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COMBINATORIAL MANIFOLDS WITH FEW VERTICES

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We show that a d-manifold M with less than $3\lceil d/2\rceil+3$ vertices is a sphere and that a d-manifold with $3d/2+3$ vertices is either a sphere or $d=2, 4, 8$ or 16 and M is a "manifold like a projective plane". There are such examples for $d=2, 4, 8$.

Furthermore we show that a d-manifold M with $2d+3-i$ vertices is i-connected [i.e. $\pi_0(M)= \ldots =\pi_i(M)=0$] where $0\leqslant i<d/2$. In particular the smallest number of vertices of a non-simply-connected d-manifold is exactly $n=2d+3$, $(d\geqslant 3)$.

1. INTRODUCTION AND RESULTS

A combinatorial manifold M^d of dimension d is a simplicial complex such that the link of every vertex is a $(d-1)$-dimensional combinatorial sphere. We denote by $|M|$ the underlying topological manifold with the induced PL structure. We write $M_1\simeq M_2$ if M_1 and M_2 are combinatorially isomorphic, and $|M_1|\approx|M_2|$ if they are PL homeomorphic. For general facts about PL manifolds compare [8], [17], [22]. One expects that the number n of vertices must be large if $|M^d|$ is topologically complicated. If n is not too much larger than $d+2$ (the number of vertices of the boundary complex of a $(d+1)$-simplex) then $|M^d|\approx S^d$. The question is: which is the minimal number $m=m(d)$ for given d, such that there exists a combinatorial d-manifold with m vertices which is not a sphere? It seems that this $m(d)$ is known so far only for $d\leqslant 4$. A lower bound for $m(d)$ has been given in [4]:

Theorem 1 [4]. *Let M^d be a combinatorial α-manifold with n vertices and assume $d\geqslant 3$, $d\neq 4$ and $n\leqslant d+5$. Then $|M^d|$ is PL homeomorphic to the sphere S^d.*

Note that the exception $d\neq 4$ is essential: in fact Theorem 1 becomes false for $d=4$ (see below). For $n\leqslant d+4$ a theorem of Mani [19] says that M is the boundary complex of a convex polytope and that this bound is sharp.

First we state our theorems and corollaries which will be proved in §3.

Our main theorem is the following which gives a better lower bound than Theorem 1 including a discussion of equality. $\lceil x\rceil$ denotes the smallest integer $k\geqslant x$. $\mathbb{R}P^2_6$ denotes the unique 6-vertex triangulation of the real projective plane $\mathbb{R}P^2$, $\mathbb{C}P^2_9$ denotes the unique 9-vertex triangulation of the complex projective plane [12].

Theorem A. *Let M^d be a combinatorial d manifold with n vertices.*

(i) *if* $n<3\left\lceil\frac{d}{2}\right\rceil+3$ *then* $|M|\approx S^d$,

(ii) *if* $n = 3\frac{d}{2} + 3$ *then either* $|M| \approx S^d$ *or* $|M|$ *must be a "manifold like a projective plane" in the sense of* [7], *in particular in this case it follows that* $d = 2, 4, 8$ *or* 16. *Moreover for* $d = 2$ $M \simeq \mathbb{R}P^2_6$, *for* $d = 4$ $M \simeq \mathbb{C}P^2_9$.

Remark. Our proof shows that in the case of a manifold like a projective plane with $n = 3d/2 + 3$ vertices opposite to each d-simplex there is the boundary complex of a $(d/2+1)$-simplex.

In the case $d = 8$ there exist at least three combinatorially different 15-vertex triangulations of the quaternionic projective plane $\mathbb{H}P^2$ (see [5]). In the case $d = 16$ it is yet undecided whether or not there exists a 27-vertex triangulation of the Cayley plane. Similar lower bounds for all projective spaces are due to A. Marin (compare the appendix of [20]).

THEOREM B. *Let* M^d *be a combinatorial d-manifold with n vertices and assume that* $|M|$ *is* $(i-1)$*-connected but not i-connected (i.e.* $\pi_1 = \pi_2 = \ldots = \pi_{i-1} = 0$, $\pi_i \neq 0$) *where* $1 \leqslant i < d/2$. *Then the inequality*

$$n \geqslant 2d + 4 - i$$

holds.

Remark. For $i = d/2$ Theorem A yields $n \geqslant 3d/2 + 4 = 2d + 4 - i$ except for "manifolds like projective planes" where $n \geqslant 3d/2 + 3$. For $i = 0$ (i.e. M not connected) we get similarly $n \geqslant 2(d+2)$ with equality iff each component is the boundary of a $(d+1)$-simplex.

COROLLARY 1. *Assume that* M *is a d-manifold with the same homology as* $S^i \times S^{d-i}$. *Then* $n \geqslant 2d + 4 - i$.

Remark. Recall that for $i = 1$ and any $d \geqslant 2$ there is such a combinatorial manifold satisfying equality $n = 2d + 3$ [11]. These "generalized Császár tori" are topologically products $S^1 \times S^{d-1}$ for even d and twisted products for odd d. If $i \geqslant 2$ no example for equality $n = 2d + 4 - i$ seems to be known. Candidates could be $(i+1)$-neighborly triangulations of $S^i \times S^{d-i}$.

COROLLARY 2. *For given* $d \geqslant 2$ *let* $n = n(d)$ *denote the smallest number n such that there exists a combinatorial d-manifold with n vertices which is not simply-connected. Then* $n(2) = 6$, $n(d) = 2d + 3$ *for* $d \geqslant 3$.

COROLLARY 3. *For given* $d \geqslant 2$ *let* $m = m(d)$ *denote the smallest number m such that there exists a combinatorial d-manifold with m vertices which is not PL homeomorphic to the sphere. Then*

$m(2) = 6$
$m(3) = 9$
$m(4) = 9$
$m(8) = 15$
$m(16) \geqslant 27$
$m(d) \geqslant 3d/2 + 4$ *for even* $d \geqslant 6, d \neq 8, 16$
$m(d) \geqslant (3d+1)/2 + 4$ *for odd* $d \geqslant 5$.

COROLLARY 4. *Let M^d be a combinatorial homology sphere with n vertices which is not PL homeomorphic to the sphere. Then*

$$n \geqslant 10 \quad \textit{for} \quad d = 3, 4$$
$$n \geqslant 2d + 3 \quad \textit{for} \quad d \geqslant 5.$$

Figure 1 visualizes our results about the possible pairs (d, n) and what is known about the topology of such combinatorial manifolds. There remains an "unknown territory" in the region $3d/2 + 4 \leqslant n \leqslant 2d + 2$.

Question. Does there exist a 12-vertex triangulation of $S^2 \times S^3$ or a 13-vertex triangulation of $S^3 \times S^3$?

2. SOME FACTS FROM *PL* MORSE THEORY

Let us consider a fixed combinatorial manifold M^d with n vertices $p_1, \ldots, p_n$. An arbitrary assignment $\{p_1, \ldots, p_n\} \to \mathbb{R}$ induces a simplex-wise linear function $f: M \to \mathbb{R}$. If $f(p_i) \neq f(p_j)$ for $i \neq j$ we call f regular simplex-wise linear. Such a function has finitely many critical points because only the vertices can be critical. A point $p \in M$ is called critical if

$$H_*(M_c, M_c \setminus \{p\}) \neq 0$$

where $c := f(p)$, $M_c := \{x \in M \mid f(x) \leq c\}$ and H_* is the ordinary singular homology with coefficients in a field. By the isomorphism $H_*(M_c, M_c \setminus \{p\}) \cong H_*(M_c, M_c \setminus (\text{star}(p))^\circ)$ we may use the simplicial homology as well. If K_c denotes the full subcomplex spanned by all vertices v with $f(v) \leq c$ then $H_*(M_c, M_c \setminus (\text{star}(p))^\circ) \cong H_*(K_c \cap \text{star}(p), K_c \cap \text{link}(p))$. If M has

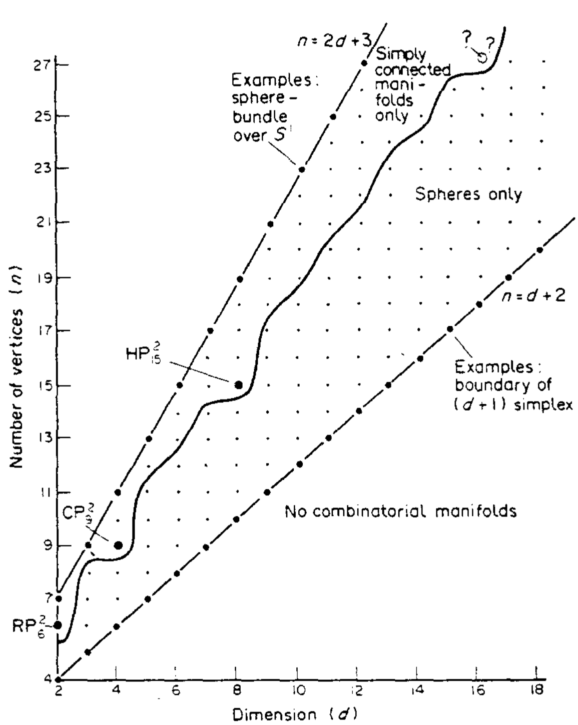

Fig. 1. (d, n)-diagram for combinatorial manifolds

few vertices then K_c has few vertices, and the last expression can be computed easily in terms of the triangulation.

We say that a critical point p has index i with multiplicity k_i if $rk\ H_i(M_c, M_c \setminus \{p\}) = k_i$. In general, this i is not uniquely determined by p, i.e. one point might be critical with several indices and total multiplicity $k = \Sigma k_i$. Let $\mu_i(f)$ be the number of critical points of index i (each counted with its multiplicity k_i), and let $\beta_i(M) = rk\ H_i(M)$. Then the Morse relations hold:

THEOREM 2 (*see* [16])

(i) $\mu_i(f) \geq \beta_i(M)$ for $i = 0, \ldots, d$.

(ii) $\sum_{i=0}^{d} (-1)^i \mu_i(f) = \sum_{i=0}^{d} (-1)^i \beta_i(M) = \chi(M)$.

Remark. If $f(p_1) < f(p_2) < \ldots < f(p_n)$ then with the notation $c_i := f(p_i)$ the inclusion

$$M_{c_i} \to M_{c_{i+1}} \setminus \{p_{i+1}\}.$$

is a homotopy equivalence, in particular

$$H_*(M_{c_i}) \simeq H_*(M_{c_{i+1}} \setminus \{p_{i+1}\}).$$

Moreover M_{c_i} collapses to the subcomplex of M spanned by the vertices $p_1, \ldots, p_i$.

Usually the notion of a *PL* Morse function refers to the case that there are charts of coordinates around each point (critical or non-critical) which behave like the ones for smooth Morse functions (compare [7] where such functions are called C^{omb}-non-degenerate functions). In particular in this case the critical points have total multiplicity 1.

DEFINITION. *The PL Morse number $\mu(M) \geq 2$ of a PL manifold M is the infimum of $\mu(f) = \sum_{i=0}^{n} \mu_i(f)$ ranging over all PL Morse functions $f: M \to \mathbb{R}$.*

THEOREM 3 [10]. *A PL manifold M^d with PL Morse number $\mu(M) = 2$ is PL homeomorphic to the sphere S^d.*

This is a *PL* version of the Reeb theorem in differential topology. Compare [14] for a topological version for continuous functions with two critical points on a topological manifold.

For *PL* Morse functions with exactly 3 critical points the following holds.

THEOREM 4 [7]. *A PL manifold M^d with PL Morse number $\mu(M) = 3$ has dimension $d = 2, 4, 8$ or 16. It is a compactification of an open d-ball by a d/2-dimensional sphere. In the cases $d = 2$ or $d = 4$ $|M|$ is homeomorphic to the real projective plane $\mathbb{R}P^2$ or the complex projective plane $\mathbb{C}P^2$, respectively.*

Remarks.

(i) In the case $d = 4$ we applied the Freedman classification of simply connected 4-manifolds (see [23]) which was not available when the paper [7] has been written.

(ii) According to [7] there are different homotopy types and many different *PL* structures of such manifolds in the cases $d = 8$ and $d = 16$.

THEOREM 5 (*see* [8] III.17). *If a d-dimensional combinatorial manifold with boundary collapses to a point then it is PL homeomorphic to a d-ball.*

COROLLARY 5. *Assume that the n vertices of a combinatorial manifold M^d can be split into two parts such that the span of each part collapses to a point. Then $|M^d|$ is PL homeomorphic to the sphere S^d.*

LEMMA 1. *If $f: M \to \mathbb{R}$ is a regular simplex-wise linear function with no critical points of index 1, 2, ..., k, then M is k-connected, i.e. the homotopy groups $\pi_1, \pi_2, \ldots, \pi_k$ vanish.*

Proof. For *PL* Morse functions Lemma 1 follows from standard arguments in Morse theory. For the case of regular simplex-wise linear functions we can use the Hurewicz isomorphism theorem for the inclusion $M_c \backslash p \to M_c$ with the extra observation that $M_c \backslash p$ simply connected implies that M_c is simply connected unless $M_c \cap \text{link}(p)$ is disconnected (which would imply that p is critical of index 1). This shows that $\pi_1, \ldots, \pi_k$ cannot change when passing through any of the vertices. On the other hand M_c is a *PL* ball if c lies between the levels of the two lowest critical points.

LEMMA 2. *Let K be a simplicial complex with vertex set $V = V_1 \dot{\cup} V_2$. Let K_i be the full subcomplex of K spanned by V_i and let $f: M \to \mathbb{R}$ be a regular simplex-wise linear function with $f(p) < f(q)$ for all $p \in V_1, q \in V_2$. Then every critical point $p \in V_1$ of f is also critical for $f|K_1$ with exactly the same indices and multiplicities.*

Proof. Let

$$f(p_1) < \ldots < f(p_k) < f(p_{k+1}) < \ldots < f(p_n),$$

$$V_1 = \{p_1, \ldots, p_k\}, \quad V_2 = \{p_{k+1}, \ldots, p_n\}.$$

Then it is easy to see that $K_1 \subseteq M_{c_k}$ is a deformation retract where $c_k := f(p_k)$ (compare also [16]). Moreover for every level $c_i := f(p_i)$, $i = 1, \ldots, k$ the subspaces

$$M_{c_i} \cap K_1 \subseteq M_{c_i}$$

and

$$M_{c_i} \cap K_1 \backslash p_i \subseteq M_{c_i} \backslash p_i$$

are deformation retracts. Consequently for $i = 1, \ldots, k$

$$H_*(M_{c_i}, M_{c_i} \backslash p_i) \cong H_*(M_{c_i} \cap K_1, M_{c_i} \cap K_1 \backslash p_i).$$

LEMMA 3. *For every regular simplex-wise linear function $f: M^d \to \mathbb{R}$ we have $\mu_i(f) = \mu_{d-i}(-f)$.*

Proof. Again the assertion is well known for *PL* Morse functions. It is trivial in the case $i = 0$ or $i = d$ because minima and maxima are interchanged when passing from f to $-f$. Now assume $1 \leqslant i \leqslant d - 1$. Then we have the Alexander duality

$$\bar{H}_{i-1}(M_c \cap \text{link}(p)) \cong \bar{H}_{d-i-1}(\text{link}(p) \backslash M_c),$$

where $\bar{H}$ denotes the reduced homology ([6], VIII, 8.15). In our case homology and cohomology are isomorphic because we use coefficients in a field. Clearly $M_c \cap \text{star}(p)$ is contractible and therefore

$$H_i(M_c \cap \text{star}(p), M_c \cap \text{link}(p)) \cong \bar{H}_{i-1}(M_c \cap \text{link}(p)).$$

Now let us denote $\tilde{M}_{-c} := \{x \in M \mid -f(x) \leqslant -c\}$. Then link $(p) \setminus M_c$ and $\tilde{M}_{-c} \cap \text{link}(p)$ are homotopy equivalent. It follows

$$\begin{aligned} H_i(M_c, M_c \setminus p) &\cong H_i(M_c \cap \text{star}(p), M_c \cap \text{link}(p)) \\ &\cong \bar{H}_{i-1}(M_c \cap \text{link}(p)) \\ &\cong \bar{H}_{d-i-1}(\tilde{M}_{-c} \cap \text{link}(p)) \\ &\cong H_{d-i}(\tilde{M}_{-c} \cap \text{star}(p), \tilde{M}_{-c} \cap \text{link}(p)) \\ &\cong H_{d-i}(\tilde{M}_{-c}, \tilde{M}_{-c} \setminus p). \end{aligned}$$

LEMMA 4. *Let M^d be a combinatorial d-manifold, $d \geqslant 5$. Assume $\mu(|M|) \geqslant 3$. Then every regular simplex-wise linear function $f: M \to \mathbb{R}$ has a critical point of index i, where $d/2 \leqslant i \leqslant d-1$.*

Proof. If $\beta(|M|) \geqslant 4$ then the assertion follows from the Poincaré duality and the Morse inequality (Theorem 2). If $\beta(|M|) = 3$ then d must be even, $d = 2m$ and $H_m(M) \neq 0$ implying $\mu_m(f) \neq 0$.

If M is not simply connected then every such function f has a critical point of index 1. The same is true for $-f$, and by Lemma 3 f has a critical point of index $d-1$. There remains the case where M is simply connected (hence orientable) and $\beta(|M|) = 2$ for every field of coefficients (homology sphere). Thus M is a homotopy sphere, and for $d \geqq 5$ M is PL homeomorphic to the sphere by the generalized PL Poincaré theorem. This contradicts our assumption $\mu(|M|) \geqq 3$.

LEMMA 5. *Assume that a regular simplexwise linear function on a simplicial complex K has a critical point p of index $i \geqslant 1$. Then K is at least i-dimensional, and the number n of vertices satisfies $n \geqslant i+2$. Moreover, if $n = i+2$ then either $K = \partial\Delta^{i+1}$ or $K = \{p\} * \partial\Delta^i$ (cone over $\partial\Delta^i$).*

Proof. By assumption $H_i(K_c, K_c \setminus p) \neq 0$ where $c = f(p)$. This means that there is a simplicial i-chain on K_c which is a non-bounding cycle modulo $K_c \setminus p$. Therefore K_c and hence K must contain at least one i-dimensional simplex containing p. A non-bounding simplicial i-cycle requires at least two i-simplices and therefore at least $i+2$ vertices. Now assume that there are exactly $n = i+2$ vertices. These cannot span a simplex Δ^{i+1} because the homology would be trivial. The only possibility to build a non-bounding $(i-1)$-cycle with $i+1$ vertices is the boundary complex of a Δ^i (with suitable coefficients). This implies that K must contain the cone from p to $\partial\Delta^i$ spanned by the other $i+1$ vertices. We denote this cone by $\{p\} * \partial\Delta^i$. Now the two possibilities are obvious: either K contains Δ^i or not. In the first case $K = \partial\Delta^{i+1}$, in the latter case $K = \{p\} * \partial\Delta^i$. In any case p is the maximum of the function.

3. PROOF OF THE THEOREMS

THEOREM A.

For $d = 1$ nothing has to be proved, for $d = 2$ everything is known (see [9], [21]), for $d = 3$ our assertion follows from the enumeration of all combinatorial 3-manifolds with at most 9 vertices (see [1]), and for $d = 4$ our assertion is contained in [4] except for the existence and uniqueness of $\mathbb{C}P^2_9$ (see below).

Now let us assume $d \geqslant 5$. If $\mu(|M|) = 2$ then $|M| \approx S^d$ by Theorem 3. Now assume $\mu(|M|) \geqslant 3$ and let $f: M \to \mathbb{R}$ be a regular simplex-wise linear function with $f(p_1) < \dots < f(p_n)$ where $p_1, \dots, p_n$ are the n vertices and $\Delta^d = \langle p_{n-d}, p_{n-d+1}, \dots, p_n \rangle$ is a d-simplex of M. Then f attains its maximum at p_n and has no other critical point on Δ^d. By Lemma 4 f has a critical point of index i with $d/2 \leqslant i \leqslant d-1$ which is one of the vertices $p_1, \ldots, p_{n_0}$ where $n_0 := n-(d+1)$. Let K_0 denote the full subcomplex of M spanned by $p_1, \ldots, p_n$. By Lemma 2 the same vertex is a critical point of index i of $f: K_0 \to \mathbb{R}$. By Lemma 5 $n_0 \geqslant i+2$ which implies $n \geqslant 3d/2+3$. Moreover $n = \lceil 3d/2 \rceil + 3$ implies that $n_0 = i+2$, and Lemma 5 says that either $K_0 = \partial\Delta^{i+1}$ or $K_0 = \{p\} * \partial\Delta^i$, the latter would imply that K_0 collapses to a point, and M is a sphere by Corollary 5. Hence $K_0 = \partial\Delta^{i+1}$, and it follows that f is a PL Morse function with exactly 3 critical points which implies that d is even and $i = d/2$. The splitting into Δ^d and $K_0 = \partial\Delta^{i+1}$ directly shows that M is the compactification of an open d-ball by a $d/2$-sphere. Now we can apply Theorem 4. This also shows that for odd d equality $n = \lceil 3d/2 \rceil + 3$ cannot occur, thus $n \geqslant 3\lceil d/2 \rceil + 3$.

In the case $d=2$ it is well known that $\mathbb{R}P^2_6$ is the only combinatorial 2-manifold with 6 vertices which is not homeomorphic to the sphere. In the case $d=4$ it was known that a non-sphere with 9 vertices must have the cohomology ring of $\mathbb{C}P^2$ (see [4]) (it follows that M is homeomorphic to $\mathbb{C}P^2$ (compare [23]), but this is not used in the following). We know that $\beta(|M|)=3$ and this implies that every regular simplex-wise linear function must have a critical point of index 2. Then Lemma 5 implies that for every 4-simplex $\Delta^4 = \langle p_1, \ldots, p_5 \rangle$ the full subcomplex K_0 spanned by $p_6, \ldots, p_9$ is a $\partial\Delta^3$ (in the other case $K_0 = \{p\} * \partial\Delta^2$ M would be a 4-sphere by Corollary 5).

The next claim is that such a 9-vertex M^4 must necessarily contain all possible $\binom{9}{2}$ edges. Assume that two vertices p and q are not joined by an edge. Then $\{p, q\}$ can never be contained in one 4-simplex nor in the complement of one 4-simplex. It follows that each 4-simplex of M^4 contains exactly one of the vertices p, q. Hence the two vertex stars of p and q cover the whole manifold M which now is the union of two PL 4-balls glued together along their common boundary (i.e. the link of p and q). Hence M is homeomorphic to the 4-sphere, a contradiction.

Now we use the well-known equation (see [24])

$$10f_0 - 4f_1 + f_2 = 10\chi(M)$$

for any combinatorial 4-manifold, where f_0, f_1, f_2 are the numbers of vertices, edges and triangles. In our special case we know $f_0 = 9, f_1 = \binom{9}{2} = 36$ and $\chi = 3$ which implies

$$f_2 = 30 - 90 + 144 = 84 = \binom{9}{3}.$$

This shows that a 9-vertex triangulation of a non-sphere must necessarily contain all possible $\binom{9}{3}$ triangles. This is also called 3-neighborliness. On the other hand it has been shown in [13] that the combinatorial type of such a 3-neighborly 4-manifold is unique. Compare [12] and [20] for a proof that there exists a 9-vertex triangulation of $\mathbb{C}P^2$.

THEOREM B.

As in the proof of Theorem A we use a regular simplex-wise linear function $f: M \to \mathbb{R}$ such that

$$f(p_1) < \ldots < f(p_n)$$

where $p_{n-d}, \ldots, p_n$ span a simplex Δ^d of M, and $p_1, \ldots, p_{n_0}$ span K_0, $n_0 := n-(d+1)$. By Lemma 1 $-f$ must have a critical point in K_0 of index $j \leqslant i$ with $1 \leqslant j < d/2$ which is critical of index $d-j$ for f. By Lemma 4 the same vertex is critical for $f: K_0 \to \mathbb{R}$ and consequently $n_0 \geqslant d-j+2$. On the other hand f must also have a critical point of index j in K_0. These two may coincide as vertices. Then we get $n_0 \geqslant d-j+2$ from the critical point of index $d-j$. Lemma 5 says that equality $n_0 \geqslant d-j+2$ would imply either $K_0 = \partial\Delta^{d-j+1}$ or

$K_0=\{p\}*\hat{c}\Delta^{d-j}$ (in the latter case K_0 collapses to a point and M is a sphere by Corollary 5, a contradiction). In the first case $K_0=\hat{c}\Delta^{d-j+1}$ no other critical point can occur. In our case we have another critical point of index j which forces K_0 to have at least one more vertex. Thus $n_0 \geqslant d-j+3$ implying $n \geqslant 2d+4-i$.

COROLLARY 1.

If M has the same homology as $S^i \times S^{d-i}$, $1 \leqslant i \leqslant d/2-1$ then it is not i-connected. We cannot claim that M is $(i-1)$-connected but in any case Theorem B leads to the inequality $n \geqslant 2d+4-i$. In the case $i=d/2$ note that M is certainly not a "manifold like a projective plane", and Theorem A yields $n \geqslant d+d/2+4=2d+4-i$.

COROLLARY 2.

The assertion is known for $d=2$. Now assume $d \geqslant 3$. Then Theorem B implies $n \geqslant 2d+3$. On the other hand there are examples of such combinatorial manifolds satisfying equality $n=2d+3$ (see [11]). These are products $S^1 \times S^{d-1}$ if d is even and twisted products if d is odd. These examples also seem to have some relationship with the lower bound conjecture for manifolds. For $d=3$ and $d=4$ compare [24].

COROLLARY 3.

Again the case $d=2$ is clear. For $d=3$ and $d=4$ the inequalities $m(3) \geqslant 9$ and $m(4) \geqslant 9$ follow from Theorem A. On the other hand there are examples satisfying equality: the unique 9-vertex triangulation of the 3-dimensional Klein bottle [24, 1, 11] and the unique 9-vertex triangulation of the complex projective plane [12, 13].

In the other cases the inequality follows from Theorem A but nothing is known about equality except for $d=8$: there is a 15-vertex triangulation of the quaternionic projective plane recently found by the authors [5].

COROLLARY 4.

For $d=3$ the assertion follows from the enumerations of all combinatorial 3-manifolds with up to 9 vertices (see [1]). For $d=4$ it follows from Theorem A, for $d \geq 5$ it follows from Theorem B and the generalized PL Poincaré theorem. Under the *PL* Poincaré hypothesis for $d=4$ we could conclude $n \geqq 11$ in the case $d=4$.

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