Linear Algebra and its Applications 437 (2012) 2308–2316

Contents lists available at SciVerse ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

On the Laplacian and signless Laplacian spectrum of a graph with k pairwise co-neighbor vertices

Nair M.M. Abreu [a], Domingos M. Cardoso [b,*], Enide A. Martins [b], Maria Robbiano [c], B. San Martín [c]

[a] PEP/COPPE, Universidade Federal do Rio de Janeiro, RJ, Brazil
[b] Departamento de Matemática, Universidade de Aveiro, Aveiro, Portugal
[c] Departamento de Matemáticas, Universidad Católica del Norte, Antofagasta, Chile

ARTICLE INFO

Article history:
Received 23 February 2012
Accepted 14 May 2012
Available online 27 June 2012

Submitted by R.A. Brualdi

AMS classification:
05C50
15A18

Keywords:
Graph eigenvalues
Signless Laplacian spectrum
Laplacian spectrum
Pairwise co-neighbor vertices
Cluster

ABSTRACT

Consider the Laplacian and signless Laplacian spectrum of a graph G of order n, with k pairwise co-neighbor vertices. We prove that the number of shared neighbors is a Laplacian and a signless Laplacian eigenvalue of G with multiplicity at least $k-1$. Additionally, considering a connected graph G_k with a vertex set defined by the k pairwise co-neighbor vertices of G, the Laplacian spectrum of G^k, obtained from G adding the edges of G_k, includes $l+\beta$ for each nonzero Laplacian eigenvalue β of G_k. The Laplacian spectrum of G overlaps the Laplacian spectrum of G^k in at least $n-k+1$ places.

1. Introduction

In this paper we deal with undirected graphs G of order n, $V(G)$ denotes the vertex set and $E(G)$ the edge set. The set of neighbors of $v \in V(G)$ is denoted by $N_G(v)$ and its degree by $d(v)$. The maximum degree of the vertices of a graph G is denoted by $\Delta(G)$. A leaf is a vertex of degree 1. The adjacency matrix of the graph G is the $n \times n$ symmetric matrix $A(G) = (a_{ij})$ where $a_{ij} = 1$ if $ij \in E(G)$ and $a_{ij} = 0$ otherwise. The Laplacian (signless Laplacian) matrix of G is the matrix $L(G) = D(G) - A(G)$

* Corresponding author.
E-mail address: dcardoso@ua.pt (D.M. Cardoso).

0024-3795/$ - see front matter
http://dx.doi.org/10.1016/j.laa.2012.05.013

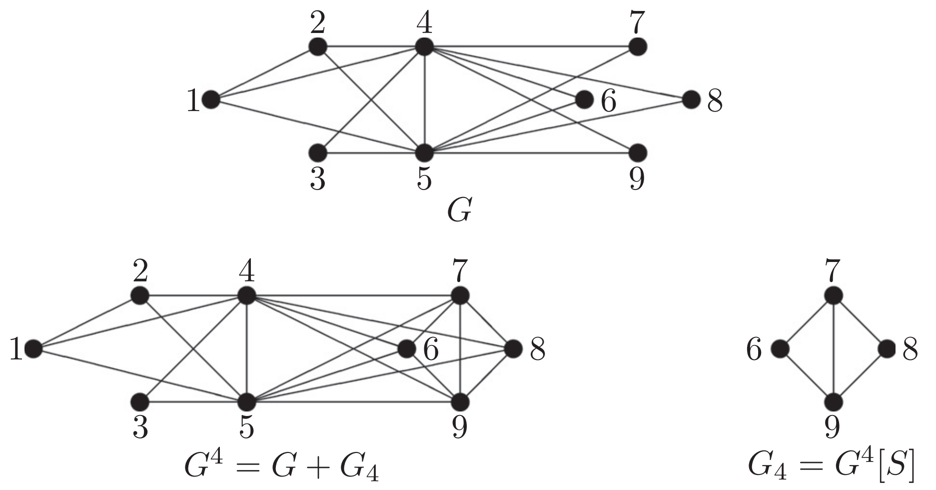

Fig. 1. A graph G with a 2-cluster of order 4, $S = \{6, 7, 8, 9\}$ and the graphs $G^4 = G + G_4$, such that $G_4 = G^4[S]$.

$(Q(G) = D(G)+A(G))$, where $D(G)$ is the $n \times n$ diagonal matrix of vertex degrees of G. The matrices $L(G)$, $Q(G)$ and $A(G)$ are all real and symmetric. From Geršgorin's theorem, it follows that the eigenvalues of $L(G)$ and $Q(G)$ are nonnegative real numbers. The spectrum of a matrix M is denoted by $\sigma(M)$ and, in particular cases of $L(G)$ and $Q(G)$, their spectra are denoted by $\sigma_L(G)$ and $\sigma_Q(G)$, respectively. Throughout the paper $\sigma_L(G) = \{\mu_1^{[i_1]}, \ldots, \mu_p^{[i_p]}\}$ $(\sigma_Q(G) = \{q_1^{[k_1]}, \ldots, q_r^{[k_r]}\})$ means that μ_j (q_j) is a Laplacian (signless Laplacian) eigenvalue with multiplicity i_j (k_j), for $j = 1, \ldots, p$ $(j = 1, \ldots, r)$. As usually, we denote the eigenvalues of $L(G)$ $(Q(G))$ in non increasing order $\mu_1(G) \geq \cdots \geq \mu_n(G)$ $(q_1(G) \geq \cdots \geq q_n(G))$. For details on the spectral properties of $L(G)$ and $Q(G)$ see, for instance, [7–10,12,1,3], respectively.

Considering a square matrix A, one of its eigenvalues λ, and a vector u of the corresponding eigenspace, the pair (λ, u) is called an eigenpair of A. A vertex subset is called independent if its elements are pairwise non-adjacent. Two vertices in $V(G)$ are *co-neighbor vertices* if they share the same neighbors. It is easy to see that if $S \subset V(G)$ is a set of pairwise co-neighbor vertices of a graph G, then S is an independent set of G. According to Merris [12], a *cluster* of order k of G is a set S of k pairwise co-neighbor vertices. The *degree of a cluster* is the cardinality of the shared set of neighbors, i.e., the common degree of each vertex in the cluster. An l-cluster is a cluster of degree l.

Regarding spectral properties of graphs with co-neighbor vertices, Faria, in [4], introduced the following result about Laplacian and signless Laplacian eigenvalues of graphs with leaves.

Theorem 1 [4]. *Let p and q be the number of leaves of G and the number of neighbors associated to these leaves, respectively. Then* 1 *is a Laplacian (signless Laplacian) eigenvalue of G with multiplicity at least* $p - q \geq 0$.

Let k be an integer greater than 1. Taking the above result as a motivation, Theorem 1 is generalized for graphs G with a cluster of order k. Moreover, considering a cluster of order k, $S \subset V(G)$, G^k is the supergraph obtained from G, adding t edges between distinct pairs of vertices in S, where $1 \leqslant t \leqslant \frac{k(k-1)}{2}$. From now on, this operation is denoted by

$$G^k = G + G_k, \tag{1}$$

where G_k is the subgraph of G^k induced by S, that is, $G_k = G^k[S]$. Notice that $V(G^k) = V(G)$ and $E(G^k) = E(G) \cup E(G_k)$. In Fig. 1 a graph G with a 2-cluster of order 4, $S = \{6, 7, 8, 9\}$, and a graph $G^4 = G + G_4$, with $G_4 = G^4[S]$ are depicted.

The main result of this paper is the determination of Laplacian and signless Laplacian eigenvalues of G^k (for which the induced subgraph G_k must be p-regular in the signless Laplacian case). In fact, assuming that S is an l-cluster of order k, and $\beta \neq 0$ ($\beta \neq 2p$) is a Laplacian (signless Laplacian) eigenvalue of G_k, it is deduced that $l+\beta$ is a Laplacian (signless Laplacian) eigenvalue of G^k. Furthermore, in the Laplacian spectrum case, we may conclude that at least $n-k+1$ Laplacian eigenvalues of G are also eigenvalues of G^k and then the Laplacian spectrum of G^k is completely characterized from the Laplacian spectrum of G and G_k.

As it is stated by the following lemma, despite the labeling of S, the operation defined by (1) produces isomorphic graphs.

Lemma 1. *Let G be a graph with a cluster $S \subset V(G)$ of order k. If G_k and G'_k are connected isomorphic graphs of order k such that $V(G_k) = V(G'_k) = S$, then $G^k = G + G_k$ is isomorphic to $G'^k = G + G'_k$.*

2. Laplacian and signless Laplacian eigenvalues of graphs with pairwise co-neighbor vertices

Now, considering a graph G, with an l-cluster S of order k, we prove that l is an eigenvalue with multiplicity at least $k-1$ for both matrices, $L(G)$ and $Q(G)$.

From now on, $\mathbf{x}_q$ denotes a vector with q entries, in particular, $\mathbf{1}_q$ denotes the all-one vector with q entries. Additionally, $G-S$ is the subgraph of G obtained by deleting the above k pairwise co-neighbor vertices of the cluster S.

Theorem 2. *Let G be a graph with an l-cluster S of order k. Then l is a Laplacian and a signless Laplacian eigenvalue of G with multiplicity at least $k-1$.*

Proof. Let $S = \{v_i\}_{i=1}^{k}$ be the l-cluster. Assuming that $N_G(v_1) = \{v_{k+i}\}_{i=1}^{l}$, we have

$$M(G) = \left(\begin{array}{c|cc} lI_k & \delta \mathbf{1}_k \mathbf{1}_l^T & \mathbf{0}_{k;n-k-l} \\ \hline \begin{array}{c} \delta \mathbf{1}_l \mathbf{1}_k^T \\ \hline \mathbf{0}_{n-k-l;k} \end{array} & \multicolumn{2}{c}{\begin{pmatrix} kI_l & \mathbf{0}_{l;n-k-l} \\ \mathbf{0}_{n-k-l;l} & \mathbf{0}_{n-k-l;n-k-l} \end{pmatrix} + M(G-S)} \end{array} \right), \qquad (2)$$

where $\delta = \begin{cases} -1, & \text{if } M(G) = L(G) \text{ and then } M(G-S) = L(G-S) \\ 1, & \text{if } M(G) = Q(G) \text{ and then } M(G-S) = Q(G-S) \end{cases}$ and $\mathbf{0}_{p;q}$ is the all zero matrix $p \times q$. Therefore, the k first rows of the matrix $M(G) - lI$ are equal and then $rank(M(G)-lI) \leq n-(k-1)$. Hence, the null space of $M(G) - lI$ has dimension not less than $k-1$ and therefore, l is an eigenvalue of $M(G)$ with multiplicity at least $k-1$. $\square$

Remark 1. Taking into account the labeling of the vertices in Theorem 2 and the matrix $M(G)$ in (2), it is immediate that $\left(l, \begin{bmatrix} \mathbf{x}_k \\ \mathbf{0}_{n-k} \end{bmatrix} \right)$ is an eigenpair for $M(G)$ for all $\mathbf{x}_k \in \mathbb{R}^k \backslash \{\mathbf{0}_k\}$, with $\mathbf{x}_k^T \mathbf{1}_k = 0$.

As immediate application of Theorem 2, considering a complete bipartite graph $K_{r,s}$, it follows that each color class of vertices is a vertex subset of pairwise co-neighbors. Therefore, despite the Laplacian spectrum of $K_{r,s}$ be well known, we may conclude that r is a Laplacian eigenvalue with multiplicity at least $s-1$ and s is a Laplacian eigenvalue with multiplicity at least $r-1$. Therefore, taking into account that

1. the trace of the Laplacian matrix is $2rs$,
2. the Laplacian matrix has 0 as eigenvalue,
3. the Laplacian and signless Laplacian matrices have the same spectrum,

then the unknown eigenvalue is $r+s$. Thus

$$\sigma_L(K_{r,s}) = \sigma_Q(K_{r,s}) = \{0, r^{[s-1]}, s^{[r-1]}, r+s\}.$$

Therefore, we have the following interesting corollary of Theorem 2.

Corollary 1. *If the complete bipartite graph $K_{r,s}$, with $\max\{r,s\} = s > 1$, is a subgraph of a graph G, then*

$$\mu_1(G) \geq r+s, \tag{3}$$

$$\mu_r(G) \geq s, \tag{4}$$

$$\mu_{r+s-1}(G) \geq r. \tag{5}$$

In particular, $\mu_2(G) \geq s$.

Proof. As above described, $\sigma_L(K_{r,s}) = \{0, r^{[s-1]}, s^{[r-1]}, r+s\}$. Assuming, without loss of generality, that the subgraph $K_{r,s}$ of G is defined by the first $r+s$ vertices of G, we may say that the Laplacian matrix $L(G)$ is such that

$$L(G) = L(H) + \begin{pmatrix} L(K_{r,s}) & 0 \\ 0 & 0 \end{pmatrix}, \tag{6}$$

where H is the subgraph of G such that $V(H) = V(G)$ and $E(H) = E(G)\backslash E(K_{r,s})$. Denoting $B = \begin{pmatrix} L(K_{r,s}) & 0 \\ 0 & 0 \end{pmatrix}$, since the eingenvalues of a matrix do not decrease whenever a positive semidefinite matrix is added and the matrix $L(H)$ is positive semidefinite, the result follows. □

Notice that since $\mu_1(G)$ is not greater than the order n of G, if G has a bipartite complete graph $K_{r,s}$ as a subgraph, then

$$r+s \leq \mu_1(G) \leq n.$$

On the other hand since $q_1(G) \geq \mu_1(G)$, see [3], the inequality (3) is also valid for $q_1(G)$. As another direct application of Corollary 1, the well know inequality $\mu_1(G) \geq \Delta(G) + 1$ (see, for instance [12]) can also be obtained, considering $\Delta = \Delta(G)$ and $K_{1,\Delta}$ as a subgraph of G.

2.1. The Laplacian case

Before the main result of this subsection it is worth to consider the following lemmas.

Lemma 2 [11]**.** *Consider the square matrices A and B of order n with spectra $\{\alpha_1, \ldots, \alpha_n\}$ and $\{\beta_1, \ldots, \beta_n\}$, respectively. If $AB = BA$, then there exists a permutation $i_1, \ldots, i_n$ of $1, \ldots, n$ such that the spectrum of $A+B$ is $\{\alpha_1 + \beta_{i_1}, \ldots, \alpha_k + \beta_{i_k}, \ldots, \alpha_n + \beta_{i_n}\}$.*

Lemma 3. *If we consider the matrix*

$$K = \left(\begin{array}{c|c} L(G_k) & \mathbf{0}_{k;n-k} \\ \hline \mathbf{0}_{n-k;k} & \mathbf{0}_{n-k;n-k} \end{array}\right), \tag{7}$$

then $L(G^k) = L(G) + K$ and $L(G)K = KL(G)$.

Proof. The proof follows immediately from the structure of the matrices $L(G)$ and K, taking into account that

$$L(G) = \left(\begin{array}{c|c|c} lI_k & -\mathbf{1}_k\mathbf{1}_l^T & \mathbf{0}_{k;n-k-l} \\ \hline -\mathbf{1}_l\mathbf{1}_k^T & \multicolumn{2}{c}{\multirow{2}{*}{X}} \\ \cline{1-1} \mathbf{0}_{n-k-l;k} & \multicolumn{2}{c}{} \end{array} \right),$$

where X is the diagonal block matrix of order $n-k$ in (2), with $M(G-S) = L(G-S)$. □

Lemmas 2 and 3 imply the following corollary.

Corollary 2. *Consider a graph G with an l-cluster of order k, $S = \{v_i\}_{i=1}^k$, sharing the l neighbors $\{v_{k+j}\}_{j=1}^l$, a graph G_k such that $V(G_k) = S$ and the matrix K defined in (7). If G^k is defined as in (1), reordering the n eigenvalues β_i in $\sigma(K)$, it follows that*

$$\sigma_L(G^k) = \{\alpha_i + \beta_i : i = 1, \ldots, n\}, \tag{8}$$

where $\alpha_1, \ldots, \alpha_n$ are the Laplacian eigenvalues of G. Moreover,

$$L(G^k)L(G) = L(G)L(G^k). \tag{9}$$

Now, we are able to deduce how the Laplacian eigenvalues of G^k are modified in function of G_k.

Theorem 3. *Let G be a graph with an l-cluster S of order k. Assume that G_k is a connected graph such that $V(G_k) = S$, $G^k = G + G_k$ and*

$$\Lambda = \{l + \beta : \beta \in \sigma_L(G_k) \setminus \{0\}\}$$

is a multiset. Then $\sigma_L(G^k)$ overlaps $\sigma_L(G)$ in $n-k+1$ places and the elements of Λ are the remaining eigenvalues in $\sigma_L(G^k)$.

Proof. Let $S = \{v_i\}_{i=1}^k$ be the cluster, sharing the l neighbors $\{v_{k+i}\}_{i=1}^l$. Let K be the matrix in (7). Since $0 \in \sigma(K)$ with multiplicity at least $n-k+1$, using (8), it is immediate that at least $n-k+1$ Laplacian eigenvalues of G overlap the Laplacian eigenvalues of G^k. We just need to prove that the elements of Λ are the remaining eigenvalues in $\sigma_L(G^k)$.
Taking into account (9), the matrices $L(G)$ and $L(G^k)$ are simultaneously diagonalizable [11, Theorem 1.3.12]. Therefore, assuming that $\mathbf{u} = \begin{pmatrix} \mathbf{x}_k \\ \mathbf{y}_l \\ \mathbf{z}_{n-k-l} \end{pmatrix}$ is one of the n chosen common eigenvectors of $L(G)$ and $L(G^k)$, from $L(G^k)\mathbf{u} = \lambda\mathbf{u}$, $L(G)\mathbf{u} = \lambda'\mathbf{u}$ and $L(G^k) = L(G) + K$, the following system of equations are obtained:

$$l\mathbf{x}_k + L(G_k)\mathbf{x}_k - (\mathbf{1}_l^T\mathbf{y}_l)\mathbf{1}_k = \lambda\mathbf{x}_k, \tag{10}$$

$$l\mathbf{x}_k - (\mathbf{1}_l^T\mathbf{y}_l)\mathbf{1}_k = \lambda'\mathbf{x}_k, \tag{11}$$

$$\begin{pmatrix} -(\mathbf{1}_k^T\mathbf{x}_k)\mathbf{1}_l + k\mathbf{y}_l \\ \mathbf{0}_{n-k-l} \end{pmatrix} + L(G-S)\begin{pmatrix} \mathbf{y}_l \\ \mathbf{z}_{n-k-l} \end{pmatrix} = \lambda\begin{pmatrix} \mathbf{y}_l \\ \mathbf{z}_{n-k-l} \end{pmatrix}, \tag{12}$$

$$\begin{pmatrix} -(\mathbf{1}_k^T\mathbf{x}_k)\mathbf{1}_l + k\mathbf{y}_l \\ \mathbf{0}_{n-k-l} \end{pmatrix} + L(G-S)\begin{pmatrix} \mathbf{y}_l \\ \mathbf{z}_{n-k-l} \end{pmatrix} = \lambda'\begin{pmatrix} \mathbf{y}_l \\ \mathbf{z}_{n-k-l} \end{pmatrix}. \quad (13)$$

Notice that $L(G)$ is the matrix $M(G)$ in (2), with $\delta = -1$. By subtracting the Eq. (11) from Eq. (10), we may conclude that

$$L(G_k)\mathbf{x}_k = (\lambda - \lambda')\mathbf{x}_k. \quad (14)$$

- If $\lambda = \lambda'$, then $\lambda \in \sigma_L(G^k) \cap \sigma_L(G)$ and from (14), $L(G_k)\mathbf{x}_k = 0$. This implies that $\mathbf{x}_k = \gamma\mathbf{1}_k$, where γ is a nonzero scalar if $\mathbf{x}_k$ is an eigenvector of $L(G_k)$ (since the connectivity of G_k implies that 0 is a simple eigenvalue of $L(G_k)$).
- If $\lambda \neq \lambda'$, then (14) is equivalent to $L(G_k)\mathbf{x}_k = \beta\mathbf{x}_k$, with $\beta = \lambda - \lambda' \neq 0$ which implies $\mathbf{1}_k^T\mathbf{x}_k = 0$. Moreover, from the equalities (12) and (13), it follows that $\begin{pmatrix} \mathbf{y}_l \\ \mathbf{z}_{n-k-l} \end{pmatrix} = \begin{pmatrix} \mathbf{0}_l \\ \mathbf{0}_{n-k-l} \end{pmatrix}$ and then $\mathbf{x}_k \neq \mathbf{0}$, that is, $\mathbf{x}_k$ is an eigenvector of $L(G_k)$ orthogonal to the all one eigenvector $\mathbf{1}_k$. Therefore, the eigenvector $\mathbf{u}$ is of the type $\mathbf{u} = \begin{pmatrix} \vartheta \\ \mathbf{0}_{n-k} \end{pmatrix}$, where ϑ is an eigenvector of $L(G_k)$, associated to an eigenvalue $\beta \neq 0$. From equality (11), it follows that $\lambda' = l$ and so $\lambda = l + \beta$.

From the above analysis, we may conclude that $L(G)$ and $L(G^k)$ are sharing two types of eigenvectors. The eigenvectors $\begin{pmatrix} \gamma\mathbf{1}_k \\ \mathbf{y}_l \\ \mathbf{z}_{n-k-l} \end{pmatrix}$ correspond to the eigenvalues $\lambda \in \sigma_L(G^k) \cap \sigma_L(G)$ and the eigenvectors $\begin{pmatrix} \vartheta \\ \mathbf{0}_{n-k} \end{pmatrix}$ correspond to the Laplacian eigenvalues $l+\beta$ of G^k ($l \in \sigma_L(G)$ and $\beta \in \sigma_L(G_k) - \{0\}$). Since $L(G_k)$ has the $k-1$ eigenpairs (β_i, ϑ^i), with $\beta_i \neq 0$, for $i = 1, \ldots, k-1$, we have $k-1$ shared eigenvectors $\begin{pmatrix} \vartheta^i \\ \mathbf{0}_{n-k} \end{pmatrix}$, for $i = 1, \ldots, k-1$. Therefore, taking into account (8) in Corollary 2, we have at least $n-k+1$ shared eigenvectors $\begin{pmatrix} \gamma\mathbf{1}_k \\ \mathbf{y}_l \\ \mathbf{z}_{n-k-l} \end{pmatrix}$ which are orthogonal to $\begin{pmatrix} \vartheta^i \\ \mathbf{0}_{n-k} \end{pmatrix}$, for $i = 1, \ldots, k-1$. Comparing both cardinalities with the order of $L(G^k)$ and $L(G)$, the Laplacian eigenvalues of G^k which can be different from the Laplacian eigenvalues of G are just the $k-1$ elements of Λ. □

Remark 2. Assuming that G is a graph with an l-cluster S of order k, consider two connected graphs G_k and G'_k defined on S. From the proof of Theorem 3, we conclude that the Laplacian spectra of $G^k = G+G_k$ and $G'^k = G+G'_k$ overlap in $n-k+1$ places. Furthermore, the remaining Laplacian eigenvalues of G^k and G'^k, $\beta + l$ (with $\beta \in \sigma_L(G_k)\backslash\{0\}$) and $\beta' + l$ (with $\beta' \in \sigma_L(G'_k)\backslash\{0\}$), respectively, replace $k-1$ of the positions of the eigenvalue l of G (see Remark 1).

Taking into account that a graph G is called Laplacian integral if its Laplacian eigenvalues are all integers, it is immediate to conclude the following corollary from Theorem 3.

Corollary 3. *Let G be a graph with an cluster S of order k, and G_k a connected graph such that $V(G_k) = S$. If G and G_k are Laplacian integral graphs, then $G^k = G + G_k$ is also Laplacian integral.*

Other interesting consequences are related with the algebraic connectivity, the largest Laplacian eigenvalue and the Laplacian spread (defined below) of some families of graphs.

Definition 1. The Laplacian spread of a graph is the difference between the largest Laplacian eigenvalue and the algebraic connectivity.

In fact, considering the graph $G = K_{r,s}$ on the vertices $\{v_i\}_{i=1}^{r+s}$ and a connected graph G_r, such that $V(G_r) = S$, where $S = \{v_1, \ldots, v_r\}$ (an s-cluster of order r in G), then we may conclude the following result.

Theorem 4. *If $r \leq s$, $G = K_{r,s}$ and G_r is a connected graph defined on the vertex subset of r pairwise co-neighbors of G, then the graphs G and $G^r = G + G_r$ have the same largest Laplacian eigenvalue $r + s$, the same algebraic connectivity r and the same Laplacian spread s.*

Proof. By Theorem 3

$$\sigma_L(G^r) = \Lambda \cup (\sigma_L(G) \backslash \{s^{[r-1]}\}),$$

where $\Lambda = \{s + \beta : \beta \in \sigma_L(G_r) \backslash \{0\}\}$, with $\beta \leq r$ (since, as it is well known, the spectral radius of the Laplacian matrix of any graph is not greater than its order) and $\{s^{[r-1]}\}$ is the multiset with s repeated $r - 1$ times (see Remark 2). Since $\sigma_L(G) = \{0, r^{[s-1]}, s^{[r-1]}, r + s\}$ we obtain $\sigma_L(G^r) = \{0, r^{[s-1]}, r+s\} \cup \Lambda$. Therefore, the largest eigenvalue of G and G^r is $r+s$ and, since $r \leq s$, the algebraic connectivity of both graphs coincide and is r. Moreover, the Laplacian spread of G and G^r is s. $\square$

2.2. The signless Laplacian case

In the next results we deal with the concept of main (non-main) eigenvalue. This concept was introduced in [2] and has been largely used in the context of adjacency matrices. A good survey on this topic was published in [13]. Herein, we extend this concept also to signless Laplacian matrices. Given a graph G, an eigenvalue $\lambda \in \sigma_Q(G)$ is *non-main* if the corresponding eigenspace is orthogonal to the all one vector, $\mathbf{1}$, otherwise it is *main*. For instance, in the graphs considered in Theorem 2, if the signless Laplacian eigenvalue l has multiplicity exactly $k - 1$, then l is non-main. In fact, taking into account the structure of the matrix $Q(G^k)$, it is immediate that $\xi_i = \epsilon_1 - \epsilon_i$, for $i = 2, \ldots, k$, where ϵ_i is the ith vector of the canonical basis of $\mathbb{R}^n$, are the eigenvectors of $Q(G^k)$ corresponding to l.

Theorem 5. *Let G be a graph with an l-cluster $S = \{v_i\}_{i=1}^{k}$ of order k. If G_k is a graph such that $V(G_k) = S$ and $G^k = G + G_k$, then $\sigma_Q(G^k)$ includes the multiset*

$$\left\{l + \beta : \beta \in \sigma_Q(G_k) \text{ and it is non-main}\right\}.$$

Furthermore, any main eigenvalue γ of $Q(G_k)$, with multiplicity $m > 1$, produces an eigenvalue $l + \gamma$ of $Q(G^k)$, with multiplicity $m - 1$.

Proof. Taking into account that

$$Q(G^k) = \left(\begin{array}{c|c|c} lI_k + Q(G_k) & \mathbf{1}_k\mathbf{1}_l^T & \mathbf{0}_{k;n-k-l} \\ \hline \begin{array}{c} \mathbf{1}_l\mathbf{1}_k^T \\ \hline \mathbf{0}_{n-k-l;k} \end{array} & \multicolumn{2}{c}{\begin{pmatrix} kI_l & \mathbf{0}_{l;n-k-l} \\ \mathbf{0}_{n-k-l;l} & \mathbf{0}_{n-k-l;n-k-l} \end{pmatrix} + Q(G-S)} \end{array}\right),$$

let us assume that $\mathbf{x_k}$ is an eigenvector of $Q(G_k)$ associated to the non-main eigenvalue β of $Q(G_k)$. Since $\mathbf{x_k}$ is orthogonal to $\mathbf{1_k}$, it is immediate that $\begin{pmatrix} \mathbf{x_k} \\ \mathbf{0_{n-k}} \end{pmatrix}$ is an eigenvector of $Q(G^k)$ associated to the eigenvalue $l+\beta$. The second part is direct consequence of the fact that if a main eigenvalue γ has multiplicity $m>1$, then the corresponding eigenspace has at least $m-1$ linearly independent eigenvectors $\mathbf{x_k}$ orthogonal to the all one vector $\mathbf{1_k}$. □

As immediate consequence of Theorem 5, we have the following corollary.

Corollary 4. *If G_k is a p-regular graph defined on an l-cluster of order k of a graph G, and $G^k = G + G_k$, then $\sigma_Q(G^k)$ includes the multiset $\{l+\beta : \beta \in \sigma_Q(G_k)\backslash\{2p\}\}$.*

Now, we recall the Fiedler's lemma which was introduced in [5] in the context of the inverse eigenvalue problem.

Lemma 4 [5]**.** *Let A and B be symmetric matrices of orders m and n, respectively, with corresponding eigenpairs (α_i, u_i), $i = 1, \ldots, m$ and (β_i, v_i), $i = 1, \ldots, n$, respectively. Suppose that $\|u_1\| = 1 = \|v_1\|$. Then, for any ρ the matrix*

$$C = \begin{pmatrix} A & \rho u_1 v_1^T \\ \rho v_1 u_1^T & B \end{pmatrix}$$

has eigenvalues $\alpha_2, \ldots, \alpha_n, \beta_2, \ldots, \beta_m, \gamma_1, \gamma_2$, where γ_1, γ_2 are eigenvalues of

$$\widehat{C} = \begin{pmatrix} \alpha_1 & \rho \\ \rho & \beta_1 \end{pmatrix}.$$

From now on the matrix $\widehat{C}$ referred in the above lemma will be called the Fiedler matrix.

Theorem 6. *Consider the complete bipartite graph $G = K_{r,s}$, with vertex set $\{v_i\}_{i=1}^{r+s}$ and let G_r be a connected p-regular graph defined on the s-cluster of order r, $\{v_1, \ldots, v_r\}$ of G. If $G^r = G + G_r$, then*

$$\{r^{[s-1]}\} \subset \sigma_Q(G) \cap \sigma_Q(G^r)$$

and the remainder signless Laplacian eigenvalues of G^r are the elements of the multiset $\{s+\gamma : \gamma \in \sigma_Q(G_r)\backslash\{2p\}\} \cup \left\{\frac{r+s+2p \pm \sqrt{(r+s+2p)^2 - 8pr}}{2}\right\}$.

Proof. Since $\{v_{r+1}, \ldots, v_{r+s}\}$ is an r-cluster of order s of G (and also in G^r), each one of them with r neighbors, according to Theorem 2, r is a signless Laplacian eigenvalue of G with multiplicity at least $s-1$. On the other hand, considering the signless Laplacian matrix $Q(G^r) = \begin{pmatrix} sI_r + Q(G_r) & \mathbf{1}_r\mathbf{1}_s^T \\ \mathbf{1}_s\mathbf{1}_r^T & rI_s \end{pmatrix}$ and applying the Fiedler's lemma, with $\rho = \sqrt{rs}$, it follows that

$$\sigma_Q(G^r) = \sigma(sI_r + Q(G_r))\backslash\,\{s+2p\} \cup \left\{r^{[s-1]}\right\} \cup \sigma(\widehat{C}),$$

where the Fiedler matrix $\widehat{C}$ is such that $\widehat{C} = \begin{pmatrix} s+2p & \sqrt{rs} \\ \sqrt{rs} & r \end{pmatrix}$. Thus the result follows. □

Given two graphs G and H, the *join* of these graphs, $G \vee H$, is such that $V(G \vee H) = V(G) \cup V(H)$ and $E(G \vee H) = E(G) \cup E(H) \cup \{xy : x \in V(G), y \in V(H)\}$. Theorem 6 can be extended to the determination of the signless Laplacian spectrum of the join of a p-regular graph of order r with a q-regular of order s, respectively, from their signless Laplacian spectra and the spectrum of the Fiedler matrix $\widehat{C} = \begin{pmatrix} s+2p & \sqrt{rs} \\ \sqrt{rs} & r+2q \end{pmatrix}$, as it is stated in the following corollary (a similar result was published in [6, Theorem 2.1]).

Corollary 5. *Let G_r and G_s be two regular graphs order r and s, respectively. Assuming that G_r is p-regular and G_s is q-regular, then*

$$\sigma_Q(G_r \vee G_s) = \{s + \theta : \theta \in \sigma_Q(G_r) \setminus \{2p\}\} \cup \{r + \zeta : \zeta \in \sigma_Q(G_s) \setminus \{2q\}\} \cup \left\{ \frac{r+s+2(p+q) \pm \sqrt{(r+s+2(p+q))^2 - 8(pr+qs+2pq)}}{2} \right\}.$$

Notice that $G_r \vee G_s$ can be obtained from $G = K_{r,s}$, defining G_r on the vertex subset of r pairwise co-neighbors of G and defining G_s on the vertex subset of s pairwise co-neighbors of G. Therefore, $G_r \vee G_s = (G + G_r) + G_s = G^r + G_s$.

From Theorem 6 and Corollary 5, we may conclude that the overlapping of the Laplacian spectrum of G and G^k, mentioned in Theorem 3, does not holds for the signless Laplacian case.

Acknowledgments

Nair M.M. Abreu partially supported by CNPq Grant PQ 300563/94-9(NV). Domingos M. Cardoso and Enide A. Martins supported by *FEDER* founds through *COMPETE* – Operational Programme Factors of Competitiveness and by Portuguese founds through the *Center for Research and Development in Mathematics and Applications* (University of Aveiro) and the Portuguese Foundation for Science and Technology ("FCT – Fundação para a Ciência e a Tecnologia"), within project PEst-C/MAT/UI4106/2011 with COMPETE number FCOMP-01-0124-FEDER-022690. M. Robbiano partially supported by Fondecyt Grant 11090211, Chile. All the authors are members of the Project PTDC/MAT/112276/2009.
The authors thank to an anonymous referee for his/her suggestions which contributed to improve the presentation of the paper.

References

[1] D.M. Cardoso, D. Cvetković, P. Rowlinson, S.K. Simić, A sharp lower bound for the least eigenvalue of the signless Laplacian of non-bipartite graph, Linear Algebra Appl. 429 (2008) 2270–2780.
[2] D. Cvetković, M. Doob, H. Sachs, Spectra of Graphs – Theory and Application, Academic Press, New York, 1980.
[3] D. Cvetković, P. Rowlinson, S. Simić, Signless Laplacians of finite graphs, Linear Algebra Appl. 423 (2007) 155–171.
[4] I. Faria, Permanental roots and the star degree of a graph, Linear Algebra Appl. 64 (1985) 255–265.
[5] M. Fiedler, Eigenvalues of nonnegative symmetric matrices, Linear Algebra Appl. 9 (1974) 119–142.
[6] M.A.A. de Freitas, N.M.M. de Abreu, R.R. Del-Vecchio, S. Jurkiewicz, Infinite families of Q-integral graphs, Linear Algebra Appl. 432 (2010) 2352–2360.
[7] R. Grone, On the geometry and Laplacian of a graph, Linear Algebra Appl. 150 (1991) 167–178.
[8] R. Grone, R. Merris, The Laplacian spectrum of a graph II, SIAM J. Discrete Math. 7 (1994) 221–229.
[9] R. Grone, R. Merris, V.S. Sunder, The Laplacian spectrum of a graph, SIAM J. Matrix Anal. Appl. 11 (1990) 218–238.
[10] R. Grone, R. Merris, V. Sunder, The Laplacian spectrum of a graph, SIAM J. Matrix Anal. Appl. 11 (1994) 218–238.
[11] R. Horn, C. Johnson, Matrix Analysis, Cambridge University Press, 1985.
[12] R. Merris, Laplacian matrices of a graph: a survey, Linear Algebra Appl. 197 & 198 (1994) 143–176.
[13] P. Rowlinson, The main eigenvalues of a graph: a survey, Appl. Anal. Discrete Math. 1 (2007) 445–471.