatrix}, \quad \text{with} \quad \Sigma_i \in M(n, m), i = 1, 2, 3, \]
taking the trace and using the above inequality, we get
\[ \text{Tr}(\Sigma_1^T \Sigma_1 X + \Sigma_2^T \Sigma_2 Y + \Sigma_3^T \Sigma_3 Z) - 2\gamma \text{Tr} \Sigma_1^T \Sigma_2 \]

\[ \leq \frac{4M}{\delta} \text{Tr} \left\{ |x - z|^2 (\Sigma_1 - \Sigma_3)' (\Sigma_1 - \Sigma_3) + |\tilde{y} - z|^2 (\Sigma_2 - \Sigma_3)' (\Sigma_2 - \Sigma_3) \right\} \]

\[ + \frac{8M}{\delta} \text{Tr} \left\{ (x - z) \otimes (x - z)(\Sigma_1 - \Sigma_3)' (\Sigma_1 - \Sigma_3) \right\} \]

\[ + (\tilde{y} - z) \otimes (\tilde{y} - z)(\Sigma_2 - \Sigma_3)' (\Sigma_2 - \Sigma_3) \]

\[ + \frac{M}{\delta} \text{Tr} (\Sigma_1 + \Sigma_2 - 2\Sigma_3)' (\Sigma_1 + \Sigma_2 - 2\Sigma_3). \quad (6.16) \]

Also, we observe that

\[ (b_1, D_x \varphi(x, y, z)) + (b_2, D_y \varphi(x, y, z)) + (b_3, D_z \varphi(x, y, z)) \]

\[ = \frac{4M}{\delta} |x - z|^2 (b_1 - b_2, x - z) + \frac{4M}{\delta} |y - z|^2 (b_2 - b_3, y - z) \]

\[ + \frac{2M}{\delta} (b_1 + b_2 - 2b_3, x + y - 2z) + 2\gamma (b_1, x) \quad (7.17) \]

for \( x, y, z \in \mathbb{R}^n \) and \( b_i \in \mathbb{R}^n, i = 1, 2, 3 \), and that

\[ c_1 u(x) + c_2 u(y) - 2c_3 u(z) \]

\[ = (c_1 - c_3) (u(x) - u(z)) + (c_2 - c_3) (u(y) - u(z)) \]

\[ + (c_1 + c_2 - 2c_3) u(z) + c_3 (u(x) + u(y) - 2u(z)) \quad (7.18) \]

for \( x, y, z \in \mathbb{R}^n \) and \( c_i \in \mathbb{R}, i = 1, 2, 3 \). Finally, we add (7.12) to (7.13), subtract twice of (7.14) from the resulting inequality, and apply (7.16)–(7.18) and the inequality

\[ |g(x) + g(y) - 2g(z)| \leq \frac{1}{2} \| g \|_{W^{2, \infty}} \left( \delta + \frac{1}{\delta} \psi(x, y, z) \right) \]

for \( x, y, z \in \mathbb{R}^n \) and \( g \in W^{2, \infty}(\mathbb{R}^n) \), we then obtain

\[ c_0 (u'(\tilde{x}) + u'(\tilde{y}) - 2u_c(\tilde{z})) \leq C(M + 1) \left( \delta + \frac{1}{\delta} \psi(\tilde{x}, \tilde{y}, \tilde{z}) \right) + o(1), \quad (7.19) \]

where \( o(1) \to 0 \) as \( \epsilon, \gamma \to 0_+ \) and \( C \) is a positive constant depending only on \( \| \Sigma_x \|_{W^{2, \infty}}, \| b_x \|_{W^{2, \infty}}, \| c_x \|_{W^{2, \infty}}, \) and \( \| f_x \|_{W^{2, \infty}}, \alpha \in \mathcal{A} \). Thus, we have

\[ c_0 M \left( \delta + \frac{1}{\delta} \psi(\tilde{x}, \tilde{y}, \tilde{z}) \right) \leq C(M + 1) \left( \delta + \frac{1}{\delta} \psi(\tilde{x}, \tilde{y}, \tilde{z}) \right) + o(1), \]
and hence, if $c_0 > C$, $M < C/(c_0 - C)$ and $\varepsilon$ and $\gamma$ are small enough, then we get a contradiction.

Remarks. (1) Techniques similar to the above but closer to the proof of Theorem II.2, of course, yield the Hölder (including Lipschitz) continuity of solutions of Isaacs–Bellman equations, i.e., Eqs. (1.1) with $F$ defined by (2.20), provided that $c_0$ (from (2.21)) is large enough, $\Sigma_{x\beta}, b_{x\beta}, c_{x\beta}, f_{x\beta}$ are bounded in $W^{1, \infty}(\mathbb{R}^n)$ uniformly in $x, \beta$ and $u$ satisfies (7.2).

(2) Of course, the same result holds for the parabolic version

$$\frac{\partial u}{\partial t} + F(x, t, u, Du, D^2u) = 0 \quad \text{in } \mathbb{R}^n \times (0, T),$$

where $F$ is still given by (2.7) ($\Sigma, f, c$ now depend on $t$). With the same assumption (7.9) but now uniformly in $t \in (0, T)$, Theorem VII.3 holds and the constant $C$ in (7.10) does not depend on $t$. Observe that $c_0$ no longer needs to be large.

References

11. L. Caffarelli, personal communication.


55. N. S. Trudinger, Personnel communication and work in preparation.