Applying multiquadric quasi-interpolation for boundary detection

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\textbf{ABSTRACT}

In this paper, we propose a novel scheme for simulating geometric active contours (geometric flow) of one kind, applying multiquadric (MQ) quasi-interpolation. We first represent the geometric flow in its parametric form. Then we obtain the numerical scheme by using the derivatives of the quasi-interpolation to approximate the spatial derivative of each dependent variable and a forward difference to approximate the temporal derivative of each dependent variable. The resulting scheme is simple, efficient and easy to implement. Also images with complex boundaries can be more easily proposed on the basis of the good properties of the MQ quasi-interpolation. Several biomedical and astronomical examples of applications are shown in the paper. Comparisons with other methods are included to illustrate the validity of the method.

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\textbf{1. Introduction}

Boundary detection is a fundamental problem in image processing and computer vision. It can be formulated by applying variational methods which in turn require solutions to partial differential equations (PDEs). There are two general kinds of active contour models in the literature: parametric active contours [1] based on energy minimization and geometric active contours [2–4] based on curve evolution. Caselles et al. [5] have established the link between the curve evolution based methods and the energy minimization methods (snakes). Here we focus on the numerical scheme of the geometric active contours, though we expect our results to have applications in snakes too.

For simulating the geometric active contours there are two main methods:

(1) The level set methods [6]: The curve evolution equation is implemented by embedding the initial curve as a contour line in a surface and allows the surface to evolve elegantly. Then one can get the final curve from the final surface. The level set methods are able to represent topological changes during the evolution of the curve; thereby they can draw the unconnected images’ boundaries without having a priori knowledge of their topology. However as these methods change a two-dimensional (2D) PDE to a 3D PDE, they require long calculating times.

(2) Curve fitting methods: The set of discrete points is fitted by a continuous curve or surface. The most general approach is to use B-spline interpolation [7]. However linear equations have to be solved every iteration, which affects the speed of curve evolution. Also the scheme is not robust, applying the B-spline interpolation method [8].

One can conclude that the validity of an boundary detection model is related to both the equation and the numerical scheme used to simulate the equation. A lot of work has been done on simulating the boundary detection models [9–12]. The geometric active contours presented in this paper are derived from what Caselles et al. [5] called geodesic active contours. We introduce a scheme in this paper that simulates the geometric active contour models with one kind of MQ quasi-interpolation [13].

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MQ functions were first proposed by Hardy [14] in 1968 for the design of features of aircraft at Boeing Co. and they performed rather well in many calculations including the numerical experiments that were reported in [15]. Micchelli [16] discussed the problem systematically in the theoretical category. Beatson and Powell proposed some quasi-interpolation schemes [17] by using MQ functions. Wu and Schaback [18] improved these schemes and discussed their approximation order and the shape preserving property. Beatson and Dyn [19] developed the theory in a wider category. Ma and Wu [20,21] have even proved the capability and stability of MQ quasi-interpolation when approximating high order derivatives. For applications in numerical solutions of PDEs using the collocation method with MQ functions, readers are referred to Kansa [22] and the symmetric method of Wu [23]. Hon and Wu [24] used MQ quasi-interpolation to solve the developing PDEs. Wu [25] used MQ quasi-interpolation to simulate the solutions of shock wave equations. Chen and Wu [13] constructed a new MQ quasi-interpolation to simulate shock wave equations (Burgers equations) with periodic boundary conditions. Research on the MQ quasi-interpolation has remained a hot topic [26,27]. Beatson used MQ quasi-interpolation instead of the B-spline as a computer aided design tool in the film “The Lord of the Rings III”.

The fundamental advantage of applying the MQ method to simulate the geodesic active contour is that one can evaluate the approximant directly without the need to solve any linear system of equations. Also with the high order and stability of the MQ quasi-interpolation, and the simple representation of MQ quasi-interpolation, more complicated non-linear PDEs can be simulated than with other methods [25]. As a result, the complex boundaries can be drawn more precisely theoretically. One can infer from the experiments that the proposed scheme is valid.

The paper is organized as follows. Section 2 gives a brief introduction to geodesic active contours. In Section 3 we propose the scheme and related properties. Section 4 presents several examples and comparisons with traditional snakes and level set methods. The paper is ended with some conclusions.

2. Geodesic active contours

We present the equations of geometric flow following the discussion of [5]. Let \( \vec{C}(s, 0) \) be an initial curve in \( \mathbb{R}^2 \), \( \sqrt{C}(s, t) \) be a parameterized curve for fixed parameter \( t \) and \( I : [0, a] \times [0, b] \rightarrow \mathbb{R}^+ \) be an underlying gray-level image function. The boundaries of the image are to be detected.

The geodesic active contours can be written as follows:

\[
\frac{\partial \vec{C}}{\partial t} = g(\nabla I(\vec{C}))(\alpha \kappa + \nu_0)\frac{\partial}{\partial s} - (\nabla g \cdot \vec{N})\frac{\partial}{\partial s} \vec{N},
\]

where \( g : [0, +\infty) \rightarrow \mathbb{R}^+ \) is a strictly decreasing function such that \( g(r) \rightarrow 0 \) as \( r \rightarrow \infty \); here we choose the function \( g(\nabla I) = \frac{1}{1 + |\nabla I|^2} \). \( \nabla I \) is the gradient of \( I \), and \( \alpha \) and \( \nu_0 \) are constant. \( \vec{N} \) is the unit normal vector of the curve \( \vec{C} \).

It is crucial in the above model to allow convex initial curves to detect non-convex objects. The ‘forces’ \( \kappa, \nu_0 \) act as the internal forces in the curve evolution, which are called the curvature evolution and constant evolution separately. The curvature evolution discards the curve’s corners and makes the curve smoother; however the constant evolution generates corners. Caselles et al. [2] has talked about the balancing of these two terms to get different effects when proposing different images. The last term attracts the contour to boundaries of high curvature. We will introduce the so called MQ quasi-interpolation method (the MQQI method) to simulate (2.1) in Section 3.

3. The multiquadric quasi-interpolation scheme

One can infer that the curve evolution is similar to a function of the Burgers equation with viscosities. As a result, when proposing images with complex boundaries applying geometric active contours, we face a difficulty similar to that encountered in the simulation of a shock wave PDEs. Because of this, the geometric flow model will go unstable when simulated with the finite difference method directly. Chen and Wu [13] constructed one kind of MQ quasi-interpolation which can simulate the shock wave equations with periodic initial data, where the shape parameter \( c \) plays a viscosity role in the Burgers equation. Ma and Wu [20,21] proved that the MQ quasi-interpolation could provide PDEs with higher derivatives and generated more stability as compared with the finite difference method.

In this section, we will adopt the MQ quasi-interpolation of [13] for the closed curves \( \vec{C}(s, t) \).

For the data \( (x_j, f_j)^m_{j=0}, f_j = f(x_j) \), the univariate quasi-interpolation on \([x_0, x_m]\) is defined by

\[
f^+(x) = \sum_{j=0}^m f_j \psi_j(x),
\]

\[
\psi_j(x) = \frac{\phi_{j+1}(x) - \phi_j(x)}{2(x_{j+1} - x_j)} - \frac{\phi_j(x) - \phi_{j-1}(x)}{2(x_j - x_{j-1})},
\]

\[
\phi_m(x) = \phi_0(x) - 2x + x_m + x_0,
\]

\[
\phi_{-1}(x) = \phi_0(x) + x_0 - x_1,
\]

\[
\phi_{m+1}(x) = \phi_m(x) + x_{m+1} - x_m,
\]

where \( \phi_j(x) \) is a univariate MQ quasi-interpolation, \( \psi_j(x) \) is the kernel function of \( \phi_j(x) \).
and
\[ \phi_j(x) = \sqrt{(x-x_j)^2 + c^2}, \quad 0 \leq j \leq m - 1. \]

Chen and Wu [13] prove the following theorem:

**Theorem 3.1.** On \([x_0, x_m]\), the quasi-interpolation \(f^*(x)\) can be rewritten as follows:
\[ f^*(x) = \frac{f_0 + f_m}{2} + \frac{1}{2} \sum_{k=0}^{m-1} \frac{\phi_k(x) - \phi_{k+1}(x)}{x_{k+1} - x_k} (f_{k+1} - f_k). \]

Moreover, on \([x_0, x_m]\) we have
\[ (f^*(x))' = \frac{1}{2} \sum_{k=0}^{m-1} \frac{\phi'_k(x) - \phi'_{k+1}(x)}{x_{k+1} - x_k} (f_{k+1} - f_k), \]
\[ (f^*(x))'' = \frac{1}{2} \sum_{k=0}^{m-1} \frac{\phi''_k(x) - \phi''_{k+1}(x)}{x_{k+1} - x_k} (f_{k+1} - f_k). \]

According to [13,20], \(f^*(x)\) possesses good approximation order and is able to simulate the Burgers equation with shock waves efficiently when \(c = O(h^{\frac{1}{m+1}})\). Here \(k\) is the order of derivatives and \(h\) is the spatial step length. Because of those properties, we try to apply the MQ quasi-interpolation to simulate Eq. (2.1).

Denoting as \(\vec{V} = (-y(s, t), x(s, t))\), the unit normal vector and the curvature of \(\vec{C} = (x(s, t), y(s, t))\) are then
\[ \vec{N} = \frac{\vec{V}}{||\vec{V}||}, \quad \kappa = \frac{\vec{C}_{ss} \cdot \vec{V}_s}{||\vec{C}_s||^2}, \]
and \(\vec{C}_s, \vec{C}_{ss}\) are the first-order and second-order derivatives of \(\vec{C}\) with respect to \(s\); \(\vec{V}_s\) is the first-order derivative of \(\vec{V}\) with respect to \(s\).

Then we get the algorithm for boundary detection applying MQ quasi-interpolation (the MQQI method):

**Algorithm 3.1.** 1. Discretizing the equation in time, we get
\[ \vec{C}^{n+1}_j = \vec{C}^n_j + \tau g(\nabla (\vec{C}^n_s)) (\alpha \kappa^n + v_0) (\vec{N})^n. \tag{3.1} \]
2. Using the derivatives of the MQ quasi-interpolation to approximate \(x_s, y_s, x_{ss}, y_{ss}\), we get
\[ (\vec{C}^n_s) = \frac{1}{2} \sum_{k=0}^{m-1} \frac{\phi'_k(s_j) - \phi'_{k+1}(s_j)}{s_{k+1} - s_k} (\vec{C}^n_{k+1} - \vec{C}^n_k), \tag{3.2} \]
\[ (\vec{C}^n_{ss}) = \frac{1}{2} \sum_{k=0}^{m-1} \frac{\phi''_k(s_j) - \phi''_{k+1}(s_j)}{s_{k+1} - s_k} (\vec{C}^n_{k+1} - \vec{C}^n_k), \tag{3.3} \]
where \(\phi_j(s) (j = 0, \ldots, m)\) are defined as above. \(\vec{C}^n_s, \kappa^n, (\vec{N})^n\) are the approximations of the values of \(\vec{C}^n_s, \kappa^n, \vec{N}^n\) at points \((s_j, t_n)\), \(t_n = n\tau\), where \(\tau\) is the step length of time.

The main idea of the proposed scheme is discretizing the equation in time and approximating the derivatives applying MQ quasi-interpolation. One can conclude from the formulas that when simulating derivatives of different functions, only the coefficients change. Compared to the MQQI method the level set methods apply finite difference to simulate the derivatives. When higher accuracy should be achieved, the meshes should be smaller and smaller, which results in long calculating time. Also, when simulating higher order derivatives, more meshes are involved, which results in low accuracy and instability. The proposed scheme escapes these problems. The advantages of the proposed scheme are its simplicity, stability and capability of simulating more complicated problems.

4. Experiments, comparisons and analysis

Experiments with both synthetic images and real images obtained from satellite and medical sources are utilized to demonstrate the performance of the MQQI method. The following figures show the performance the level set method and the MQQI method when simulating the geodesic active contours.

Table 1 shows the time consumption and iterations of the level set method and the MQQI method. We choose \(\tau = 0.001, h = \text{spatial length}/100\) and \(\alpha = 1\) in all the following experiments. One can infer from the last column of Table 1 that when proposing high curvature or complex images, the value of the parameter \(v_0\) is larger, which is consistent with the theoretical analysis in Section 2.
Table 1
Time consumption of the level set method and the MQQI method.

<table>
<thead>
<tr>
<th>Time consuming</th>
<th>Level set method (s)</th>
<th>Iterations</th>
<th>MQQI method (s)</th>
<th>Iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fig. 1 (140 × 139)</td>
<td>177.824617</td>
<td>5000</td>
<td>22.475278</td>
<td>201</td>
</tr>
<tr>
<td>Fig. 2 (150 × 152)</td>
<td>88.072285</td>
<td>2500</td>
<td>14.983319</td>
<td>41</td>
</tr>
<tr>
<td>Fig. 3 (179 × 179)</td>
<td>17.787556</td>
<td>3000</td>
<td>2.0</td>
<td></td>
</tr>
<tr>
<td>Fig. 4 (319 × 276)</td>
<td>549.988195</td>
<td>3000</td>
<td>30.205674</td>
<td>201</td>
</tr>
</tbody>
</table>

4.1. Synthetic images

Fig. 1 shows a 64 × 64-pixel line-drawing of a U-shaped object (shown in gray) with a boundary concavity at the top. Xu and Prince [28] show that when applying normal methods the final solution splits across the concave region and provides the reason for the poor convergence. The MQQI method can approximate the non-convex boundary after 201 steps, which costs only \( \frac{1}{25} \) iterations of the level set method.
4.2. Medical images

A coronal slice from a head scan was used to demonstrate the capabilities of different methods in extracting a real object with complex boundaries: the brain contour. Fig. 2 illustrates the results of applying different algorithms to a representative head scan. The process of extraction of the boundary of the brain was nontrivial as the mask had both convex and concave parts. The proposed MQQI method obtains more details of the deep sulci in head scans and costs only 2% iterations compared with the level set method.

A prostate ultrasound image is shown in Fig. 3. The MQQI scheme gets to the boundary after 201 steps and the resulting contour is more similar to the doctor’s delineation [29]. This figure is provided to show the ability of the proposed method to draw the boundaries of images with noisy background. The results show its higher accuracy and greater robustness in comparison with the finite difference method.

4.3. Astronomical images

Fig. 4 shows the performance of the proposed method on a moon with fuzzy and concave boundaries on the left side. The results show the capability of the MQQI method when dealing with real images with fuzzy and complex boundaries.
The level set method can get into the local minimization easily and evolves to the concave and fuzzy boundaries slowly. The MQQI scheme can locate the true boundary after far less time.

5. Conclusion

We have constructed a new algorithm for dealing with deformable contours based on MQ quasi-interpolation. Compared to other methods, the MQQI method inherits the unique property that it can simulate high derivatives of a function efficiently [20] and be more stable. Equations are solved in 2D spaces and there is no need to solve any large scale system of linear equations. As a result, wider scopes of contours can be simulated. From the figures and tables above, we conclude that the MQQI method has more advantages over the level set method when proposing connected images. In addition to this, far less time (about $\frac{1}{6}$) and far fewer iterations are needed than for the level set method. The proposed method is feasible, valid and easy to implement. The MQQI method can also be applied to other active contour models such as 3D image segmentation and fast video segmentation, which we will do in the future.

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