A Taylor matrix method for the solution of a two-dimensional linear hyperbolic equation

Berna Bülbül, Mehmet Sezer
Department of Mathematics, Faculty of Science, Mugla University, Mugla, Turkey

ARTICLE INFO
Article history:
Received 14 December 2010
Received in revised form 19 April 2011
Accepted 20 April 2011
Keywords:
Taylor polynomial solutions
Two-dimensional linear hyperbolic equation
Taylor matrix method

ABSTRACT
A Taylor matrix method is proposed for the numerical solution of the two-space-dimensional linear hyperbolic equation. This method transforms the equation into a matrix equation and the unknown of this equation is a Taylor coefficients matrix. Solutions are easily acquired by using this matrix equation, which corresponds to a system of linear algebraic equations. As a result, the finite Taylor series approach with three variables is obtained. The accuracy of the proposed method is demonstrated with one example.

© 2011 Elsevier Ltd. All rights reserved.

1. Introduction

In this work, we consider the two-dimensional linear hyperbolic equation

\[ \frac{\partial^2 u}{\partial t^2}(x, y, t) + 2\alpha \frac{\partial u}{\partial t}(x, y, t) + \beta^2 u(x, y, t) + A \frac{\partial^2 u}{\partial x^2}(x, y, t) + B \frac{\partial^2 u}{\partial y^2}(x, y, t) = f(x, y, t), \]

\[(x, y, t) \in [0, 1] \times [0, 1] \times (0, T), \]

with the initial conditions

\[ u(x, y, 0) = \psi_1(x, y), \quad \frac{\partial u}{\partial t}(x, y, 0) = \psi_2(x, y), \]

and the boundary conditions

\[ u(x, 0, t) = f_0(x, t), \quad u(x, 1, t) = f_1(x, t), \quad t \geq 0 \]
\[ u(0, y, t) = g_0(y, t), \quad u(1, y, t) = g_1(y, t), \quad t \geq 0 \]

and the solution is expressed as the Taylor polynomial

\[ u(x, y, t) = \sum_{p=0}^{N} \sum_{r=0}^{N} \sum_{s=0}^{N} a_{p, r, s} x^{p} y^{r} t^{s}, \quad a_{p, r, s} = \frac{1}{p! r! s!} u^{(p,r,s)}(0, 0, 0) \]

so the Taylor coefficients to be determined are \( a_{p, r, s} \) \( (p, r, s = 0, \ldots, N) \), where \( f, \psi_1, \psi_2, f_0, f_1, g_0, g_1 \) are known functions.

Partial differential equations arise in connection with various physical and geometrical problems in which the functions involved depend on two or more independent variables, on time \( t \) and on one or several space variables \([1]\). For \( \alpha > 0, \beta > 0 \) and \( A = B = -1 \), Eq. (1) represents a two-space-dimensional damped wave equation (with a source term). The numerical solution of damped wave equations is of great importance in studying wave phenomena.
2. Fundamental relations

To obtain the numerical solution of the hyperbolic partial differential equation with the method presented, we first evaluate the Taylor coefficients of the unknown function. For convenience, the solution function (4) can be written in the matrix form

\[ u(x, y, t) = \mathbf{X}(x, y, t) \mathbf{A} \]  

where \( \mathbf{X}(x, y, t) \) is the \( 1 \times (N + 1)^3 \) matrix defined as

\[
\mathbf{X}(x, y, t) = \begin{bmatrix}
\mathbf{X}_0(x, y, t) & \mathbf{X}_1(x, y, t) & \ldots & \mathbf{X}_N(x, y, t)
\end{bmatrix}_{1 \times (N+1)^3},
\]

\[
\mathbf{X}(x, y) = \begin{bmatrix}
\mathbf{X}_0(x, y) & \mathbf{X}_1(x, y) & \ldots & \mathbf{X}_N(x, y)
\end{bmatrix}_{1 \times (N+1)^2},
\]

\[
\mathbf{X}(x) = \begin{bmatrix}
1 & x & x^2 & \ldots & x^N
\end{bmatrix}
\]

and

\[
\mathbf{A} = \begin{bmatrix}
\mathbf{A}_0 & \mathbf{A}_1 & \ldots & \mathbf{A}_N
\end{bmatrix}^T
\]

or in detail

\[
\mathbf{A}_i = \begin{bmatrix}
a_{0,0,i} & a_{1,0,i} & \ldots & a_{N,0,i} & a_{0,1,i} & a_{1,1,i} & \ldots & a_{N,1,i} & \ldots & a_{0,N,i} & a_{1,N,i} & \ldots & a_{N,N,i}
\end{bmatrix}^T.
\]

On the other hand, the relations between the matrix \( \mathbf{X}(x, y, t) \) and its derivatives are as follows:

\[
u(x, y, t)^{(n,0,0)} = \mathbf{X}^{(n,0,0)}(x, y, t) \mathbf{A} = \mathbf{X}(x, y, t)(\mathbf{B}^n) \mathbf{A}, \quad n = 0, 1, 2, \ldots
\]

\[
u(x, y, t)^{(0,m,0)} = \mathbf{X}^{(0,m,0)}(x, y, t) \mathbf{A} = \mathbf{X}(x, y, t)(\mathbf{M}^m) \mathbf{A}, \quad m = 0, 1, 2, \ldots
\]

\[
u(x, y, t)^{(0,0,s)} = \mathbf{X}^{(0,0,s)}(x, y, t) \mathbf{A} = \mathbf{X}(x, y, t)(\mathbf{S}^s) \mathbf{A}, \quad s = 0, 1, 2, \ldots
\]

where

\[
\mathbf{B} = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & N \\
0 & 0 & 0 & \cdots & 0
\end{bmatrix}, \quad \tilde{\mathbf{B}} = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
0 & \mathbf{B} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \mathbf{B} \\
0 & 0 & \cdots & \mathbf{B}
\end{bmatrix},
\]

\[
\mathbf{M} = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 2I & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & N \mathbf{I} \\
0 & 0 & 0 & \cdots & 0
\end{bmatrix}, \quad \tilde{\mathbf{M}} = \begin{bmatrix}
\mathbf{M} & 0 & \cdots & 0 \\
0 & \mathbf{M} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \mathbf{M} \\
0 & 0 & \cdots & N \mathbf{I}
\end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix}
0 & \tilde{\mathbf{I}} & 0 & \cdots & 0 \\
0 & 0 & 2\tilde{\mathbf{I}} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & N \tilde{\mathbf{I}} \\
0 & 0 & 0 & \cdots & 0
\end{bmatrix}
\]

where \( \mathbf{I} \) and \( \tilde{\mathbf{I}} \) are \((N + 1) \times (N + 1)\) and \((N + 1)^2 \times (N + 1)^2\) identity matrices respectively.

By using the relations (6)–(8) we have

\[
u^{(m,n,s)}(x, y, t) = \mathbf{X}^{(m,n,s)}(x, y, t) \mathbf{A} = \mathbf{X}(x, y, t)(\mathbf{B}^m)(\mathbf{M}^n)(\mathbf{S}^s) \mathbf{A}, \quad m, n, s = 0, 1, 2, \ldots
\]

We can also expand the function \( f(x, y, t) \) as a Taylor series:

\[
f(x, y, t) = \sum_{p=0}^{N} \sum_{r=0}^{N} \sum_{s=0}^{N} f_{p, r, s} x^p y^r t^s, \quad f_{p, r, s} = \frac{1}{p! r! s!} f^{(p, r, s)}(0, 0, 0)
\]

or from (9) get the matrix form

\[
f(x, y, t) = \mathbf{X}(x, y, t) \mathbf{F}
\]

where

\[
\mathbf{F} = \begin{bmatrix}
\mathbf{F}_0 & \mathbf{F}_1 & \cdots & \mathbf{F}_N
\end{bmatrix}^T
\]

where

\[
\mathbf{F}_i = \begin{bmatrix}
f_{0,0,i} & f_{1,0,i} & \cdots & f_{N,0,i} & f_{0,1,i} & f_{1,1,i} & \cdots & f_{N,1,i} & \cdots & f_{0,N,i} & f_{1,N,i} & \cdots & f_{N,N,i}
\end{bmatrix}^T.
\]

Substituting the expressions (5)–(18) and (10) into Eq. (1) and simplifying the result, we have the matrix equation

\[
\{(\mathbf{S})^2 + 2\alpha \mathbf{S} + \beta^2 \mathbf{I} + A(\mathbf{B})^2 + B(\mathbf{M})^2\} \mathbf{A} = \mathbf{F}.
\]
Briefly, we can write Eq. (11) in the form

\[
WA = G
\]

(12)

where \( \mathbf{I} \) is the \((N + 1)^3 \times (N + 1)^3\) identity matrix and \( \mathbf{W} = [w_{ij}], \ i, j = 1, \ldots, (N + 1)^3\).

We now present the alternative forms for \(u(x, y, t)\) which are important for simplifying matrix forms of the conditions. The simplification in conditions is done only with respect to the variables \(x, y\) and \(t\). Therefore we must use different forms for initial and boundary conditions. For the initial conditions (2),

\[
u(x, y, t) = X(x, y)Q(t)
\]

(13)

while for the boundary conditions (3),

\[
u(x, y, t) = \bar{X}(x, t)H^*(y)
\]

(14)

where

\[
X(x, t) = \begin{bmatrix} X_0(x, t) & \bar{X}_1(x, t) & \cdots & \bar{X}_N(x, t) \end{bmatrix}_{1 \times (N+1)^2}; \quad \bar{X}_i(x, y) = t\bar{X}(x),
\]

\[
H(y) = \begin{bmatrix} H(y) & 0 & \cdots & 0 \\
0 & H(y) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & H(y) \end{bmatrix}_{(N+1)^2 \times (N+1)^2},
\]

\[
H^*(y) = \begin{bmatrix} H(y) & 0 & \cdots & 0 \\
0 & H(y) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & H(y) \end{bmatrix}_{(N+1)^2 \times (N+1)^2},
\]

and also we use

\[
u(x, y, t) = X(y, t)L(x)
\]

(15)

where

\[
X(y) = \begin{bmatrix} X_0(y, t) & \bar{X}_1(y, t) & \cdots & \bar{X}_N(y, t) \end{bmatrix}_{1 \times (N+1)^2}; \quad \bar{X}_i(y, t) = t\bar{Y}(y)
\]

\[
Y(y) = \begin{bmatrix} 1 & y & y^2 & \cdots & y^N \end{bmatrix}
\]

\[
L(x) = \begin{bmatrix} X(x) & 0 & \cdots & 0 \\
0 & X(x) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & X(x) \end{bmatrix}_{(N+1)^2 \times (N+1)^2}
\]

Notice also that the matrices involved in the right-hand side of Eqs. (2) and (3) are of the form

\[
\psi_1(x, y) = X(x, y)\psi_1, \quad \psi_1 = [\tau_{0,0,0}, \tau_{0,1,0}, \cdots \tau_{0,N,1}, \cdots \tau_{N,0,1}, \cdots \tau_{N,N,1}]^T,
\]

(16)

\[
\tau_{m,n} = \frac{\psi_1^{(m,n)}(x, y)}{m!n!}
\]

\[
\psi_2(x, y) = X(x, y)\psi_2, \quad \psi_2 = [\eta_{0,0}, \eta_{0,1}, \cdots \eta_{0,N}, \eta_{1,0}, \cdots \eta_{N,0}, \eta_{1,N}, \cdots \eta_{N,N}]^T,
\]

(17)

\[
\eta_{m,n} = \frac{\psi_2^{(m,n)}(x, y)}{m!n!}
\]

\[
f_0(x, t) = X(x, t)F_0, \quad F_0 = [\mu_{0,0}, \mu_{0,1}, \cdots \mu_{0,N}, \mu_{1,0}, \cdots \mu_{1,N}, \cdots \mu_{N,N}]^T,
\]

(18)

\[
\mu_{m,n} = \frac{f_0^{(m,n)}(x, t)}{m!n!}
\]

\[
f_1(x, t) = X(x, t)F_1, \quad F_1 = [\xi_{0,0}, \xi_{0,1}, \cdots \xi_{0,N}, \xi_{1,0}, \cdots \xi_{1,N}, \cdots \xi_{N,N}]^T,
\]

(19)

\[
\xi_{m,n} = \frac{f_1^{(m,n)}(x, t)}{m!n!}
\]

\[
g_0(y, t) = X(y, t)G_0, \quad G_0 = [\zeta_{0,0}, \zeta_{0,1}, \cdots \zeta_{0,N}, \zeta_{1,0}, \cdots \zeta_{1,N}, \cdots \zeta_{N,N}]^T,
\]

(20)

\[
\zeta_{m,n} = \frac{g_0^{(m,n)}(y, t)}{m!n!}
\]

\[
g_1(y, t) = X(y, t)G_1, \quad G_1 = [\sigma_{0,0}, \sigma_{0,1}, \cdots \sigma_{0,N}, \sigma_{1,0}, \cdots \sigma_{1,N}, \cdots \sigma_{N,N}]^T.
\]

(21)
Consider the problem of finding a solution for a differential equation. We illustrate our work with the following example. The numerical computations were done using MAPLE 9.

4. A numerical example

We can easily check the accuracy of the method. Since the truncated Taylor series (4) is an approximate solution of Eq. (1), when the function \( u_{n,N,N}(x, y, t) \) and its derivatives are substituted in Eq. (1), the resulting equation must be satisfied approximately; that is, for

\[
-x = x_p, \quad y = y_q, \quad t = t_r \in [0, 1] \times [0, 1] \times (0, T), \quad p, q, r = 0, 1, 2, \ldots
\]

\[
E_{N,N,N}(x_p, y_q, t_r) = \left| \frac{\partial^2 u}{\partial t^2}(x_p, y_q, t_r) + 2 \frac{\partial u}{\partial t}(x_p, y_q, t_r) + \beta^2 u(x_p, y_q, t_r) \right| + \left| A \frac{\partial^2 u}{\partial x^2}(x_p, y_q, t_r) + \beta \frac{\partial^2 u}{\partial y^2}(x_p, y_q, t_r) - f(x_p, y_q, t_r) \right| \lesssim 0
\]

and \( E_{N,N,N}(x_p, y_q, t_r) \leq 10^{-k_{pq}} (k_{pq} \text{ positive integer}) \). If max \( 10^{-k_{pq}} \) is prescribed, then the truncation limit \( N \) is increased until the difference \( E_{N,N,N}(x_p, y_q, t_r) \) at each of the points becomes smaller than the prescribed \( 10^{-k} \).

On the other hand, the error, can be estimated by \( L_{\infty} \) and \( L_2 \) errors, and the root mean square error (RMS). We calculate the RMS error using the following formula [2]:

\[
\text{RMS error} = \sqrt{\frac{\sum_{i=1}^{n} \sum_{j=1}^{m} (u(x_i, y_j, \tau) - \hat{u}(x_i, y_j, \tau))^2}{m \cdot n}}
\]

where \( u \) and \( \hat{u} \) are the exact and approximate solutions of the problem, respectively, and \( \tau \) is an arbitrary time \( t \) in \([0, T]\).

3. Accuracy of the solution and error analysis

4. A numerical example

This section is devoted to a computational result. We applied the method presented in this work and solved one example. We illustrate our work with the following example. The numerical computations were done using MAPLE 9.

Example. Consider the problem

\[
u_{tt} + 2u_t + u - u_{xx} - u_{yy} = -2 + x^2 + y^2 + t.
\]
The initial conditions are given by
\[
 u(x, y, 0) = x^2 + y^2 \\
 u_t(x, y, 0) = x^2 + y^2 + 1.
\]

By applying the technique described in the preceding section we find the matrix representation of the equation as follows:
\[
 ((S)^2 + 2S + \tilde{I} - (\tilde{B})^2 - (\tilde{M})^2)A = F.
\]

The solution of this system gives the Taylor coefficients
\[
 A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.
\]

Thus, the solution of this problem becomes
\[
 u(x, y, t) = x^2 + y^2 + t
\]

which is the exact solution [3].

5. Conclusion

Two-space-dimensional linear hyperbolic equations with constant coefficients are usually difficult to solve analytically. For this reason, the present method has been proposed for approximate solutions and also analytical solutions. It is observed that the method has a great advantage when the known functions in the equation can be expanded in Taylor series. Moreover, this method is applicable for the approximate solution of parabolic and elliptic type partial differential equations in two space dimensions. The method can also be extended to nonlinear partial differential equations, but some modifications are required.

References