A Preconditioned Multigrid Method for Efficient Simulation of Three-dimensional Compressible and Incompressible Flows

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Abstract

To develop an efficient and robust aerodynamic analysis method for numerical optimization designs of wing and complex configuration, a combination of matrix preconditioning and multigrid method is presented and investigated. The time derivatives of three-dimensional Navier-Stokes equations are preconditioned by Choi-Merkle preconditioning matrix that is originally designed for two-dimensional low Mach number viscous flows. An extension to three-dimensional viscous flow is implemented, and a method improving the convergence for transonic flow is proposed. The space discretization is performed by employing a finite-volume cell-centered scheme and using a central difference. The time marching is based on an explicit Runge-Kutta scheme proposed by Jameson. An efficient FAS multigrid method is used to accelerate the convergence to steady-state solutions. Viscous flows over ONERA M6 wing and M100 wing are numerically simulated with Mach numbers ranging from 0.010 to 0.839. The inviscid flow over the DLR-F4 wing-body configuration is also calculated to preliminarily examine the performance of the presented method for complex configuration. The computed results are compared with the experimental data and good agreement is achieved. It is shown that the presented method is efficient and robust for both compressible and incompressible flows and is very attractive for aerodynamic optimization designs of wing and complex configuration.

Keywords: Navier-Stokes equations; preconditioning method; multigrid method; numerical simulation

1 Introduction

Over the past two decades, the time-marching algorithms have been widely used for compressible flow simulation solving Euler and Navier-Stokes (N-S) equations. However, it is proved by practice that these “standard” numerical schemes for the compressible equations are not applicable for incompressible flow, and do not converge to the solution of the incompressible equations as the Mach number approaches zero[1].

To overcome this difficulty, several methods were developed for solving nearly incompressible flow problems. Among them, the pseudo-compres-
sibility[2] method and preconditioning method[1,3-6] may be the most popularly used approaches. Due to the assumption of incompressibility, the pseudo-compressibility can only be used for low-speed flow simulation, whereas the preconditioning method is appropriate for both compressible and incompressible flows. The development of the preconditioning method is motivated by two main considerations. First, the actual flow can contain both compressible and incompressible flows simultaneously. Second, it is preferable to develop such a method that is suit for flows at all flow regimes.
Several inviscid and viscous preconditioning methods were available in the past few years, such as Choi-Merkle’s[3], Turkle’s[1], Van Leer’s[4] and Allmaras’s[5] methods, etc. The viscous preconditioning method developed by Choi and Merkle has been proved to be efficient for two-dimensional low Mach number flows. However, for compressible flows, this method seems to have little effect on accelerating the convergence.

In present work, Choi-Merkle preconditioning method is extended to solve three-dimensional flows. An improvement is proposed for accelerating the calculation of three-dimensional transonic flows. Furthermore, the incorporation of matrix preconditioning with an FAS multigrid method is implemented, which results in a very efficient approach for the simulations of both compressible and incompressible flows. The developed method is expected to use as an efficient and robust method for aerodynamic optimization designs of wing and wing-body configuration.

2 Computation Method

2.1 Governing equation

After introducing the matrix preconditioning, the non-dimensional form of the three-dimensional compressible N-S equations can be written as

\[
p\frac{\partial W}{\partial t} + \frac{\partial \left(E - E_{0}\right)}{\partial x} + \frac{\partial \left(F - F_{0}\right)}{\partial y} + \frac{\partial \left(G - G_{0}\right)}{\partial z} = 0
\]

where \(W\) is the preconditioning matrix (or preconditioner) and will take various forms depending on different choices. When \(P\) is an identity matrix, Eq.(1) recovers to the standard (non-preconditioned) form. The additional vectors in Eq.(1) are

\[
W = \begin{pmatrix} \rho & \rho u & \rho v & \rho w & \rho E \end{pmatrix}^T
\]

\[
E = \begin{pmatrix} \rho u \rho u^2 + p \rho u v \rho w \rho Hu \end{pmatrix}^T
\]

\[
F = \begin{pmatrix} \rho v \rho u v^2 + p \rho v w \rho Hv \end{pmatrix}^T
\]

\[
G = \begin{pmatrix} \rho w \rho w u \rho w v + p \rho w \rho Hw \end{pmatrix}^T
\]

\[
E_v = \begin{pmatrix} 0 & \tau_{xx} & \tau_{xy} & \tau_{xz} & \beta_x \end{pmatrix}^T
\]

\[
F_v = \begin{pmatrix} 0 & \tau_{yx} & \tau_{yy} & \tau_{yz} & \beta_y \end{pmatrix}^T
\]

\[
G_v = \begin{pmatrix} 0 & \tau_{zx} & \tau_{zy} & \tau_{zz} & \beta_z \end{pmatrix}^T
\]

where \(\rho\) denotes the density, \(u, v\) and \(w\) denote the components of velocity vector, and \(E\) denotes the total energy per unit mass. Pressure and temperature are given by the equations of state for perfect gas:

\[
P = \rho(\gamma - 1)(E - 0.5(u^2 + v^2 + w^2))
\]

\[
T = \frac{P}{\rho}
\]

where \(\gamma\) is the ratio of specific heat and is taken as 1.4 for air. The viscous shear stresses and the heat fluxes are of the form:

\[
\tau_{xx} = 2\mu u_x + \lambda (u_x + v_y + w_z)
\]

\[
\tau_{yx} = 2\mu v_y + \lambda (u_x + v_y + w_z)
\]

\[
\tau_{zx} = 2\mu w_z + \lambda (u_x + v_y + w_z)
\]

\[
\tau_{yy} = \tau_{yx} = \mu (u_x + v_y)
\]

\[
\tau_{zz} = \tau_{yx} = \mu (v_y + w_z)
\]

\[
\beta_x = u\tau_{xx} + v\tau_{xy} + w\tau_{xz} + kT
\]

\[
\beta_y = u\tau_{yx} + v\tau_{yy} + w\tau_{yz} + kT
\]

\[
\beta_z = u\tau_{zx} + v\tau_{zy} + w\tau_{zz} + kT
\]

where \(k\) is the coefficient of thermal conductivity and is determined by using the assumption of constant Prandtl number. The bulk viscosity \(\lambda\) is taken to be \(-2\mu/3\) according to Stokes’s hypothesis. For turbulent flows, the total viscosity \(\mu\) is calculated as

\[
\mu = \mu_1 + \mu_2
\]

where \(\mu_1\) is the molecular viscosity calculated by Sutherland law, and \(\mu_2\) is the eddy viscosity determined by turbulence model. Then, Eq.(1) can be called preconditioned Reynolds averaged N-S equation. For the aerodynamic design of wing, the attached flows with small separation region are frequently considered. In considerations of simplicity and efficiency, the Baldwin-Lomax algebraic turbulence model is used for all the calculations of the present work.

Rather than using the conservative variables

\[
W = (\rho \rho u \rho v \rho w \rho E)^T
\]

the primitive variables

\[
Q = (p u v w T)^T
\]

are frequently used for viscous flows, especially for low Mach number flows. Then Eq.(1) can be written as

\[
\Gamma \frac{\partial Q}{\partial t} + \frac{\partial \left(E - E_{0}\right)}{\partial x} + \frac{\partial \left(F - F_{0}\right)}{\partial y} + \frac{\partial \left(G - G_{0}\right)}{\partial z} = 0
\]

where \(\Gamma\) is the time step.
where $\Gamma = P \partial W \partial Q$ represents the preconditioning matrix for primitive variables. Thus, the flux term keeps the form of conservation law, and the shock wave in transonic flow can be correctly captured.

Although the above governing equations are described in Cartesian coordinates, all computations are conducted in generalized coordinates to allow the treatment of arbitrary geometries.

2.2 Preconditioning method and improvement

The preconditioning method proposed by Choi and Merkle[3] is directly extended to three-dimensional case. The preconditioning matrix takes the form

$$
\Gamma^{-1} = \begin{bmatrix}
\frac{1}{c^2 Ma^2_c} & 0 & 0 & 0 \\
\frac{u}{c^2 Ma^2_c} & \rho & 0 & 0 \\
\frac{v}{c^2 Ma^2_c} & 0 & \rho & 0 \\
\frac{w}{c^2 Ma^2_c} & 0 & 0 & \rho \\
H & \frac{\delta \rho \mu \rho v \rho w \gamma p}{c^2 Ma^2_c - 1}
\end{bmatrix}
$$

(7)

where $c$ is the sound speed and $\delta$ equals to 0 or 1 ($\delta = 1$ in this paper). To demonstrate the characteristics of the preconditioned system in curvilinear coordinates, a condition number (CN) proposed by Turkel[1] is defined as

$$
CN = \frac{\max \lambda_i}{\min \lambda_i}, \quad i = 1, \cdots, 5
$$

(8)

where $\lambda_i$ denotes the eigenvalue. The eigenvalues of the preconditioned system in $\xi$ direction for curvilinear coordinates are

$$
\lambda_{1,2,3} = q_\xi, \lambda_{4,5} = \frac{q_\xi (1 + \beta^2 Ma^2_c / c^2) \pm c'}{2}
$$

and

$$
q_\xi = q_{\xi \xi} + q_{\xi \eta} + q_{\xi \zeta} + Q_{\xi \xi} = \frac{q_{\xi \xi}}{c^2} - \frac{q_{\xi}}{c^2}
$$

(9)

When $Ma \to 0$, $\lambda_{4,5} \approx 0.5 q_{\xi} (1 \pm \sqrt{5})$ and CN = $|1 + \sqrt{5}| / |1 - \sqrt{5}| \approx 2.6$. For non-preconditioned system, CN is infinite when $Ma \to 0$. It is clearly shown that the eigenvalue stiffness for low Mach number flow or flow region is effectively eliminated.

To avoid the singular of Choi-Merkle preconditioner near the stagnation points, a technique was proposed by Choi and Merkle. That is,

$$
Ma^2_0 = \begin{cases}
\varepsilon^2, & \text{if } Ma < \varepsilon \\
Ma^2_c, & \text{if } \varepsilon < Ma < 1 \\
1, & \text{if } Ma > 1
\end{cases}
$$

(10)

where the value of $Ma_c$ is controlled by the flow conditions. In Choi and Merkle’s study, the typical value of $\varepsilon$ is taken as $10^{-5}$.

According to the authors’ experience, when local Mach number $Ma > 0.6$, the disparities of different eigenvalues can be reasonably neglected, hence, the effect of preconditioning can be gradually removed. For transonic flow over wing or wing-body configuration, the local Mach numbers of most flow regions are larger than 0.600, whereas a part of flow regions (such as the flow in the boundary layer or near the stagnation point) is incompressible. It is preferable to switch on the preconditioning in these low-speed regions and smoothly reduce the effects of preconditioning outside these regions. Base on the above viewpoint, an alternative method is proposed. It takes the form

$$
Ma^2_0 = \min \{ \max [K_1 Ma^2_c (1 + \frac{1 - Ma^2_c}{Ma^2_0} - Ma^2_c), K_2 Ma^2_c], 1] \}
$$

(11)

where $K_1$ and $K_2$ are free parameters. Typically $K_1$ equals to 1.0-1.1, and $K_2$ equals to 0.5-1.0. $K_2$ is adjusted according to the difficulty of convergence. For some flow conditions which are extremely difficult for convergence, $K_2$ is taken as large as 3.0 or 4.0. However, from Eq.(11), one can find out that large value of $K_2$ will reduce the effect of preconditioning. For $K_2$ approach infinite, the computation will recover none preconditioning state. From Eq.(11), it is also shown that $Ma_c$ return to the non-preconditioned value “1” when the local Mach number $Ma \geq Ma_0$. Therefore, preconditioning will be removed automatically when local Mach number is larger than $Ma_0$. In this study, $K_1 = K_2 = 1.0$ and $Ma_0 = 0.600$ are used for all computations.
2.3 Multigrid method

The governing equations are solved based on a finite volume method \cite{6,7} developed by Jameson within the framework of an efficient FAS multigrid method. The spatial discretization is a cell-centered finite volume method using central difference, with nearly second-order accuracy on smooth structured grid. A blend of second and fourth order artificial dissipations based on preconditioned system is added to avoid the spurious oscillations and stabilize the scheme. An explicit multi-stage Runge-Kutta method is used for time stepping.

Runge-Kutta method has good damping characteristics for high wave number error. Effective removal of low wave number error is accomplished using a multigrid scheme originally developed by Jameson\cite{8}. A coarser grid is created by removing every other grid line from the finer grid, essentially doubling the grid spacing in each direction. The multigrid cycle begins by iterating a fixed number of full Runge-Kutta cycles. Then the solution is restricted to the coarse grid using a restrict operator. The residual of fine grid is also restricted for calculating a forcing function that drives the solution on coarse grid. The forcing function is defined as the difference between the restricted fine-grid residuals and the coarse-grid residuals of the first step of Runge-Kutta cycle. The procedure is then repeated recursively. Once the coarsest grid is reached, an interpolation operator accomplishes the correction procedure. The restriction and interpolation operators of the multigrid cycle are both based on volume-weighted averaging. Computational experience suggests that the performance of this multigrid algorithm highly depends on the number of iterations on each level and the type of the multigrid cycle (such as “V” cycle or “W” cycle etc.). In the present work, a 3-level V-cycle multigrid method is implemented as a simple but extremely effective method for accelerating the convergence of the Runge-Kutta time stepping. At each multigrid cycle, the numbers of iteration for each level are 5, 10 and 15, respectively. It is shown by the authors’ practice that the “5-10-15 V” cycle provides an efficient and robust way for a wide variety of flow simulations.

3 Numerical Examples

Three-dimensional compressible and incompressible flows over ONERA M6 wing, M100 wing and DLR F4 wing-body configuration are simulated as numerical examples to check the accuracy and efficiency of the proposed method. All computation use structured grid, B-L turbulence model and a FAS multigrid scheme of “5-10-15V” cycle.

3.1 Low-speed and transonic flow over ONERA M6 wing

Incompressible and transonic viscous flows over ONERA M6 wing are employed as the first test case. The Reynolds number based on mean aerodynamic chord is $11.7 \times 10^6$, and the angle of attack $\alpha$ is $3.060^\circ$. A C-H type grid (see Fig.1) consisting of $209 \times 49 \times 49$ points is used for these computations.

![Schematics of C-H type grid for viscous flow over M6 wing.](image)

The effects of preconditioning on convergence rate for incompressible flows are indicated in Fig.2. Without preconditioning, the residual divergent for $Ma=0.010$ and stalls after dropping 3.5 orders for $Ma=0.100$. When the preconditioning is switched on, good convergences are found for both $Ma=0.010$ and $Ma=0.100$. Fig.3 gives the comparison of pressure distributions with and without preconditioning at 80% span location. It is shown that the convergence and accuracy for incompressible three-dimensional flows over wing are markedly improved by using the preconditioning method and multigrid scheme.
To demonstrate the effects of the combination of preconditioning method and multigrid method, the comparison of convergence histories with and without preconditioning for single grid calculation is demonstrated in Fig.4. It is shown that the preconditioning has little effect on accelerating the computation for single grid calculation. Extensive computation practices show that the combination of preconditioning and multigrid method is a key point of this paper to obtain a highly efficient and robust method for viscous flow simulations.

Figs.5-8 show the case of transonic flow simulation ($Ma=0.839$). This is a well-known benchmark for transonic flow over wing. Fig.5 indicates the comparison of the convergence histories with and without preconditioning. The convergence rate is significantly improved by using the improved Choi-Merkle preconditioning method. Figs.6-8 demonstrate the comparisons of pressure distributions at three typical stations along the wingspan. The results computed by preconditioning are in good agreement with the experimental data and those of the non-preconditioning, which validate the developed method.
3.2 Subsonic flow over M100 wing

The second case considered here is the flow over M100 wing, which is typical transport wing. The present computations are performed at a subsonic case with $Ma=0.600$, $\alpha=1.733^\circ$ and $Re=3.2\times10^6$ based on the mean aerodynamic chord. In computation a C-H type grid consisting of $209 \times 49 \times 73$ points is used (see Fig.9).

The convergence histories for this case are presented in Fig.10. The results from the original non-preconditioned and current preconditioned schemes are compared. The residual in the non-preconditioned scheme indicates slowdown in convergence at approximately 4-orders, whereas the residual in current scheme shows much better convergence. Note that although the Choi-Merkle preconditioning matrix is designed for low Mach number flow, it can be also used to accelerate the calculation of the 3-D subsonic flow through proper modification.

The computed pressure distributions at 81.7% span location are compared in Fig.11. As expected, little difference is observed for the results from non-preconditioning and preconditioning method. Furthermore, these computed results show excellent agreement with the experimental data.

![Fig.7 Comparison of pressure distributions at 80.0% span location.](image7)

![Fig.8 Comparison of pressure distributions at 95.0% span location.](image8)

![Fig.9 Schematics of C-H type grid for viscous flow over M100 wing at Ma=0.600.](image9)

![Fig.10 Comparison of convergence histories for subsonic flow simulation.](image10)

![Fig.11 Comparison of pressure distributions at 81.7% span location.](image11)
3.3 Low-speed flow over DLR F4 wing-body configuration

The final test case presented here involves inviscid low-speed flow over DLR F4 wing-body configuration, which is also a benchmark for the validation of CFD codes. The computation is performed at a Mach number of 0.300 and an angle of attack of −1.200°. The computation uses a C-H type single-block grid that consists of 269×43×105 points (see Fig.12).

![Fig.12 Schematics of C-H type grid for viscous flow over DLR F4 wing-body configuration.](image)

The convergence histories for this case are shown in Fig.13. The results from the original non-preconditioned and current preconditioned schemes are compared. The residual in the non-preconditioned scheme indicates slowdown in convergence at approximately 3-orders, whereas the residual in current scheme shows much better convergence. For preconditioning computation, the residual drops 5.0 orders within 90 multigrid cycles, which shows that the convergence rate for simulation of 3-D low-speed flow over complex aerodynamic geometry is markedly improved.

4 Conclusions

An effective and robust method for simulation of flows over wing and wing-body configuration are presented in the framework of preconditioning applied to compressible flow equations. The compressible and incompressible flows over ONERA M6 Wing, M100 wing and DLR F4 wing-body configuration are numerically simulated and excellent convergence are obtained at Mach numbers ranging from 0.010 to 0.839. The resulting pressure distributions agree well with the experimental data or known solutions for these cases. The presented method offers the advantage of being able to computing both compressible and incompressible flows. In addition, it can improve the convergence rate for three-dimensional viscous transonic flows by preconditioning the embedded low-speed flows in the boundary layers. The developed method is expected to use as an efficient and robust aerodynamic analysis method for numerical optimization design of wing and wing-body configuration.

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