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Quantum gauge fields and flat connections in 2-dimensional BF theory

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ABSTRACT

The 2-dimensional BF theory is both a gauge theory and a topological Poisson σ-model corresponding to a linear Poisson bracket. In [3], Torossian discovered a connection which governs correlation functions of the BF theory with sources for the B-field. This connection is flat, and it is a close relative of the KZ connection in the WZW model. In this Letter, we show that flatness of the Torossian connection follows from (properly regularized) quantum equations of motion of the BF theory.

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1. Introduction

The 2-dimensional BF theory is a an interesting example of a model which is at the same time a gauge theory and a (topological) Poisson σ-model corresponding to a linear Poisson bracket. Hence, we have an interesting opportunity to compare two different approaches to quantization of the model.

As a Poisson σ-model, the BF theory gives rise to a star product on the dual space of a Lie algebra $\mathcal{G}$ (see [1]). The Kontsevich approach to quantization is to fix the gauge and to study the Feynman graphs of the model [2]. In this context, Torossian [3] discovered a very interesting flat connection which governs the behavior of correlation functions of exponentials of the B-field. This connection is a close relative of the Knizhnik–Zamolodchikov connection [4] in the WZW model.

Our aim in this Letter is to better understand the origin of the Torossian connection from the point of view of gauge theory. To this end, we consider the BF theory with source terms for the B-field placed at the points $z_1, \ldots, z_n$, and we study the expectations of the quantum gauge field $\mathcal{A}$ and of the quantum $\mathcal{B}$-field. In terms of Feynman diagrams, we obtain tree contributions for the field $\mathcal{A}$ and one-loop (wheel) contributions for $\mathcal{B}$. Quantum fields $\mathcal{A}$ and $\mathcal{B}$ satisfy quantum equations of motion which actually coincide with the classical ones.

In order to control the behavior of correlators, we need to specify the quantum gauge field $\mathcal{A}$ at the points $z_1, \ldots, z_n$ where the source terms are located. Since $\mathcal{A}$ diverges at these points, we regularize it by subtracting the pole. At the level of Feynman diagrams, this corresponds to excluding one particular length-one tree from summation (the choice of this short tree depends on the point z_i). The set of regularized values $\mathcal{A}^{reg}(z_1), \ldots, \mathcal{A}^{reg}(z_n)$ form a connection $\mathbb{A}$ on the space of configurations of points $z_1, \ldots, z_n$. This connection governs the behavior of correlation functions, and it takes values in the Lie algebra of vector fields on n copies of $\mathcal{G}$.

It turns out that the connection $\mathbb{A}$ is flat [5]. We explain the flatness of $\mathbb{A}$ as a consequence of the quantum equations of motion for the fields $\mathcal{A}$ and $\mathcal{B}$.

The Letter is organized as follows. In Section 2, we briefly recall the basics of the BF theory, the Feynman diagrams and classical and quantum equations of motion. In Section 3, we study the dependence of the correlation functions on the sources, introduce the regularized gauge field and consider the flatness property of the connection $\mathbb{A}$.

2. Classical and quantum BF theory

2.1. Classical action and equations of motion

Topological field theories [6] (see [7] for a review) were introduced about 20 years ago as a novel class of field theories whose partition functions are independent of the metric. In particular, the BF theory is a topological gauge theory which can be defined in any dimension. Let G be a connected Lie group, $\mathcal{G}$ its Lie algebra, and denote by $\mathrm{tr}(ab)$ an invariant scalar product on $\mathcal{G}$ (for instance, the Killing form if G is semisimple). For $\mathcal{M}$ an oriented manifold of dimension n (the space–time of the model) and P a principal G-bundle over $\mathcal{M}$, fields of the BF theory are the gauge field A on

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the bundle P and the $\mathcal{G}$-valued $(n-2)$-form B. The action is given by

$$S_{BF} = \mathrm{tr}\int BF, \quad F = dA + \frac{1}{2}[A, A]. \tag{1}$$

Its quadratic part is of the first order in derivatives, so the theory has no physical degrees of freedom (it is a topological theory of Schwarz type, [8]). Setting the variation of the action equal to zero, we obtain the field equations:

$$dB + [A, B] = D_A B = 0, \tag{2}$$

$$dA + \frac{1}{2}[A, A] = F = 0. \tag{3}$$

The gauge transformations are of the form

$$A^g = g^{-1}dg + g^{-1}Ag, \qquad B^g = g^{-1}Bg. \tag{4}$$

Since F is the curvature form, Eq. (3) states that the connection A is flat. It is this feature that we shall investigate below in the context of quantum gauge theory.

2.2. Feynman diagrams

It is convenient to rewrite the classical action in the form

$$S_{BF} = \mathrm{tr}\int\left(B\,dA + \frac{1}{2}B[A, A]\right), \tag{5}$$

where the first term can be viewed as a free part of the action (in fact, it corresponds to an Abelian BF theory) while the second term represents the interaction. Feynman diagrams in this theory are built of oriented edges pointing from A to B and of trivalent vertices with one incoming B-field and two outgoing A-fields, see Fig. 1.

Depending on the choice of $\mathcal{M}$, the propagator corresponding to an oriented edge can be chosen in various ways. For the BF theory on a plane, one can choose

$$\langle A_a(u)B_b(v)\rangle = \frac{\delta_{ab}}{2\pi}d\arg(u-v),$$

where u and v are complex coordinates on the plane, and the right-hand side is viewed as a 1-form with respect to u. Note that the choice of propagator corresponds to a particular gauge fixing in the theory. The triple vertex corresponds to structure constants f_{abc} of the Lie algebra $\mathcal{G}$.

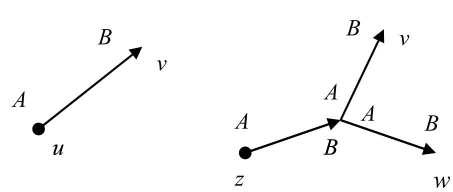

Fig. 1. Diagram building blocks: (a) single edge; (b) vertex.

Connected Feynman graphs of the BF theory are tree diagrams with one external A-field and an arbitrary number of B-fields (see Fig. 2(a)), and one-loop (or wheel-type) diagrams with only B-fields on the external lines (see Fig. 2(b)).

2.3. BF theory with sources

We shall be interested in the BF theory with source terms for B-field added. For the classical action, we have

$$S_\eta = \mathrm{tr}\left(\int_{\mathcal{M}} BF + \sum_{i=1}^{n}\eta_i B(z_i)\right), \tag{6}$$

where we added classical sources η_i at n fixed points, $(z_1, \ldots, z_n)$. The partition function is then given by

$$K_\eta(z_1, \ldots, z_n) = \int e^{S_\eta} = \int e^{S_{BF}+\sum_{i=1}^{n}\mathrm{tr}(\eta_i B(z_i))}, \tag{7}$$

and it can be viewed as a correlation function of the operators $\exp\mathrm{tr}(\eta_i B(z_i))$ in the theory without sources [9],

$$K_\eta(z_1, \ldots, z_n) = \left\langle e^{\mathrm{tr}(\eta_i B(z_i))} \cdots e^{\mathrm{tr}(\eta_i B(z_i))}\right\rangle. \tag{8}$$

For an operator $\mathcal{O}$, the expectation value is defined by formula

$$\langle\mathcal{O}\rangle_\eta = \left(\int \mathcal{O}e^{S_\eta}\right)\Big/\left(\int e^{S_\eta}\right). \tag{9}$$

Thus,

$$\langle\mathcal{O}\rangle_\eta = \frac{\langle\mathcal{O}e^{\sum_{i=1}^{n}\mathrm{tr}(\eta_i B(z_i))}\rangle}{\langle e^{\sum_{i=1}^{n}\mathrm{tr}(\eta_i B(z_i))}\rangle}. \tag{10}$$

In particular, we shall study two cases: when $\mathcal{O}$ is the gauge field $A(u)$ and when $\mathcal{O}$ is the B-field $B(u)$. Note that these are not gauge invariant observables, and that the source terms explicitly break the gauge invariance of the action.

First, we observe that the expectation value of the A-field obtains contributions only from tree-type diagrams. This defines the quantum gauge field $\mathcal{A}$,

$$\mathcal{A}(u) = \langle A(u)\rangle_\eta = \sum_{\text{all trees}}\big(\text{Fig. 2(a)}\big). \tag{11}$$

For a B-field, it is slightly more complicated: we obtain all possible wheel-type diagrams hanging on a branch of a tree-type diagram, see Fig. 3.

$$\mathcal{B}(u) = \langle B(u)\rangle_\eta = \sum_{\text{all [TW] compositions}}(\text{Fig. 3}). \tag{12}$$

Note that both trees and wheels may have arbitrary lengths, and this is taken into account in the infinite sums of (11) and (12). In particular, among tree diagrams there are short trees (containing only one edge, $[T(l=1)]$), see Fig. 1. It is convenient to rewrite Eq. (11) as a sum of two terms

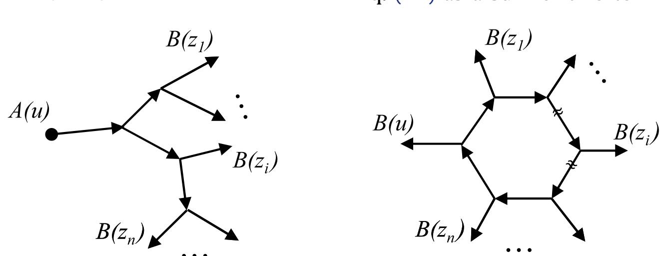

Fig. 2. Basic diagrams: (a) Tree-type diagram, [T]; (b) Wheel-type diagram, [W].

$$\mathcal{A}(u) = \sum_{i=1}^{n} \eta_i d\arg(u - z_i) + a(u; z_1, \ldots, z_n), \tag{13}$$

where $a(u; z_1, \ldots, z_n)$ is the sum over all trees with length $l > 1$, $[T(l > 1)]$.

2.4. Quantum equations of motion

We aim at obtaining quantum equations of motion for the BF theory with sources. The canonical way of doing it is by applying the BRST technique, or rather its generalization – the Batalin–Vilkovisky method, as the BRST operator does not provide a well defined cohomology needed to define physical observables of the theory. This method implies introducting ghosts and anti-fields with complimentary ghost numbers and degrees (see, e.g. [10,11]). We shall instead make use of the graphical representation of the quantum fields – Eqs. (11), (12), resp. Fig. 2, Fig. 3, where all terms in the field expansions are present, thus the equations obtained should account for all quantum corrections, including those coming from the gauge-fixing terms.

On Fig. 4 we show the differential of the quantum $\mathcal{B}$-field. By taking the derivative with respect to the root-point u, the corresponding diagram splits into two subgraphs. The first subgraph is a wheel-type diagram, and the second subgraph is a tree. Two subgraphs are related by a Lie bracket corresponding to the vertex where they meet. Thus, the quantum equation of motion for $\mathcal{B}$ reads

$$d\mathcal{B} = -[\mathcal{A}, \mathcal{B}]. \tag{14}$$

In fact, it coincides with the classical equation of motion, Eq. (2).

For the differential of the quantum gauge field $\mathcal{A}$, we use the splitting (13) to obtain the singular and the regular parts of the result. The singular part (one-edge graphs) generates a sum-over-sources term, Fig. 5. As seen from Fig. 6, the derivative of the regular part, similarly to the case of the B-field, splits into two tree-type subgraphs rooted at u.

Thus, the quantum equation for $\mathcal{A}$ takes the form

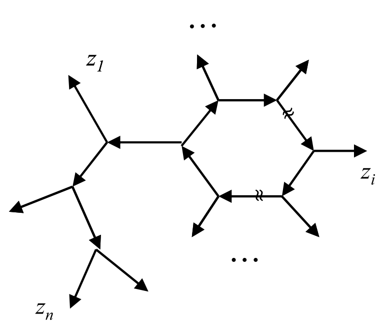

Fig. 3. A typical B-field diagram – a [TW] composition.

$$d\mathcal{A} = -\frac{1}{2}[\mathcal{A}, \mathcal{A}] + \sum_{i=1}^{n} \eta_i \delta(u - z_i), \tag{15}$$

which is again of the same form as the corresponding classical equation of motion.

3. Equations for correlators and quantum flat connection

In this section, we give a physical interpretation of the equations for correlation functions constructed in [3]. These equations fit into a flat connection studied in a more mathematical framework in [5].

For this purpose, we shall investigate the dependence of the generating functional of the B-field correlators $K_\eta(z_1, \ldots, z_n)$ on the positions of the sources $z_1, \ldots, z_n$. That is, we will be interested in the derivatives of the quantum fields $\mathcal{A}$ and $\mathcal{B}$ with respect to coordinates z_i.

Note that the quantum field (13) is singular at the points where the sources are placed. In order to regularize this singularity, it is convenient to introduce for each i a new splitting of $\mathcal{A}(u)$ in the form

$$\mathcal{A}_{(i)}(u) = \frac{\eta_i}{2\pi} d\arg(u - z_i) + \mathcal{A}^{reg}_{(i)}(u), \tag{16}$$

where all the unit-length trees but one (connecting the points u and z_i) are now kept in the regular part:

$$\begin{aligned} \mathcal{A}^{reg}_{(i)}(u) &= \sum_{j \neq i} \big[T(l = 1); \{u, z_j\}\big] + \sum_{\text{all trees},\, l>1} [T] \\ &= \sum_{j \neq i} \frac{\eta_j}{2\pi} d\arg(u - z_j) + a(u; z_1, \ldots, z_n). \end{aligned} \tag{17}$$

Observe, that $\mathcal{A}^{reg}_{(i)}(u)$ has no singularity at $u = z_i$. Let us denote its value by

$$a_i := \mathcal{A}^{reg}_{(i)}(u; z_1, \ldots, z_i, \ldots, z_n)\big|_{u = z_i}. \tag{18}$$

The quantum equation of motion for the $\mathcal{B}$-field leads to the following relation:

$$\begin{aligned} d\,\mathrm{tr}\big(\eta \mathcal{B}(u)\big) &= -\,\mathrm{tr}\big(\eta\big[\mathcal{A}(u), \mathcal{B}(u)\big]\big) = -\,\mathrm{tr}\big(\big[\eta, \mathcal{A}(u)\big]\mathcal{B}(u)\big) \\ &= -\,\mathrm{tr}\big[\eta, \mathcal{A}(u)\big] \frac{\partial}{\partial \eta} \mathrm{tr}\big(\eta \mathcal{B}(u)\big). \end{aligned}$$

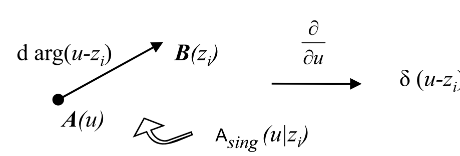

Fig. 5. Equation of motion for $\mathcal{A}$: singular terms.

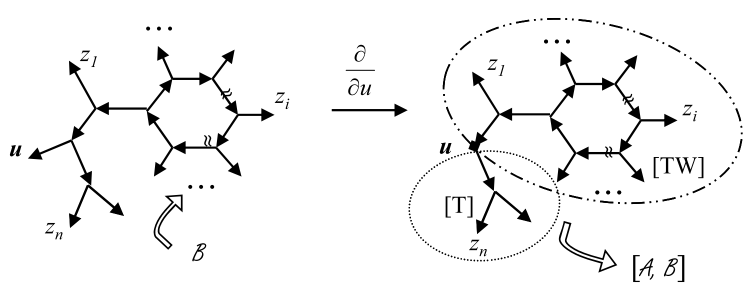

Fig. 4. Equation of motion for $\mathcal{B}$-field.

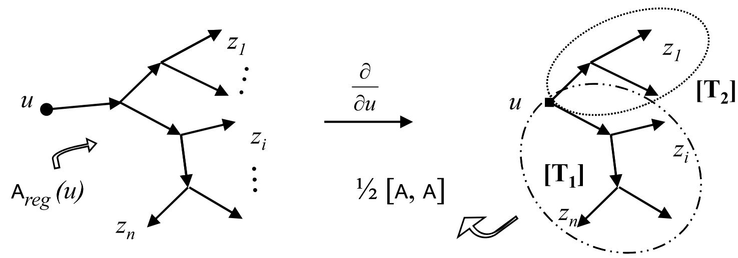

Fig. 6. Equation of motion for $\mathcal{A}$: regular terms.

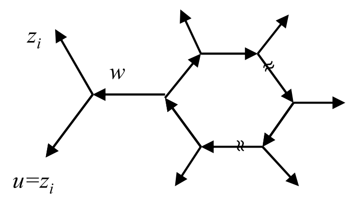

Fig. 7. Vanishing B-field diagram.

Naively, we should expect the following equation for $K_\eta(z_1, \ldots, z_n)$ to hold:

$$d_{z_i} K_\eta(z_1, \ldots, z_n) + \mathrm{tr}\big[\eta_i, \mathcal{A}(z_i)\big] \frac{\partial}{\partial \eta_i} K_\eta(z_1, \ldots, z_n) = 0. \tag{19}$$

Here d_{z_i} stands for the de Rham differential with respect to the coordinate z_i (note that it includes both holomorphic and anti-holomorphic differentials). Since $\mathcal{A}(z_i)$ is ill-defined, we need to re-examine the Feynman graphs which contribute in the right-hand side of Eq. (19).

The only interesting (different from the naive approach) case is the diagram shown on Fig. 7. Its contribution vanishes because of the factor $(d \arg(w - z_i))^2 = 0$ in the integrand of the corresponding Feynman integral. Hence, the one-edge tree connecting w and z_i does not contribute in the derivative of K_η, and the renormalized quantum formula replacing Eq. (19) is

$$d_{z_i} K_\eta + \mathrm{tr}[\eta_i, a_i] \frac{\partial}{\partial \eta_i} K_\eta = 0. \tag{20}$$

Eq. (20) for different i can be put together in one equation

$$dK_\eta + \mathrm{tr} \sum_{i=1}^{n} [\eta_i, a_i] \frac{\partial}{\partial \eta_i} K_\eta = 0, \tag{21}$$

where d is the total de Rham differential for all variables $z_1, \ldots, z_n$. For functions $\alpha_i(\eta_1, \ldots, \eta_n) \in \mathcal{G}, i = 1, \ldots, n$, operators

$$D_\alpha = \mathrm{tr} \sum_{i=1}^{n} [\eta_i, \alpha_i] \frac{\partial}{\partial \eta_i} \tag{22}$$

form an interesting Lie algebra

$$[D_\alpha, D_\beta] = D_{\{\alpha,\beta\}} \tag{23}$$

with Lie bracket

$$\{\alpha, \beta\}_i = D_\alpha \beta_i - D_\beta \alpha_i + [\alpha_i, \beta_i]. \tag{24}$$

One can view the collection of 1-forms $(a_1, \ldots, a_n)$ as components of a connection $\mathbb{A} = (a_1, \ldots, a_n)$ with values in this Lie algebra. Then, Eq. (21) for correlation functions simply reads

$$dK_\eta + D_{\mathbb{A}} K_\eta = 0.$$

Similarly, for the differential of gauge field $\mathcal{A}(u)$ with respect to the source positions we obtain

$$d_{z_i} \mathcal{A}(u) = -\mathrm{tr}[\eta_i, a_i] \frac{\partial}{\partial \eta_i} \mathcal{A}(u) = -D_{a_i} \mathcal{A}(u). \tag{25}$$

Note that for $j \neq i$ we can replace $\mathcal{A}(u)$ by $\mathcal{A}^{reg}_{(j)}$. Indeed, the one-edge tree which is subtracted from $\mathcal{A}(u)$ to get $\mathcal{A}^{reg}_{(j)}$ (the edge connecting u to z_j) does not contribute neither to the left-hand side nor to the right-hand side of Eq. (25). Then, putting $u = z_j$ yields

$$d_{z_i} a_j = -D_{a_i} a_j. \tag{26}$$

We will now show that the curvature $\mathbb{F}$ of $\mathbb{A}$ vanishes [5]. The curvature is defined as

$$\mathbb{F} = d\mathbb{A} + \frac{1}{2}\{\mathbb{A}, \mathbb{A}\}. \tag{27}$$

We will first compute its components $\mathbb{F}_{ij}$ corresponding to two different coordinates $z_i \neq z_j$ (note that the curvature has holomorphic, anti-holomorphic and mixed components). The curvature $\mathbb{F}_{ij}$ has n components $(\mathbb{F}_{ij})_k$ for $k = 1, \ldots, n$. The components with $k \neq i, j$ vanish identically. For the remaining components, we have

$$(\mathbb{F}_{ij})_i = d_{z_j} a_i + D_{a_j} a_i = 0, \qquad (\mathbb{F}_{ij})_j = d_{z_i} a_j + D_{a_i} a_j = 0. \tag{28}$$

The curvature F_{ii} has only one nonvanishing component,

$$(\mathbb{F}_{ii})_i = d_{z_i} a_i + D_{a_i} a_i + \frac{1}{2}[a_i, a_i]. \tag{29}$$

In more detail, put $a_i = \alpha_i dz_i + \bar{\alpha}_i d\bar{z}_i$ to obtain

$$(\mathbb{F}_{ii})_i = \partial_{z_i} \bar{\alpha}_i - \bar{\partial}_{z_i} \alpha_i + D_{\alpha_i} \bar{\alpha}_i - D_{\bar{\alpha}_i} \alpha_i + [\alpha_i, \bar{\alpha}_i]. \tag{30}$$

In order to compute this expression, we consider the differential $d_{z_i} a_i$. There are several types of diagrams which contribute (see Fig. 8). Note that graphs of type (a) vanish, as in the derivative of K_η. Graphs of types (b) and (c) generate source terms and covariant derivative terms. Graphs of type (d) accounts for an extra z_i dependence due to the root of the tree. The result is

$$d_{z_i} a_i(z_i) + D_{a_i} a_i + \frac{1}{2}[a_i, a_i] = \sum_{j \neq i} \eta_j \delta(z_i - z_j). \tag{31}$$

That is, away from the sources positions, the connection is flat,

$$d\mathbb{A} + \frac{1}{2}\{\mathbb{A}, \mathbb{A}\} = 0. \tag{32}$$

With sources taken into account, we have $\mathbb{F} = (\mathbb{F}_1, \ldots, \mathbb{F}_n)$, where

$$\mathbb{F}_i = \sum_{j \neq i} \eta_j \delta(z_i - z_j). \tag{33}$$

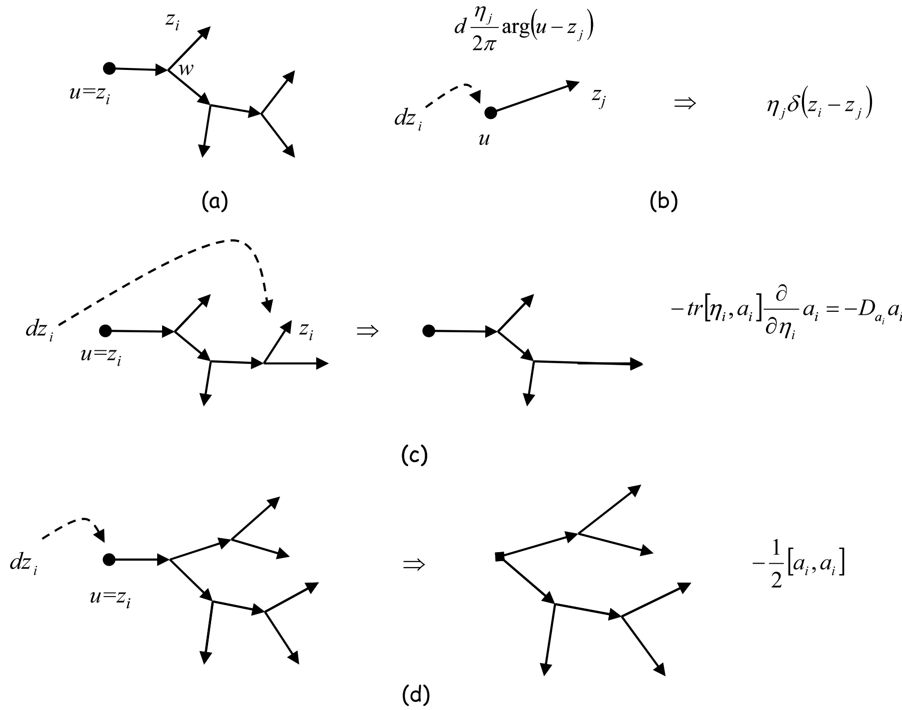

Fig. 8. Graphic z_i differentiation of a_i.

4. Outlook

The Torossian connection discussed in Section 3 is a close relative of the Knizhnik–Zamolodchikov (KZ) connection in the WZW theory. Recall that the KZ connection describes correlators of primary fields, and that it has the form

$$d\Psi+\mathbb{A}_{KZ}\Psi=0,\qquad \mathbb{A}_{KZ}=\frac{1}{2\pi i}\sum_{i,j}t_{i,j}d\ln(z_i-z_j),\tag{34}$$

where $t_{i,j}=\sum_a e_a^i\otimes e_a^j$ are operators acting on the product of irreducible representation of $\mathcal{G}$ carried by primary fields placed at the points $z_1,\dots,z_n$. Note that operators $t_{i,j}$ play the role of one-edge trees, and the propagator has the form $d\ln(z_i-z_j)/2\pi i$.

The KZ connection admits the second interesting interpretation: one can view it as an equation on the wave function of the Chern–Simons topological field theory with n time-like Wilson lines (corresponding to primary fields) [12]. From this perspective, holonomy matrices of the flat connection $\mathbb{A}_{KZ}$ correspond to braiding of Wilson lines in the Chern–Simons theory.

It would be very interesting to find a three-dimensional topological field theory which has the Torossian connection as an equation on the wave function. Of course, such a theory must have nonlocal observables (similar to Wilson lines) which will correspond to insertions of operators $\exp(\mathrm{tr}\,\eta_i B(z_i))$ in the 2-dimensional theory.

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