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Positive solutions for three-point boundary value problem on the half-line[☆]

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Abstract

In this paper we consider the existence of positive solutions for the following boundary value problem on the half-line

$$\begin{cases}(\rho(t)x'(t))' + f(t, x(t), x'(t)) = 0, & t \in [0, +\infty),\\ x(0) = \alpha x(\xi), & \lim\limits_{t\to\infty} x(t) = 0.\end{cases}$$

By applying fixed-point theorems, we obtain a variety of existence results. In particular, the nonlinear term is involved with the first-order derivative.

Keywords: Three-point boundary value problem; Half-line; Positive solutions; Existence criteria; Green's function

1. Introduction

This paper is concerned with the existence of solutions to three-point boundary value problem

$$\begin{cases}(\rho(t)x'(t))' + f(t, x(t), x'(t)) = 0, & t \in I = [0, +\infty),\\ x(0) = \alpha x(\xi), & \lim\limits_{t\to\infty} x(t) = 0,\end{cases} \tag{1.1}$$

where $\rho \in C[0, +\infty) \cap C^1(0, +\infty)$, $\rho(t) > 0$ for $t \in [0, +\infty)$, $\int_0^\infty \frac{1}{\rho(t)} dt < \infty$, $\alpha \geq 0$, $0 \leq \xi < \infty$, $f : I \times I \times R \to I$.

Definition 1.1. A function $x \in C^2(I, I)$ is said to be a positive solution of boundary value problem (1.1), if $x(t) \geq 0$, and x satisfies (1.1) for $t \in I$.

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The motivation for the present work stems from both practical and theoretical aspects. In fact, boundary value problems (BVPs) on the half-line occur naturally in the study of radially symmetric solutions of nonlinear elliptic equations, see [5,10], and various physical phenomena [2,8], such as unsteady flow of gas through a semi-infinite porous media, the theory of drain flows, plasma physics, in determining the electrical potential in an isolated neutral atom. In all these applications, it is frequent that only solutions that are positive are useful. Agarwal and O'Regan studied infinite interval problems for differential, difference and integral equations and obtained a series of results [1]. Recently there have been many papers that investigated the positive solutions of boundary value problems on the half-line, see [3,12–14]. Zima [14] studied the existence of at least one positive solution to the following boundary value problem on the half-line:

$$\begin{cases} x''(t) - k^2 x(t) + f(t, x(t)) = 0, & t \in [0, \infty), \\ x(0) = 0, & \lim_{t\to\infty} x(t) = 0, \end{cases} \tag{1.2}$$

where $k > 0$ and f is a continuous, nonnegative function. Most linear assumption is imposed on nonlinearity f, that is,

$$f(t, x) \le a(t) + b(t)x, \quad t, x \in [0, \infty)$$

where $a, b : [0, \infty) \to [0, \infty)$ are nonnegative continuous functions. Yan [12] considered the following two-point boundary value problem

$$\begin{cases} \dfrac{1}{p(t)}(p(t)x'(t))' + f(t, x(t)) = 0, & t \in [0, \infty), \\ x(0) = a \ge 0, & \lim_{t\to\infty} p(t)x'(t) = b \ge 0. \end{cases}$$

By utilizing fixed-point index theory, the existence of multiple unbounded solutions was obtained. The assumptions of superlinear at 0 and ∞ are imposed on nonlinearity, that is

$$f(t, y) \ge q_0(t)\Psi(y)$$

and $\lim_{y\to+\infty} \frac{\Psi(y)}{y} = +\infty$, $\lim_{y\to 0^+} \frac{\Psi(y)}{y} = +\infty$, $q_0 \in C((0, +\infty), [0, +\infty))$. Bai and Fang [3] applied the fixed-point theorem on cone to obtain the positive solutions for the following boundary value problem on infinite interval for second-order functional differential equations:

$$\begin{cases} x''(t) - px'(t) - qx(t) + f(t, x(t)) = 0, & t \in [0, \infty), \\ \alpha x(0) - \beta x'(0) = \xi(t), \quad t \in [-\tau, 0] & \lim_{t\to\infty} x(t) = 0, \end{cases}$$

here $x(t) \in C([-\tau, 0], I)$. Most linear assumption is imposed on nonlinearity f.

From the above results, we can see three problems: (i) only two-point BVPs on the half-line were studied; (ii) nonlinearity is only dependent on t, x; (iii) the assumptions imposed on nonlinearities are always superlinear, sublinear or most linear.

It is well known that the study of multi-point BVPs is very important. For finite interval, there are many results, see [6,7,9]. So it is necessary to discuss the existence of the multi-point boundary value problems on the half-line. We transform BVP into the integral equations to establish the existence results for solutions. In particular, the corresponding Green's function and some useful inequalities are obtained. Nonlinearity f is dependent on the first derivative, which brings about much trouble, such as, the verification of the compactness and continuity of the operator and norm of Banach space. In Section 4, we apply fixed-point theory [4] and obtain the existence of triple positive solutions. The restriction on nonlinear term f is different from superlinear, sublinear or most linear assumptions, which are always imposed on the nonlinearity f in the literature [1,3,12–14]. This is the first time that fixed-point theory [4] is applied to BVP on the half-line. The assumptions on f are novel and not seen before in the literature.

2. Related lemmas

Lemma 2.1 (*Nonlinear Alternative, See [11]*). *Let C be a convex subset of a normed linear space E, and U be an open subset of C, with $p^* \in U$. Then every compact, continuous map $N : \overline{U} \to C$ has at least one of the following two properties:*

(a) *N has a fixed point;*
(b) *there is an $x \in \partial U$, with $x = (1-\overline{\lambda})p^* + \overline{\lambda}Nx$ for some $0 < \overline{\lambda} < 1$.*

Definition 2.1. The map ψ is said to be a nonnegative continuous concave functional on cone P provided that $\psi : P \to [0, \infty)$ is continuous and

$$\psi(tx + (1-t)y) \geq t\psi(x) + (1-t)\psi(y)$$

for all $x, y \in P$ and $0 \leq t \leq 1$. Similarly, we say the map α is a nonnegative continuous convex functional on P provided that: $\alpha : P \to [0, \infty)$ is continuous and

$$\alpha(tx + (1-t)y) \leq t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in P$ and $0 \leq t \leq 1$.

Let $r > a > 0, L > 0$ be constants, ψ is a nonnegative continuous concave functional and α, β nonnegative continuous convex functionals on the cone P. Define convex sets

$$P(\alpha, r; \beta, L) = \{y \in P | \alpha(y) < r, \beta(y) < L\},$$
$$\overline{P}(\alpha, r; \beta, L) = \{y \in P | \alpha(y) \leq r, \beta(y) \leq L\},$$
$$P(\alpha, r; \beta, L; \psi, a) = \{y \in P | \alpha(y) < r, \beta(y) < L, \psi(y) > a\},$$
$$\overline{P}(\alpha, r; \beta, L; \psi, a) = \{y \in P | \alpha(y) \leq r, \beta(y) \leq L, \psi(y) \geq a\}.$$

The following assumptions about the nonnegative continuous convex functionals α, β will be used:

(A1) there exists $M > 0$ such that $\|x\| \leq M \max\{\alpha(x), \beta(x)\}$, for all $x \in P$;
(A2) $P(\alpha, r; \beta, L) \neq \emptyset$ for all $r > 0, L > 0$.

Lemma 2.2 (*Bai and Ge [4]*). *Let E be a Banach space, $P \subset E$ a cone and $r_2 \geq d > b > r_1 > 0, L_2 \geq L_1 > 0$. Assume that α, β are nonnegative continuous convex functionals satisfying* (A1) *and* (A2)*, ψ is a nonnegative continuous concave functional on P such that $\psi(y) \leq \alpha(y)$ for all $y \in \overline{P}(\alpha, r_2; \beta, L_2)$, and $T : \overline{P}(\alpha, r_2; \beta, L_2) \to \overline{P}(\alpha, r_2; \beta, L_2)$ is a completely continuous operator. Suppose*

(B1) $\{y \in P(\alpha, d; \beta, L_2; \psi, b) | \psi(y) > b\} \neq \emptyset$, $\psi(Ty) > b$ *for $y \in \overline{P}(\alpha, d; \beta, L_2; \psi, b)$;*
(B2) *$\alpha(Ty) < r_1, \beta(Ty) < L_1$ for all $y \in \overline{P}(\alpha, r_1; \beta, L_1)$;*
(B3) *$\psi(Ty) > b$ for all $y \in \overline{P}(\alpha, r_2; \beta, L_2; \psi, b)$ with $\alpha(Ty) > d$.*

Then T has at least three fixed points y_1, y_2 and y_3 in $\overline{P}(\alpha, r_2; \beta, L_2)$ with

$$y_1 \in P(\alpha, r_1; \beta, L_1), \qquad y_2 \in \{\overline{P}(\alpha, r_2; \beta, L_2; \psi, b) | \psi(y) > b\}$$

and

$$y_3 \in \overline{P}(\alpha, r_2; \beta, L_2) \setminus (\overline{P}(\alpha, r_2; \beta, L_2; \psi, b) \cup \overline{P}(\alpha, r_1; \beta, L_1)).$$

Definition 2.2. We say $f : I \times I \times R \to [0, +\infty)$ is an L^1-Carathédory function if

(i) $t \to f(t, x, y)$ is measurable for any $(x, y) \in R \times R$,
(ii) $(x, y) \to f(t, x, y)$ is continuous for a.e. $t \in I$,
(iii) for each $r_1, r_2 > 0$ there exists $l_{r_1,r_2} \in L^1[0, \infty)$ such that $|x| \leq r_1, |y| \leq r_2$ implies $|f(t, x, y)| \leq l_{r_1,r_2}(t)$ for almost all $t \in I$.

For convenience, we denote $v(t) = \int_t^\infty \frac{1}{\rho(s)} \mathrm{d}s$.

Lemma 2.3. *Assume that $\sigma \in C(I, I)$ with $\int_0^\infty \sigma(s) \mathrm{d}s < \infty$. Then $x \in C^2(I)$ is a solution of the boundary value problem*

$$\begin{cases} (\rho(t)x'(t))' + \sigma(t) = 0, \quad t \in I = [0, +\infty), \\ x(0) = \alpha x(\xi), \qquad \lim\limits_{t\to\infty} x(t) = 0, \end{cases} \tag{2.1}$$

if and only if $x \in C(I)$ *is a solution of the following integral equation*

$$x(t) = \int_0^{\infty} G(t,s)\sigma(s)\mathrm{d}s, \quad t \in I, \tag{2.2}$$

where

$$G(t,s) = \begin{cases} \dfrac{v(t)(v(0)-v(s))}{v(0)-\alpha v(\xi)}, & s \le t, s \le \xi; \\ v(s) - \dfrac{v(t)[v(s)-\alpha v(\xi)]}{v(0)-\alpha v(\xi)}, & t \le s \le \xi; \\ v(t) - \dfrac{(1-\alpha)v(t)v(s)}{v(0)-\alpha v(\xi)}, & \xi \le s \le t; \\ v(s)\left[1 - \dfrac{(1-\alpha)v(t)}{v(0)-\alpha v(\xi)}\right], & s \ge t, s \ge \xi. \end{cases}$$

Proof. If $x \in C(I)$ is a solution of (2.1), integrating equation in (2.1) from 0 to t, one has

$$\rho(t)x'(t) = \rho(0)x'(0) - \int_0^t \sigma(s)\mathrm{d}s. \tag{2.3}$$

Dividing $\rho(t)$ on both sides of (2.3) and integrating from t to ∞, one has

$$x(t) = -\int_t^{\infty} \frac{1}{\rho(s)}\left(\rho(0)x'(0) - \int_0^s \sigma(\theta)\mathrm{d}\theta\right)\mathrm{d}s. \tag{2.4}$$

Boundary value condition $x(0) = \alpha x(\xi)$ means

$$\int_0^{\infty} \frac{1}{\rho(s)}\left(\rho(0)x'(0) - \int_0^s \sigma(\theta)\mathrm{d}\theta\right)\mathrm{d}s = \alpha\int_{\xi}^{\infty} \frac{1}{\rho(s)}\left(\rho(0)x'(0) - \int_0^s \sigma(\theta)\mathrm{d}\theta\right)\mathrm{d}s.$$

Thus

$$\rho(0)x'(0) = \frac{1}{v(0)-\alpha v(\xi)}\left[\int_0^{\infty}\left(\frac{1}{\rho(s)}\int_0^s \sigma(\theta)\mathrm{d}\theta\right)\mathrm{d}s - \alpha\int_{\xi}^{\infty}\left(\frac{1}{\rho(s)}\int_0^s \sigma(\theta)\mathrm{d}\theta\right)\mathrm{d}s\right]. \tag{2.5}$$

Substituting (2.5) into (2.4) we have

$$\begin{aligned} x(t) &= -\frac{v(t)}{v(0)-\alpha v(\xi)}\left[\int_0^{\infty}\left(\frac{1}{\rho(s)}\int_0^s \sigma(\theta)\mathrm{d}\theta\right)\mathrm{d}s - \alpha\int_{\xi}^{\infty}\left(\frac{1}{\rho(s)}\int_0^s \sigma(\theta)\mathrm{d}\theta\right)\mathrm{d}s\right] \\ &\quad + \int_t^{\infty}\left(\frac{1}{\rho(s)}\int_0^s \sigma(\theta)\mathrm{d}\theta\right)\mathrm{d}s \\ &= \left[1 - \frac{(1-\alpha)v(t)}{v(0)-\alpha v(\xi)}\right]\int_0^{\infty}\left(\frac{1}{\rho(s)}\int_0^s \sigma(\theta)\mathrm{d}\theta\right)\mathrm{d}s - \frac{\alpha v(t)}{v(0)-\alpha v(\xi)}\int_0^{\xi}\left(\frac{1}{\rho(s)}\int_0^s \sigma(\theta)\mathrm{d}\theta\right)\mathrm{d}s \\ &\quad - \int_0^{t}\left(\frac{1}{\rho(s)}\int_0^s \sigma(\theta)\mathrm{d}\theta\right)\mathrm{d}s. \end{aligned}$$

By integration by parts, (2.2) holds.

If $x \in C(I)$ is a solution of integral equation (2.2), then it is easy to know from the following Lemma 2.4 and condition $\int_0^{\infty}\sigma(s)\mathrm{d}s < \infty$ that $x \in C^2(I)$ is a solution of problem (2.1). □

Lemma 2.4. *If* $v(0) > \alpha v(\xi)$ *and* $G(t,s)$ *is given in Lemma 2.3, then the following inequalities hold:*

(a) $0 \le G(t,s) \le \Delta$, $(t,s) \in I \times I$, *where* $\Delta = \frac{(1+\alpha)[v(0)]^2}{v(0)-\alpha v(\xi)}$;
(b) $\lim_{t\to\infty} G(t,s) = 0$ *for* $s \in I$;
(c) $\left|\frac{\partial G(t,s)}{\partial t}\right| \le \frac{\Gamma}{\rho(t)}$ *and* $\lim_{t\to\infty}\frac{\partial G(t,s)}{\partial t} = 0$, *where* $\Gamma = (3+\alpha)\frac{v(0)}{v(0)-\alpha v(\xi)}$.
Besides, if $0 \le \alpha < 1$, *then*

(d) $\gamma G(s,s) \le G(t,s) \le G(s,s)$ *for* $(t,s) \in [l_1, l_2] \times I$, $0 < l_1 < \xi < l_2 < \infty$, *here*

$$0 < \gamma = \min\left\{\frac{v(l_2)}{v(0)}, \frac{v(\xi)[v(0)-v(l_1)]}{v(0)[v(0)-v(\xi)]}, \frac{v(l_2)}{v(\xi)}, 1 - \frac{(1-\alpha)v(l_1)}{v(0)-\alpha v(\xi)}\right\} < 1.$$

Proof. If $v(0) > \alpha v(\xi)$. Now we will show that (a) holds.

For $s \le t$, $s \le \xi$,

$$0 \le G(t,s) = \frac{v(t)[v(0)-v(s)]}{v(0)-\alpha v(\xi)} \le \frac{[v(0)]^2}{v(0)-\alpha v(\xi)}.$$

For $t \le s \le \xi$,

$$\begin{aligned} G(t,s) &= v(s) - \frac{v(t)[v(s)-\alpha v(\xi)]}{v(0)-\alpha v(\xi)} \\ &= \frac{v(s)[v(0)-v(t)] + \alpha v(\xi)[v(t)-v(s)]}{v(0)-\alpha v(\xi)} \ge 0 \end{aligned}$$

and

$$G(t,s) = \frac{v(s)[v(0)-v(t)] + \alpha v(\xi)[v(t)-v(s)]}{v(0)-\alpha v(\xi)} \le \frac{(1+\alpha)[v(0)]^2}{v(0)-\alpha v(\xi)}.$$

For $\xi \le s \le t$,

$$\begin{aligned} G(t,s) &= v(t) - \frac{(1-\alpha)v(t)v(s)}{v(0)-\alpha v(\xi)} \\ &= v(t)\left[1 - \frac{(1-\alpha)v(s)}{v(0)-\alpha v(\xi)}\right] \\ &= v(t)\left[\frac{v(0)-v(s)-\alpha(v(\xi)-v(s))}{v(0)-\alpha v(\xi)}\right], \end{aligned}$$

if $0 \le \alpha < 1$, then $v(0) - v(s) - \alpha(v(\xi) - v(s)) \ge v(0) - v(\xi) \ge 0$;
if $\alpha \ge 1$, then $v(0) - v(s) - \alpha(v(\xi) - v(s)) = v(0) - \alpha v(\xi) + (\alpha - 1)v(s) \ge v(0) - \alpha v(\xi) > 0$. So for $\xi \le s \le t$, $G(t,s) \ge 0$.

$$G(t,s) \le v(t) \times \frac{v(0)+\alpha v(s)}{v(0)-\alpha v(\xi)} \le \frac{(1+\alpha)[v(0)]^2}{v(0)-\alpha v(\xi)}.$$

For $s \ge t$, $s \ge \xi$,

$$G(t,s) = v(s)\left[1 - \frac{(1-\alpha)v(t)}{v(0)-\alpha v(\xi)}\right] = \frac{v(s)}{v(0)-\alpha v(\xi)}[v(0)-\alpha v(\xi)-(1-\alpha)v(t)],$$

if $0 < \alpha < 1$, then

$$v(0) - \alpha v(\xi) - (1-\alpha)v(t) \ge v(0) - \alpha v(\xi) - (1-\alpha)v(0) = \alpha[v(0) - v(\xi)] \ge 0;$$

if $\alpha \ge 1$, then $v(0) - \alpha v(\xi) - (1-\alpha)v(t) \ge v(0) - \alpha v(\xi) > 0$.

So for $s \ge t$, $s \ge \xi$, $G(t,s) \ge 0$.

$$G(t,s) = \frac{v(s)}{v(0)-\alpha v(\xi)}[v(0)-\alpha v(\xi)-(1-\alpha)v(t)] \le \frac{(1+\alpha)[v(0)]^2}{v(0)-\alpha v(\xi)}.$$

In the following we show that (b) holds. By $v(0) = \int_0^\infty \frac{1}{\rho(s)} \mathrm{d}s < +\infty$, we have $\lim_{t\to+\infty} v(t) = 0$, so $\lim_{t\to+\infty} G(t,s) = 0$ for $s \in I$.

Next we show that (c) holds. By the expression of $G(t,s)$, we have

$$\frac{\partial G(t,s)}{\partial t}=\begin{cases}-\dfrac{v(0)-v(s)}{\rho(t)[v(0)-\alpha v(\xi)]}, & s\le t, s\le \xi;\\ \dfrac{v(s)-\alpha v(\xi)}{\rho(t)[v(0)-\alpha v(\xi)]}, & t\le s\le \xi;\\ -\dfrac{1}{\rho(t)}\left[1-\dfrac{(1-\alpha)v(s)}{v(0)-\alpha v(\xi)}\right], & \xi\le s\le t;\\ \dfrac{(1-\alpha)v(s)}{\rho(t)[v(0)-\alpha v(\xi)]}, & s\ge t, s\ge \xi.\end{cases}$$

For $s\le t, s\le\xi$, $\left|\frac{\partial G(t,s)}{\partial t}\right|\le\frac{2v(0)}{\rho(t)[v(0)-\alpha v(\xi)]}$.
For $t\le s\le\xi$, $\xi\le s\le t$ and $s\ge t, s\ge\xi$, $\left|\frac{\partial G(t,s)}{\partial t}\right|\le\frac{(1+\alpha)v(0)}{\rho(t)[v(0)-\alpha v(\xi)]}$.
So $\left|\frac{\partial G(t,s)}{\partial t}\right|\le\frac{\Gamma}{\rho(t)}$, which together with $\int_0^\infty\frac{1}{\rho(t)}<+\infty$, yields $\lim_{t\to+\infty}\frac{\partial G(t,s)}{\partial t}=0$.

If $0<\alpha<1$, we will show that (d) holds.
For $s\le t, s\le\xi$, it is clear that $G(t,s)\le G(s,s)$ and $\frac{G(t,s)}{G(s,s)}=\frac{v(t)}{v(s)}\ge\frac{v(l_2)}{v(0)}$, $t\in[l_1,l_2]$.
For $t\le s\le\xi$, $G(t,s)=v(s)-\frac{v(t)[v(s)-\alpha v(\xi)]}{v(0)-\alpha v(\xi)}\le G(s,s)$ and

$$\frac{G(t,s)}{G(s,s)}=\frac{v(s)[v(0)-v(t)]+\alpha v(\xi)[v(t)-v(s)]}{v(s)[v(0)-v(s)]}\ge\frac{v(\xi)[v(0)-v(l_1)]}{v(0)[v(0)-v(\xi)]},\quad t\ge l_1.$$

For $\xi\le s\le t$, $G(t,s)=v(t)\left[\frac{v(0)-v(s)-\alpha[v(\xi)-v(s)]}{v(0)-\alpha v(\xi)}\right]\le G(s,s)$ and $\frac{G(t,s)}{G(s,s)}=\frac{v(t)}{v(s)}\ge\frac{v(l_2)}{v(\xi)}$, $t\le l_2$.
For $s\ge t, s\ge\xi$,

$$G(t,s)=\frac{v(s)}{v(0)-\alpha v(\xi)}[v(0)-\alpha v(\xi)-(1-\alpha)v(t)]\le G(s,s)$$

and

$$\frac{G(t,s)}{G(s,s)}=\frac{v(0)-\alpha v(\xi)-(1-\alpha)v(t)}{v(0)-\alpha v(\xi)-(1-\alpha)v(s)}\ge\frac{v(0)-\alpha v(\xi)-(1-\alpha)v(l_1)}{v(0)-\alpha v(\xi)}=1-\frac{(1-\alpha)v(l_1)}{v(0)-\alpha v(\xi)},\quad t\ge l_1.$$

Therefore, $G(t,s)\ge\gamma G(s,s)$ for $(t,s)\in[l_1,l_2]\times I$. □

Define the space $X=\{x\in C^1[0,+\infty):\lim_{t\to+\infty}x(t)=0, \|x'\|_\infty<\infty\}$ with the norm $\|x\|=\max\{\|x\|_\infty,\|x'\|_\infty\}$, where $\|x\|_\infty=\sup_{t\in[0,+\infty)}|x(t)|$. Evidently, X is a Banach space.

Define the operator $T:X\to X$ by

$$(Tx)(t)=\int_0^\infty G(t,s)f(s,x(s),x'(s))\mathrm{d}s.$$

Lemma 2.3 means that $x\in C^2[0,\infty)$ is a solution of BVP (1.1) if and only if x is a fixed point of the operator T.

Choose $P\subseteq X$ be a cone defined by $P=\{x\in X:x(t)\ge 0, t\in I\}$.

Lemma 2.5. *If $v(0)>\alpha v(\xi)$, then $T:P\to P$ is completely continuous.*

Proof. (1) First we show that the operator T is continuous. For this let $\{x_n\}\subseteq P$, $x\in P$ and $x_n\to x$ in X as $n\to\infty$. Then there exists an $M>0$ such that $\|x_n\|\le M$. By Lemma 2.4 we have

$$\begin{aligned}|Tx_n(t)-Tx(t)| &\le \int_0^\infty G(t,s)|f(s,x_n(s),x_n'(s))-f(s,x(s),x'(s))|\mathrm{d}s\\ &\le \Delta\int_0^\infty |f(s,x_n(s),x_n'(s))-f(s,x(s),x'(s))|\mathrm{d}s,\\ |(Tx_n)'(t)-(Tx)'(t)| &\le \int_0^\infty\left|\frac{\partial G(t,s)}{\partial t}[f(s,x_n(s),x_n'(s))-f(s,x(s),x'(s))]\right|\mathrm{d}s\\ &\le \frac{\Gamma}{\rho(t)}\int_0^\infty |f(s,x_n(s),x_n'(s))-f(s,x(s),x'(s))|\mathrm{d}s.\end{aligned}$$

Since f is an L^1-Carathédory function we have

$$\int_0^\infty |f(s,x_n(s),x_n'(s)) - f(s,x(s),x'(s))|\mathrm{d}s \le 2\int_0^\infty |l_{M,M}(s)|\mathrm{d}s < \infty \tag{2.6}$$

and

$$\lim_{n\to\infty} f(t,x_n(t),x_n'(t)) = f(t,x(t),x'(t)). \tag{2.7}$$

According to the Dominated Convergence Theorem, (2.6) and (2.7) imply

$$\begin{aligned}\lim_{n\to\infty}\|Tx_n - Tx\|_\infty &= \lim_{n\to\infty}\sup_{t\in I}|Tx_n(t) - Tx(t)|\\ &\le \Delta \lim_{n\to\infty}\int_0^\infty |f(s,x_n(s),x_n'(s)) - f(s,x(s),x'(s))|\mathrm{d}s = 0\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty}\|(Tx_n)' - (Tx)'\|_\infty &= \lim_{n\to\infty}\sup_{t\in I}|(Tx_n)'(t) - (Tx)'(t)|\\ &\le \sup_{t\in[0,+\infty)}\left|\frac{\Gamma}{\rho(t)}\right| \lim_{n\to\infty}\int_0^\infty |f(s,x_n(s),x_n'(s)) - f(s,x(s),x'(s))|\mathrm{d}s = 0.\end{aligned}$$

Furthermore, by $G(t,s)\ge 0$ for $(t,s)\in I\times I$ and $f(t,x,y)\ge 0$, $(t,x,y)\in I\times I\times R$, we have $(Tx)(t)\ge 0$, $t\in I$. So $T: P\to P$ is continuous.

(2) We need to show $T: P\to P$ is relatively compact.

Given a bounded set $D\subseteq P$. Choose $M>0$ such that $\|x\|\le M$ for all $x\in D$. By Lemma 2.4

$$\begin{aligned}|Tx(t)| &= \int_0^\infty G(t,s)f(s,x(s),x'(s))\mathrm{d}s\\ &\le \Delta\int_0^\infty |l_{M,M}(s)|\mathrm{d}s < \infty,\end{aligned}$$

and

$$\begin{aligned}|(Tx)'(t)| &\le \int_0^\infty \left|\frac{\partial G(t,s)}{\partial t}\right| f(s,x(s),x'(s))\mathrm{d}s\\ &\le \sup_{t\in[0,+\infty)}\left|\frac{\Gamma}{\rho(t)}\right|\int_0^\infty |l_{M,M}(s)|\mathrm{d}s < \infty\end{aligned}$$

since $\rho\in C(I)$ and $\int_0^\infty \frac{1}{\rho(t)}\mathrm{d}t < \infty$. So $\{TD(t)\}$ and $\{(TD)'(t)\}$ are uniformly bounded. At the same time, the fact that $\{(TD)'(t)\}$ is uniformly bounded implies that $\{TD(t)\}$ is locally equicontinuous on $[0,\infty)$.

Now we show that $\{(TD)'(t)\}$ is locally equicontinuous on $[0,\infty)$. For any $R>0$, $t_1,t_2\in[0,R]$ and $x\in D$, then

$$\begin{aligned}|(Tx)'(t_1) - (Tx)'(t_2)| &= \int_0^\infty |(\partial/\partial t)G(t_1,s) - (\partial/\partial t)G(t_2,s)|\, f(s,x(s),x'(s))\mathrm{d}s\\ &\le \int_0^\infty |(\partial/\partial t)G(t_1,s) - (\partial/\partial t)G(t_2,s)|\, l_{M,M}(s)\mathrm{d}s.\end{aligned}$$

Since $\frac{1}{\rho(t)}\in C([0,\infty))$, for any $\varepsilon>0$, there exists $\delta>0$ such that $\left|\frac{1}{\rho(t_1)} - \frac{1}{\rho(t_2)}\right| < \varepsilon$ for $|t_1 - t_2|<\delta$, $t_1,t_2\in[0,T]$. By the expression of Green's function, for any $\varepsilon>0$ there exists $\delta>0$, such that if $|t_1-t_2|<\delta$,

$$\int_0^\infty |(\partial/\partial t)G(t_1,s) - (\partial/\partial t)G(t_2,s)|\, l_{M,M}(s)\mathrm{d}s < \varepsilon \quad \text{for all } x\in D.$$

Since R is arbitrary, $\{(TD)'(t)\}$ is locally equicontinuous on $[0,\infty)$.

(3) $T: P\to P$ is equiconvergent at ∞.

Now for $x\in D$, similar to the proof in (2) one has

$$\sup_{t\in[0,+\infty)}\int_0^\infty G(t,s)f(s,x(s),x'(s))\mathrm{d}s<\infty \quad\text{and}\quad \sup_{t\in[0,+\infty)}\int_0^\infty\left|\frac{\partial G(t,s)}{\partial t}\right|f(s,x(s),x'(s))\mathrm{d}s<\infty.$$

By Lemma 2.4,

$$\lim_{t\to\infty}G(t,s)f(s,x(s),x'(s))=0 \quad\text{and}\quad \lim_{t\to\infty}\left|\frac{\partial G(t,s)}{\partial t}\right|f(s,x(s),x'(s))=0 \quad \text{for } s\in I.$$

Thus Dominated Convergence Theorem guarantees that

$$\lim_{t\to\infty}\int_0^\infty|G(t,s)-G(\infty,s)|f(s,x(s),x'(s))\mathrm{d}s=\int_0^\infty\lim_{t\to\infty}G(t,s)f(s,x(s),x'(s))\mathrm{d}s=0$$

and

$$\lim_{t\to\infty}\int_0^\infty\left|\frac{\partial G(t,s)}{\partial t}-(\partial/\partial t)G(\infty,s)\right|f(s,x(s),x'(s))\mathrm{d}s=\int_0^\infty\lim_{t\to\infty}\left|\frac{\partial G(t,s)}{\partial t}\right|f(s,x(s),x'(s))\mathrm{d}s=0.$$

So $T:P\to P$ is equiconvergent at ∞. □

3. The existence of one positive solution

Theorem 3.1. *Let $v(0)>\alpha v(\xi)$. Suppose that f is an L^1-Carathédory function and $f(t,0,0)\neq 0$ for a.e. $t\in I$, there exists functions $a,b,c\in L^1([0,\infty),[0,\infty))$ satisfying*

$$\|b\|_{L^1}+\|c\|_{L^1}<\min\left\{\frac{1}{\Delta},\frac{1}{\sup\limits_{t\in[0,+\infty)}\left\{\frac{\Gamma}{\rho(t)}\right\}}\right\}$$

such that

$$f(t,x,y)\le a(t)+b(t)x+c(t)y.$$

Then problem (1.1) has at least one nontrivial positive solution.

Proof. From Lemma 2.5 $T:P\to P$ is a completely continuous operator. Let

$$R>\max\left\{\frac{\Delta\|a\|_{L^1}}{1-\Delta(\|b\|_{L^1}+\|c\|_{L^1})},\frac{\sup\limits_{t\in I}\left\{\frac{\Gamma}{\rho(t)}\right\}\|a\|_{L^1}}{1-(\|b\|_{L^1}+\|c\|_{L^1})\sup\limits_{t\in[0,\infty)}\left\{\frac{\Gamma}{\rho(t)}\right\}}\right\}.$$

Now we define $\Omega=\{x\in P:\|x\|<R\}$. For any $x\in\partial\Omega$, then $\|x\|=R$, so $\|x\|_\infty\le R$, $\|x'\|_\infty\le R$,

$$\begin{aligned}|Tx(t)|&=\int_0^\infty G(t,s)f(s,x(s),x'(s))\mathrm{d}s\\&\le\Delta\int_0^\infty a(s)+b(s)|x(s)|+c(s)|x'(s)|\mathrm{d}s\\&\le\Delta[\|a\|_{L^1}+(\|b\|_{L^1}+\|c\|_{L^1})R]<R=\|x\|_\infty\end{aligned}$$

and

$$\begin{aligned}|(Tx)'(t)|&=\left|\int_0^\infty\frac{\partial G(t,s)}{\partial t}f(s,x(s),x'(s))\mathrm{d}s\right|\\&\le\sup_{t\in I}\left\{\frac{\Gamma}{\rho(t)}\right\}\int_0^\infty a(s)+b(s)|x(s)|+c(s)|x'(s)|\mathrm{d}s\\&\le\sup_{t\in I}\left\{\frac{\Gamma}{\rho(t)}\right\}[\|a\|_{L^1}+(\|b\|_{L^1}+\|c\|_{L^1})R]<R=\|x\|_\infty.\end{aligned}$$

So $\|Tx\|<\|x\|$, i.e. taking $p^*=0$ in Lemma 2.1, for any $x\in\partial\Omega$, $x=\overline{\lambda}Tx(0<\overline{\lambda}<1)$ does not hold. Thus Lemma 2.1 implies that the operator T has at least a fixed point, by

$$\int_0^\infty F(s,x(s),x'(s))\mathrm{d}s \le \int_0^\infty a(s)+b(s)x(s)+c(s)x'(s)\mathrm{d}s \le \|a\|_{L^1}+\|b\|_{L^1}R+\|c\|_{L^1}R<\infty,$$

then problem (1.1) has at least one positive solution. Besides, by $f(t,0,0)\ne 0$ for a.e. $t\in[0,\infty)$, then problem (1.1) has at least one nontrivial positive solution. □

4. The existence of triple positive solutions

Define the functionals

$$\alpha(x)=\sup_{t\in[0,\infty)}|x(t)|,\qquad \beta(x)=\sup_{t\in[0,\infty)}|x'(t)|,\qquad \psi(x)=\min_{t\in[l_1,l_2]}|x(t)|.$$

Then $\alpha,\beta:P\to[0,\infty)$ are nonnegative continuous convex functionals satisfying (A1) and (A2); ψ is a nonnegative continuous concave functional with $\psi(x)\le\alpha(x)$ for all $x\in P$.

Theorem 4.1. *Let* $0\le\alpha<1$. *Suppose that* f *is an* L^1*-Carathédory function and there exist* $q\in L^1([0,\infty),[0,\infty))$, $g\in C(I\times I\times R,I)$ *such that* $f(t,x,y)=q(t)g(t,x,y)$. *Assume there exist constants* $r_2\ge\frac{b}{\gamma}>b>r_1>0$, $L_2\ge L_1>0$ *such that* $\frac{b}{K}\le\min\{\frac{r_2}{M},\frac{L_2}{L}\}$. *If the following assumptions hold:*

(C1) $g(t,x,y)\le\min\{\frac{r_1}{M},\frac{L_1}{L}\}$, $(t,x,y)\in[0,\infty)\times[0,r_1]\times[-L_1,L_1]$;
(C2) $g(t,x,y)>\frac{b}{K}$, $(t,x,y)\in[l_1,l_2]\times[b,\frac{b}{\gamma}]\times[-L_1,L_1]$;
(C3) $g(t,x,y)\le\min\{\frac{r_2}{M},\frac{L_2}{L}\}$, $(t,x,y)\in[0,\infty)\times[0,r_2]\times[-L_2,L_2]$,

then problem (1.1) *has at least three positive solutions* x_1,x_2,x_3 *with*

$$0\le x_i(t)\le r_i,\qquad \|x_i'\|_\infty\le L_i,\quad i=1,2,\qquad r_1\le x_3(t)\le r_2,\qquad -L_1\le x_3'(t)\le L_2,\quad t\in[0,\infty),$$
$$x_2(t)>b,\qquad x_3(t)\le b,\quad t\in[l_1,l_2]$$

where $0<l_1<\xi<l_2$, γ *is defined in Lemma* 2.4 *and*

$$M=\Delta\|q\|_{L^1},\qquad L=\sup_{t\in[0,\infty)}\left\{\frac{\Gamma}{\rho(t)}\right\}\|q\|_{L^1},\qquad K=\min_{t\in[l_1,l_2]}\int_{l_1}^{l_2}G(t,s)q(s)\mathrm{d}s.$$

Proof. We will apply Lemma 2.2 to verify the existence of fixed points of the operator T. Lemma 2.5 has showed that $T:P\to P$ is completely continuous. Now we will verify that all the conditions of Lemma 2.2 are satisfied. First we show $T:\overline{P}(\alpha,r_2;\beta,L_2)\to\overline{P}(\alpha,r_2;\beta,L_2)$. If $x\in\overline{P}(\alpha,r_2;\beta,L_2)$, then $\alpha(x)\le r_2$, $\beta(x)\le L_2$ and assumption (C3) implies

$$\begin{aligned}
\alpha(Tx) &= \sup_{t\in[0,\infty)}\int_0^\infty G(t,s)f(s,x(s),x'(s))\mathrm{d}s\\
&\le \Delta\int_0^\infty q(s)\mathrm{d}s\sup_{(t,x,y)\in I\times[0,r_2]\times[-L_2,L_2]}g(t,x,y)\\
&\le r_2,\\
\beta(Tx) &= \sup_{t\in I}\left|\int_0^\infty\frac{\partial G(t,s)}{\partial t}f(s,x(s),x'(s))\mathrm{d}s\right|\\
&\le \sup_{t\in[0,\infty)}\left\{\frac{\Gamma}{\rho(t)}\right\}\|q\|_{L^1}\sup_{(t,x,y)\in I\times[0,r_2]\times[-L_2,L_2]}g(t,x,y)\\
&\le L_2.
\end{aligned}$$

Hence $T:\overline{P}(\alpha,r_2;\beta,L_2)\to\overline{P}(\alpha,r_2;\beta,L_2)$. In the same way we can show $T:\overline{P}(\alpha,r_1;\beta,L_1)\to\overline{P}(\alpha,r_1;\beta,L_1)$, so the condition (B2) is satisfied.

To check the condition (B1) in Lemma 2.2, we choose $x(t)=\frac{b}{\gamma}$, $t\in I$. It is easy to see that $x(t)=\frac{b}{\gamma}\in\overline{P}(\alpha,\frac{b}{\gamma};\beta,L_2;\psi,b)$, $\psi(x)=\frac{b}{\gamma}>b$, and consequently, $\{x\in\overline{P}(\alpha,\frac{b}{\gamma};\beta,L_2;\psi,b):\psi(x)>b\}\ne\emptyset$. For

$x \in \overline{P}(\alpha, \frac{b}{\gamma}; \beta, L_2; \psi, b)$, then $\|x\|_\infty \le \frac{b}{\gamma}, \|x'\|_\infty \le L_2, x(t) \ge b, t \in [l_1, l_2]$. Now we show $\psi(Tx) > b$. By (C2)

$$\begin{aligned}\psi(Tx) &= \min_{t\in[l_1,l_2]} \int_0^\infty G(t,s) f(s, x(s), x'(s)) \mathrm{d}s \\ &\ge \min_{t\in[l_1,l_2]} \int_{l_1}^{l_2} G(t,s) q(s) \mathrm{d}s \min_{(t,x,y)\in[l_1,l_2]\times[b,\frac{b}{\gamma}]\times[-L_2,L_2]} g(t,x,y) \\ &> b.\end{aligned}$$

Finally, we verify that the condition (B3) in Lemma 2.2 holds. For $x \in \overline{P}(\alpha, r_2; \beta, L_2; \psi, b)$ with $\alpha(Tx) > \frac{b}{\gamma}$, then by the definition ψ and Lemma 2.4 we have

$$\begin{aligned}\psi(Tx) &= \min_{t\in[l_1,l_2]} (Tx)(t) = \min_{t\in[l_1,l_2]} \int_0^\infty G(t,s) f(s, x(s), x'(s)) \mathrm{d}s \\ &\ge \gamma \int_0^\infty G(s,s) f(s, x(s), x'(s)) \mathrm{d}s \\ &\ge \gamma\alpha(Tx) > b.\end{aligned}$$

Therefore, the operator T has three fixed points $x_i \in \overline{P}(\alpha, r_2; \beta, L_2), i = 1, 2, 3$, with

$$x_1 \in P(\alpha, r_1; \beta, L_1), \qquad x_2 \in \{\overline{P}(\alpha, r_2; \beta, L_2; \psi, b) | \psi(y) > b\}$$

and

$$x_3 \in \overline{P}(\alpha, r_2; \beta, L_2) \setminus (\overline{P}(\alpha, r_2; \beta, L_2; \psi, b) \cup \overline{P}(\alpha, r_1; \beta, L_1)).$$

Besides

$$\int_0^\infty f(s, x_i(s), x_i'(s)) \mathrm{d}s \le \int_0^\infty q(s) g(s, x_i(s), x_i'(s)) \mathrm{d}s \le \|q\|_{L^1} \min\left\{\frac{r_2}{M}, \frac{L_2}{L}\right\} < \infty,$$

by Lemma 2.3, problem (1.1) has three positive solutions $x_i \in \overline{P}(\alpha, r_2; \beta, L_2), i = 1, 2, 3$ with

$$\begin{aligned}&0 \le x_i(t) \le r_i, \quad \|x_i'\|_\infty \le L_i, \quad i = 1, 2, \quad r_1 \le x_3(t) \le r_2, \quad -L_1 \le x_3'(t) \le L_2, \quad t \in [0, \infty), \\ &x_2(t) > b, \quad x_3(t) \le b, \quad t \in [l_1, l_2]. \quad \square\end{aligned}$$

Example 4.2. Consider the following boundary value problem on the half-line

$$\begin{cases} \left((1+t)^2 x'(t)\right)' + \dfrac{1}{(1+t)^2} g(t, x(t), x'(t)) = 0, \quad t \in [0, +\infty) \\ x(0) = \dfrac{1}{2} x(2), \quad \lim\limits_{t\to\infty} x(t) = 0, \end{cases} \tag{4.1}$$

where $\rho(t) = (1+t)^2, \alpha = \frac{1}{2}, \xi = 2, q(t) = \frac{1}{(1+t)^2}$,

$$g(t,x,y) = \begin{cases} \dfrac{1}{3}x^3 + \dfrac{|y|}{10^4}, \quad (t,x,y) \in I \times [0,1] \times [-2100, 2100]; \\ \dfrac{449}{78}x^3 + \dfrac{|y|}{10^4} + \dfrac{1}{3} - \dfrac{449}{78}, \quad (t,x,y) \in I \times [1,3] \times [-2100, 2100]; \\ 150 + \dfrac{|y|}{10^4}, \quad (t,x,y) \in I \times [3,15] \times [-2100, 2100]; \\ \dfrac{2}{3}x + \dfrac{|y|}{10^4} + 140, \quad (t,x,y) \in I \times [15,300] \times [-2100, 2100]; \\ \varphi(x) + \dfrac{|y|}{10^4}, \quad (t,x,y) \in I \times [300,\infty) \times [-2100, 2100]; \\ \varphi(x) + \dfrac{y^2}{21 \times 10^6}, \quad (t,x,y) \in I \times [300,\infty) \times ((-\infty, -2100] \cup [2100, \infty)), \end{cases}$$

where $\varphi \in C(I, I)$, $\varphi(300) = 340$. By computing, $M = \frac{9}{5}$, $L = \frac{21}{5}$, $K = 0.02$.

Let $r_1 = 1$, $b = 3$, $r_2 = 300$, $L_1 = 10$, $L_2 = 2100$, $l_1 = 1$, $l_2 = 4$. Theorem 4.1 are satisfied. So problem (4.1) has at least three positive solutions.

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