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Existence and uniqueness results for a third-order implicit differential equation☆

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ABSTRACT

The purpose of this paper is to give sufficient conditions for the existence and uniqueness of solutions to the boundary value problem of a third-order implicit differential equation as follows:

$$\begin{cases} f(t, u(t), u'(t), u'''(t)) = 0, \quad 0 \le t \le 1 \\ u(0) = u'(0) = u'(1) = 0. \end{cases}$$

The lower and upper solutions method, maximum principle together with an iterative technique are employed. An example illustrates the application of results obtained.

1. Introductions

Third-order differential equations arise in an important number of physical problems, such as the deflection of a curved beam having a constant or varying cross section, three layer beam, electromagnetic waves and gravity-driven flows [1]. Hence, third-order differential equations have attracted considerable attention over the last three decades, and many techniques for such problems have appeared, such as differential inequality [2,3], topological transversality [4], the shooting method [5], the lower and upper solutions method [6], comparable analysis with classical equations [7], the Lyapunov-Schmidt procedure and the continuum theory for O-epi maps [8].

Recently in [9], we used the upper and lower solutions method to prove some existence results of the following third-order two-point boundary value problem:

$$\begin{cases} u'''(t) + f(t, u(t)) = 0, \quad 0 \le t \le 1, \\ u(0) = u'(0) = u'(1) = 0. \end{cases}$$

Then in [10], we extended the study to the more general case as follows:

$$\begin{cases} u'''(t) + f(t, u(t), u'(t)) = 0, \quad 0 \le t \le 1, \\ u(0) = u'(0) = u'(1) = 0. \end{cases}$$

By the use of a new maximum principle and the upper and lower solutions method, we established the existence results of a solution and a positive solution for the above problem.

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Implicit systems of differential equations arise in a natural way in control theory. For that reason there is an extensive body of research on implicit systems of differential equations (see [11–15] and references therein). In this paper, we are concerned with an implicit third order two-point boundary value problem as follows:

$$\begin{cases} f(t, u(t), u'(t), u'''(t)) = 0, \quad 0 \le t \le 1, \\ u(0) = u'(0) = u'(1) = 0. \end{cases} \tag{IBVP}$$

It is known that the methods for the explicit equation cannot easily be extended and directly applied to the implicit equation. However, there still many methods, such as the constant rank theorem [11], the algebraic geometry method [12] and a differential-geometric approach [13], which have been used to deal with implicit equations. Motivated by the above important work and the study on fixed point of decreasing operators [16], in this paper, we try to establish some results concerning the existence and uniqueness of a solution and a positive solution for problem (IBVP) via the upper and lower solutions method, maximum principle and an iterative technique. These functional methods can be extended to implicit systems arising in control theory.

This paper is organized as follows: In Section 2 some notations and preliminaries are introduced. The existence and uniqueness results of problem (IBVP) are verified in Section 3. An example presented in the last section illustrates the application of results obtained.

2. Preliminaries

Definition 2.1. $u(t) \in C^3[0, 1]$ is called a positive solution if u is a solution of (IBVP) and $u(t) > 0$, $\forall t \in (0, 1)$.

Similarly, we can give the definition of a negative solution for (IBVP).

To obtain the main results, the following lemmas are crucial.

Lemma 2.2. *If* $m(t) \in C^3[0, 1]$ *satisfies*

(i) $m'''(t) \le 0$ $(0 < t < 1)$;
(ii) $m(0) \ge 0$, $m'(0) \ge 0$, $m'(1) \ge 0$.

Then $m'(t) \ge 0$, $m(t) \ge 0$, $\forall t \in [0, 1]$.

Proof. If $m'''(t) \le 0$ $(0 < t < 1)$, then $m'(t)$ is concave on $[0, 1]$. Hence for $\forall t \in [0, 1]$

$$m'(t) = m'((1-t) \times 0 + t \times 1) \ge (1-t)m'(0) + tm'(1) \ge 0.$$

Noting that $m(t)$ is increasing and $m(0) \ge 0$, we get $m(t) \ge 0$ for $t \in [0, 1]$ and $\max_{0 \le t \le 1} m(t) = m(1)$. ■

Corollary 2.3. *If* $\alpha, \beta \in C^3[0, 1]$ *satisfy*

(H) $\alpha'''(t) \le \beta'''(t)$, $t \in [0, 1]$, $\alpha(0) \ge 0 \ge \beta(0)$, $\alpha'(0) \ge 0 \ge \beta'(0)$, $\alpha'(1) \ge 0 \ge \beta'(1)$

then $\alpha'(t) \ge \beta'(t)$, $\alpha(t) \ge \beta(t)$ *for all* $t \in [0, 1]$.

Lemma 2.4. *If* $m(t) \in C^3[0, 1]$ *satisfies*

(i) $m'''(t) \le 0$ $(0 < t < 1)$;
(ii) $m(0) = m'(0) = m'(1) = 0$.

then $m(t) \ge t^2 \max_{0 \le t \le 1} m(t)$, $t \in [0, 1]$

Proof. In fact, let $g(t) = m(t) - t^2 m(1)$, then $g(0) = g(1) = g'(0) = 0$.

Thanks to Taylor's Theorem, there exists $\eta \in (0, 1)$ such that

$$m(1) = m(0) + m'(0) + \frac{m''(0)}{2} + \frac{m'''(\eta)}{6} \le \frac{m''(0)}{2},$$

hence $g''(0) = m''(0) - m(1) \ge 0$.

$g'''(t) = m'''(t) \le 0$ implies $g''(t)$ is nonincreasing on $[0, 1]$, then $g''(t) \ge 0$ on $[0, 1]$ or there exists $\delta \in [0, 1]$ such that $g''(t) \ge 0$ on $[0, \delta]$ and $g''(t) \le 0$ on $[\delta, 1]$.

(1) if $g''(t) \ge 0$ on $[0, 1]$, then $g(0) = g(1) = g'(0) = 0$ gives $g(t) \equiv 0$ on $[0, 1]$.

(2) if $g''(t) \ge 0$ on $[0, \delta]$, then $g(0) = g'(0) = 0$ implies $g(t) \ge 0$ on $[0, \delta]$; while $g''(t) \le 0$ means $g(t)$ is concave on $[\delta, 1]$, hence

$$g(t) = g\left(\frac{1-t}{1-\delta}\delta + \frac{t-\delta}{1-\delta}1\right) \ge \frac{1-t}{1-\delta}g(\delta) + \frac{t-\delta}{1-\delta}g(1) \ge (1-t)g(\delta) \ge 0$$

for arbitrary $t \in [\delta, 1]$.

From the above proof, we get $g(t) \ge 0$, i.e., $m(t) \ge t^2 m(1)$ for $t \in [0, 1]$. ■

3. Main results

Throughout this section, we assume that $f : [0,1] \times R \times R \times R \longrightarrow R$ is continuous. Let $u, v \in C[0,1]$ and $u(t) \le v(t)$, $t \in [0,1]$, denote

$$[u, v] = \{x \in C[0,1] | u(t) \le x(t) \le v(t), t \in [0,1]\}.$$

Theorem 3.1. *Assume there exist $\alpha, \beta \in C^3[0,1]$ satisfying condition* (H) *in Corollary 2.3 and a positive constant l such that*

(A1) $f(t, \beta(t), \beta'(t), \beta'''(t)) \ge -\frac{l}{2}(\beta'''(t) - \alpha'''(t)),\ t \in [0,1]$;
(A2) $f(t, \alpha(t), \alpha'(t), \alpha'''(t)) \le l(\beta'''(t) - \alpha'''(t)),\ t \in [0,1]$.
(A3) *For arbitrary* $t \in [0,1], u \in [\beta, \alpha], v \in [\beta', \alpha'], \beta'''(t) \ge w_1 \ge w_2 \ge \alpha'''(t)$,

$$f(t, u, v, w_1) - f(t, u, v, w_2) \le -l(w_1 - w_2).$$

(A4) *For arbitrary* $t \in [0,1], \alpha(t) \ge u_1 \ge u_2 \ge \beta(t), \alpha'(t) \ge v_1 \ge v_2 \ge \beta'(t), w \in [\alpha''', \beta''']$,

$$f(t, u_1, v_1, w) - f(t, u_2, v_2, w) \ge 0.$$

(A5) *For arbitrary* $t, \lambda \in [0,1], u_1, u_2 \in [\beta, \alpha], v_1, v_2 \in [\beta', \alpha'], w_1, w_2 \in [\alpha''', \beta''']$,

$$f(t, \lambda u_1 + (1-\lambda)u_2, \lambda v_1 + (1-\lambda)v_2, \lambda w_1 + (1-\lambda)w_2) \le \lambda f(t, u_1, v_1, w_1) + (1-\lambda) f(t, u_2, v_2, w_2).$$

Then the implicit problem (IBVP) has an unique solution $x^* \in C^3[0,1]$, which satisfies $\alpha'''(t) \le x^{*'''}(t) \le \beta'''(t)$, $t \in [0,1]$.

Proof. Problem (*IBVP*) is equivalent to the following boundary value problem

$$\begin{cases} u'''(t) = \dfrac{1}{l} f(t, u(t), u'(t), u'''(t)) + u'''(t), & 0 \le t \le 1 \\ u(0) = u'(0) = u'(1) = 0. \end{cases}$$

Let x_0 be the solution of the following boundary value problem:

$$\begin{cases} u'''(t) = \dfrac{1}{l} f(t, \beta(t), \beta'(t), \beta'''(t)) + \beta'''(t), & 0 \le t \le 1, \\ u(0) = u'(0) = u'(1) = 0, \end{cases}$$

then

$$x_0 = \int_0^1 G(t,s) \left[\frac{1}{l} f(s, \beta(s), \beta'(s), \beta'''(s)) + \beta'''(s) \right] \mathrm{d}s,$$

$$\text{where } G(t,s) = \begin{cases} \dfrac{1}{2}(t-s)^2 - \dfrac{1}{2}t^2(1-s), & 0 \le s \le t \le 1, \\ -\dfrac{1}{2}t^2(1-s), & 0 \le t \le s \le 1. \end{cases}$$

Construct iterative sequences $\{x_n\}$ as follows:

$$\begin{cases} x'''_{n+1}(t) = \dfrac{1}{l} f(t, x_n(t), x'_n(t), x'''_n(t)) + x'''_n(t), \\ x_{n+1}(0) = x'_{n+1}(0) = x'_{n+1}(1) = 0. \\ n = 0, 1, 2, \ldots. \end{cases}$$

Since $\beta \in C^3[0,1]$ and f is a continuous function, $\{x_n\}$ are well-defined and $\{x_n\} \subset C^3[0,1]$.

We shall show that $\{x_n\}$ converges to the unique solution of Problem (*IBVP*) in $\{u \in C^3[0,1] | \alpha'''(t) \le u'''(t) \le \beta'''(t)\}$. The proof is given in several steps.

Conclusion 1.

$$\begin{cases} \alpha'''(t) \le \dfrac{\alpha'''(t) + \beta'''(t)}{2} \le x'''_0(t) \le \beta'''(t), \\ \alpha'(t) \ge x'_0(t) \ge \beta'(t), \\ \alpha(t) \ge x_0(t) \ge \beta(t). \end{cases}$$

In virtue of the assumptions (A3), (A4), we have

$$\frac{1}{l} f(t, \beta(t), \beta'(t), \beta'''(t)) + \beta'''(t) \le \frac{1}{l} f(t, \alpha(t), \alpha'(t), \alpha'''(t)) + \alpha'''(t).$$

Then, by the condition (A1), (A2) and the definition of x_0, it is easy to get

$$\begin{aligned}\alpha'''(t) &\le \frac{\alpha'''(t)+\beta'''(t)}{2}\\ &\le x_0'''(t) = \frac{1}{l}f(t,\beta(t),\beta'(t),\beta'''(t))+\beta'''(t)\\ &\le \frac{1}{l}f(t,\alpha(t),\alpha'(t),\alpha'''(t))+\alpha'''(t)\\ &\le \beta'''(t).\end{aligned}$$

Due to the boundary condition

$$\alpha(0)\ge 0 = x_0(0)\ge \beta(0),\qquad \alpha'(0)\ge 0 = x_0'(0)\ge\beta'(0),\qquad \alpha'(1)\ge 0 = x_0'(1)\ge \beta'(1),$$

and Lemma 2.2, we get

$$\alpha'(t)\ge x_0'(t)\ge\beta'(t),\qquad \alpha(t)\ge x_0(t)\ge\beta(t).$$

Conclusion 2. For $n = 0, 1, 2, \ldots,$

$$\begin{cases}\alpha'''(t)\le \dfrac{\alpha'''(t)+\beta'''(t)}{2}\le x_{2n}'''(t)\le x_{2n+1}'''(t)\le\beta'''(t),\\ \alpha'(t)\ge x_{2n}'(t)\ge x_{2n+1}'(t)\ge\beta'(t),\\ \alpha(t)\ge x_{2n}(t)\ge x_{2n+1}(t)\ge\beta(t).\end{cases}$$

We shall prove the conclusion via the mathematical induction method.

In fact, since

$$x_1'''(t) = \frac{1}{l}f(t,x_0(t),x_0'(t),x_0'''(t))+x_0'''(t).$$

Conclusion 1 gives

$$\frac{1}{l}f(t,\beta(t),\beta'(t),\beta'''(t))+\beta'''(t)\le x_1'''(t)\le\frac{1}{l}f(t,\alpha(t),\alpha'(t),\alpha'''(t))+\alpha'''(t).$$

So we have

$$\alpha'''(t)\le\frac{\alpha'''(t)+\beta'''(t)}{2}\le x_0'''(t)\le x_1'''(t)\le\beta'''(t).$$

Thanks to Lemma 2.2, there hold

$$\begin{cases}\alpha'(t)\ge x_0'(t)\ge x_1'(t)\ge\beta'(t),\\ \alpha(t)\ge x_0(t)\ge x_1(t)\ge\beta(t),\end{cases}$$

which implies Conclusion 2 holds for $n = 0$.

Assume the conclusion holds for $n = k$ either, i.e.,

$$\begin{cases}\alpha'''(t)\le \dfrac{\alpha'''(t)+\beta'''(t)}{2}\le x_{2k}'''(t)\le x_{2k+1}'''(t)\le\beta'''(t),\\ \alpha'(t)\ge x_{2k}'(t)\ge x_{2k+1}'(t)\ge\beta'(t),\\ \alpha(t)\ge x_{2k}(t)\ge x_{2k+1}(t)\ge\beta(t).\end{cases}$$

Then due to assumptions (A3), (A4), we get for $t\in[0,1]$

$$\begin{aligned}\frac{1}{l}f(t,\alpha(t),\alpha'(t),\alpha'''(t))+\alpha'''(t) &\ge x_{2k+1}'''(t)\\ &= \frac{1}{l}f(t,x_{2k}(t),x_{2k}'(t),x_{2k}'''(t))+x_{2k}'''(t)\\ &\ge \frac{1}{l}f(t,x_{2k+1}(t),x_{2k+1}'(t),x_{2k+1}'''(t))+x_{2k+1}'''(t)\\ &= x_{2k+2}'''(t)\\ &\ge \frac{1}{l}f(t,\beta(t),\beta'(t),\beta'''(t))+\beta'''(t),\end{aligned}$$

which means

$$\begin{cases} \alpha'''(t) \le \dfrac{\alpha'''(t)+\beta'''(t)}{2} \le x'''_{2k+2}(t) \le x'''_{2k+1}(t) \le \beta'''(t), \\ \alpha'(t) \ge x'_{2k+2}(t) \ge x'_{2k+1}(t) \ge \beta'(t), \\ \alpha(t) \ge x_{2k+2}(t) \ge x_{2k+1}(t) \ge \beta(t). \end{cases}$$

Repeating this process we obtain

$$\begin{cases} \alpha'''(t) \le \dfrac{\alpha'''(t)+\beta'''(t)}{2} \le x'''_{2k+2}(t) \le x'''_{2k+3}(t) \le \beta'''(t), \\ \alpha'(t) \ge x'_{2k+2}(t) \ge x'_{2k+3}(t) \ge \beta'(t), \\ \alpha(t) \ge x_{2k+2}(t) \ge x_{2k+3}(t) \ge \beta(t), \end{cases}$$

which implies the conclusion holds for $n = k + 1$.

Hence Conclusion 2 holds for all $n \in \{0\} \bigcup N$.

Conclusion 3. $\{x'''_{2n}(t)\}$ is increasing, while $\{x'''_{2n+1}(t)\}$ is decreasing.

In fact, by the proof of Conclusion 2, we know that

$$x'''_1(t) \le \beta'''(t), \qquad x'_1(t) \ge \beta'(t), \qquad x_1(t) \ge \beta(t),$$

then due to properties (i–ii) of f, we obtain

$$x'''_0 = \frac{1}{l} f(t, \beta(t), \beta'(t), \beta'''(t)) + \beta'''(t) \le \frac{1}{l} f(t, x_1(t), x'_1(t), x'''_1(t)) + x'''_1(t) = x'''_2(t).$$

By a mathematical induction process similarly to the proof of Conclusion 2, we can verify that $\{x'''_{2n}(t)\}$ is increasing. The decreasing of $\{x'''_{2n+1}(t)\}$ can be proved in the same manner.

Conclusion 4. $\{x_n(t)\}$ is convergent to the solution of problem *(IBVP)*.

Conclusions 1–3 tell us that for $n = 0, 1, 2, \ldots$,

$$\alpha'''(t) \le \frac{\alpha'''(t)+\beta'''(t)}{2} \le x'''_{2n}(t) \le x'''_{2n+2}(t) \le x'''_{2n+3}(t) \le x'''_{2n+1}(t) \le \beta'''(t).$$

Let $y_n(t) = x_n(t) - \alpha(t)$, then for $n = 0, 1, 2, \ldots$,

$$0 \le \frac{\beta'''(t)-\alpha'''(t)}{2} \le y'''_{2n}(t) \le y'''_{2n+2}(t) \le y'''_{2n+3}(t) \le y'''_{2n+1}(t) \le \beta'''(t) - \alpha'''(t).$$

Let $r_n = \sup\{r \in R | y'''_{2n}(t) \ge r y'''_{2n+1}(t)\}$, then $\{r_n\}$ are well-defined. It is easy to check that $\frac{1}{2} \le r_n \le 1$ and $\{r_n\}$ is increasing, so $\{r_n\}$ is convergent.

Noting assumption (A5), we have

$$\begin{aligned} y'''_{2n+3}(t) &\le y'''_{2n+1}(t) \\ &= x'''_{2n+1}(t) - \alpha'''(t) \\ &= \frac{1}{l} f(t, x_{2n}(t), x'_{2n}(t), x'''_{2n}(t)) + x'''_{2n}(t) - \alpha'''(t) \\ &= \frac{1}{l} f(t, (y_{2n}+\alpha)(t), (y_{2n}+\alpha)'(t), (y_{2n}+\alpha)'''(t)) + (y_{2n}+\alpha)'''(t) - \alpha'''(t) \\ &\le \frac{1}{l} f(t, (r_n y_{2n+1}+\alpha)(t), (r_n y_{2n+1}+\alpha)'(t), (r_n y_{2n+1}+\alpha)'''(t)) + r_n y'''_{2n+1}(t) \\ &= \frac{1}{l} f(t, r_n(x_{2n+1}-\alpha)(t) + \alpha(t), r_n(x_{2n+1}-\alpha)'(t) + \alpha'(t), r_n(x_{2n+1}-\alpha)'''(t) \\ &\quad + \alpha'''(t)) + r_n(x_{2n+1}-\alpha)'''(t) \\ &\le \frac{1}{l} [r_n f(t, x_{2n+1}(t), x'_{2n+1}(t), x'''_{2n+1}(t)) + (1-r_n) f(t, \alpha(t), \alpha'(t), \alpha'''(t))] + r_n x'''_{2n+1}(t) - r_n \alpha'''(t) \\ &= r_n x'''_{2n+2}(t) + (1-r_n)\left[\frac{1}{l} f(t, \alpha(t), \alpha'(t), \alpha'''(t)) + \alpha'''(t)\right] - \alpha'''(t) \\ &\le r_n (y'''_{2n+2}(t) + \alpha'''(t)) + (1-r_n)\beta'''(t) - \alpha'''(t) \\ &\le r_n y'''_{2n+2}(t) + 2(1-r_n) y'''_{2n+2}(t) \\ &= (2-r_n) y'''_{2n+2}(t), \end{aligned}$$

then we have

$$r_{n+1} = \sup\{r \in R | y'''_{2n+3}(t) \geq r y'''_{2n+2}(t)\} \geq \frac{1}{2 - r_n}.$$

Let $\lim_{n\to\infty} r_n = r$, the above inequality implies $r \geq \frac{1}{2-r}$, then $r = 1$.

For arbitrary even number $p > 0$, we have

$$0 \leq y'''_{2n+p} - y'''_{2n} \leq y'''_{2n+1} - y'''_{2n} \leq (1 - r_n) y'''_{2n+1} \leq (1 - r_n)(\beta''' - \alpha''').$$

Since $r_n \to 1$, then $\{y'''_{2n}\}$ is a Cauchy sequence in $C[0, 1]$. Similarly, we can prove $\{y'''_{2n+1}\}$ is a Cauchy sequence in $C[0, 1]$ and $\lim_{n\to\infty} y'''_{2n} = \lim_{n\to\infty} y'''_{2n+1}$, so $\{y'''_n\}$ is convergent.

Let $y^* = \lim_{n\to\infty} y'''_n$ and x^* be the solution of the following boundary value problem

$$\begin{cases} u'''(t) = y^*(t) + \alpha'''(t), \\ u(0) = u'(0) = u'(1) = 0, \end{cases}$$

then it is easy to check that

$$\lim_{n\to\infty} x_n^{(k)}(t) = x^{*(k)}(t), \quad k = 0, 1, 3.$$

Hence we obtain

$$\begin{aligned} x^{*'''}(t) &= \lim_{n\to\infty} x'''_{n+1}(t) \\ &= \lim_{n\to\infty} \frac{1}{l} f(t, x_n(t), x'_n(t), x'''_n(t)) + x'''_n(t) \\ &= \frac{1}{l} f(t, x^*(t), x^{*'}(t), x^{*'''}(t)) + x^{*'''}(t). \end{aligned}$$

Which implies x^* is the solution of problem (IBVP).

Conclusion 5. x^* is the unique solution of problem (IBVP) in $\{u \in C^3[0, 1] | \alpha'''(t) \leq u'''(t) \leq \beta'''(t)\}$.

Let $v \in C^3[0, 1]$ be a solution of (IBVP) satisfying $\alpha'''(t) \leq v'''(t) \leq \beta'''(t)$, then

$$\begin{aligned} \frac{1}{l} f(t, \beta, \beta'(t), \beta'''(t)) + \beta'''(t) &\leq \frac{1}{l} f(t, v(t), v'(t), v'''(t)) + v'''(t) \\ &\leq \frac{1}{l} f(t, \alpha(t), \alpha'(t), \alpha'''(t)) + \alpha'''(t). \end{aligned}$$

which means $x'''_0(t) \leq v'''(t) \leq \beta'''(t)$.

Hence $x'_0(t) \geq v'(t) \geq \beta'(t)$, $x_0(t) \geq v(t) \geq \beta(t)$ and

$$\begin{aligned} \frac{1}{l} f(t, \beta(t), \beta'(t), \beta'''(t)) + \beta'''(t) &\leq \frac{1}{l} f(t, v(t), v'(t), v'''(t)) + v'''(t) \\ &\leq \frac{1}{l} f(t, x_0(t), x'_0(t), x'''_0(t)) + x'''_0(t). \end{aligned}$$

i.e. $x'''_0(t) \leq v'''(t) \leq x'''_1(t)$.

By use of the mathematical induction method, it is easy to verify

$$x'''_{2n}(t) \leq v'''(t) \leq x'''_{2n+1}(t), \quad n = 0, 1, 2, \ldots.$$

Then

$$v'''(t) = \lim_{n\to\infty} x'''_n(t) = x^{*'''}(t),$$

which implies $v(t) = x^*(t)$.

This ends the proof. ■

Remark. In this theorem, we constructed an iterative sequence initialed by β, and verified that this sequence is convergent to a solution of problem (IBVP). Other assumptions assured the uniqueness of this solution. Our uniqueness result is a local result, which means that the problem may have multiple solutions in $C[0, 1]$.

Now let us consider the special case when

$$f(t, u, v, w) = f_1(t, u) + f_1(t, v) + f_3(t, w).$$

Thanks to Theorem 3.1, we can get the following useful results:

Corollary 3.2. *If there exist positive constants $c > 0, l > 0$ such that*

(1) $\min_{0\le t\le 1} f(t,0,0,0) \ge \frac{-lc}{2}$, $\max_{0\le t\le 1} f(t, \frac{1}{12}c, \frac{1}{8}c, -c) \le lc$;
(2) $f_1(t,.)$ *is convex and nondecreasing on* $[0, \frac{1}{12}c]$ *for* $\forall t \in [0,1]$;
(3) $f_2(t,.)$ *is convex and nondecreasing on* $[0, \frac{1}{8}c]$ *for* $\forall t \in [0,1]$;
(4) $f_3(t,.)$ *is convex on* $[-c, 0]$ *for* $\forall t \in [0,1]$;
(5) *for arbitrary* $t \in [0,1]$, $0 \ge w_1 \ge w_2 \ge -c$

$$f_3(t,w_1) - f_3(t,w_2) \le -l(w_1 - w_2).$$

Then (IBVP) has a unique solution u^* satisfying $0 \ge u^{*'''}(t) \ge -c$ for $t \in [0,1]$. In addition, if there exists $t_0 \in [0,1]$ such that $f(t_0,0,0,0) \ne 0$, then u^* is a positive solution.

Proof. Under assumptions of Corollary 3.2, let $\beta(t) \equiv 0$, $\alpha(t) = (\frac{t^2}{4} - \frac{t^3}{6})c$, then it is easy to compute

(i) $\beta(t) = \beta'(t) \equiv 0, t \in [0,1]$;
(ii) $\alpha'(t) = \frac{c}{2}(t - t^2), \alpha'''(t)(t) = -c, \alpha(t) \ge 0, \alpha'(t) \ge 0, t \in [0,1]$;
(iii) $\alpha(0) = \alpha'(0) = \alpha'(1) = 0, \max_{0\le t\le 1}\alpha(t) = \frac{1}{12}c, \max_{0\le t\le 1}\alpha'(t) = \frac{1}{8}c$.

We can check that the conditions (A1)–(A5) of Theorem 3.1 are satisfied under the hypothesis of Corollary 3.2. Thus an application of Theorem 3.1 asserts the existence of the unique solution u^* in $\{u \in C^3[0,1] \mid \alpha'''(t) \le u'''(t) \le \beta'''(t), t \in [0,1]\}$.

Since $(u^*)'''(t) \le \beta'''(t) = 0, t \in [0,1]$, then Lemma 2.2 implies $u^{*'} \ge 0$ and $u^* \ge 0$, i.e., u^* is nonnegative and increasing. Assume there exists $t_0 \in (0,1]$ such that $f(t_0,0,0,0) \ne 0$, then $u^*(t)$ is not identically zero in $[0,1]$, hence $u^*(1) = \max_{0\le t\le 1} u^*(t) > 0$. Lemma 2.4 implies $u^*(t) > 0, t \in (0,1)$. ■

The proof of the following corollaries follows the same pattern.

Corollary 3.3. *If there exist positive constants $c > 0, l > 0$ such that*

(1) $\max_{0\le t\le 1} f(t,0,0,0) \le lc$, $\min_{0\le t\le 1} f(t, -\frac{1}{12}c, -\frac{1}{8}c, c) \ge \frac{-lc}{2}$;
(2) $f_1(t,.)$ *is convex and nondecreasing on* $[0, \frac{1}{12}c]$ *for* $\forall t \in [0,1]$;
(3) $f_2(t,.)$ *is convex and nondecreasing on* $[0, \frac{1}{8}c]$ *for* $\forall t \in [0,1]$;
(4) $f_3(t,.)$ *is convex on* $[-c, 0]$ *for* $\forall t \in [0,1]$;
(5) *for arbitrary* $t \in [0,1]$, $0 \ge w_1 \ge w_2 \ge -c$

$$f_3(t,w_1) - f_3(t,w_2) \le -l(w_1 - w_2).$$

Then (IBVP) has a unique solution u^* satisfying $0 \le u^{*'''}(t) \le c$ for $t \in [0,1]$. In addition, if there exists $t_0 \in [0,1]$ such that $f(t_0,0,0,0) \ne 0$, then u^* is a negative solution.

Corollary 3.4. *If there exist positive constants $c > 0, l > 0$ such that*

(1) $\max_{0\le t\le 1} f(t, \frac{1}{12}c, \frac{1}{8}c, -c) \le 2lc$, $\min_{0\le t\le 1} f(t, -\frac{1}{12}c, -\frac{1}{8}c, c) \ge lc$;
(2) $f_1(t,.)$ *is convex and nondecreasing on* $[-\frac{1}{12}c, \frac{1}{12}c]$ *for* $\forall t \in [0,1]$;
(3) $f_2(t,.)$ *is convex and nondecreasing on* $[-\frac{1}{8}c, \frac{1}{8}c]$ *for* $\forall t \in [0,1]$;
(4) $f_3(t,.)$ *is convex on* $[-c, c]$ *for* $\forall t \in [0,1]$;
(5) *for arbitrary* $t \in [0,1]$, $c \ge w_1 \ge w_2 \ge -c$

$$f_3(t,w_1) - f_3(t,w_2) \le -l(w_1 - w_2).$$

Then (IBVP) has a unique solution u^* such that $-c \le u^{*'''}(t) \le c$ for $t \in [0,1]$.

4. Example

Let $c = 1, l = 2, f_1(t,u) = tu^2, f_2(t,v) = v^4, f_3(t,w) = e^{-w-3} - 2w - 1$, then the conditions (1)–(5) of Corollary 3.2 are satisfied.

Thus an application of Corollary 3.2 asserts the following problem

$$\begin{cases} t(u(t))^2 + (u'(t))^4 + e^{-u'''(t)-3} - 2u'''(t) - 1 = 0, & 0 \le t \le 1, \\ u(0) = u'(0) = u'(1) = 0, \end{cases}$$

has a unique solution u^* satisfies $-1 \le u^{*'''}(t) \le 0, t \in [0,1]$. Furthermore, $f(t,0,0,0) \ne 0$ shows that $u^*(t)$ is a positive solution.

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