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Enumerating $(\mathbf{2}+\mathbf{2})$-free posets by the number of minimal elements and other statistics

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ABSTRACT

An unlabeled poset is said to be $(\mathbf{2}+\mathbf{2})$-free if it does not contain an induced subposet that is isomorphic to $\mathbf{2}+\mathbf{2}$, the union of two disjoint 2-element chains. Let p_n denote the number of $(\mathbf{2}+\mathbf{2})$-free posets of size n. In a recent paper, Bousquet-Mélou et al. [1] found, using the so called ascent sequences, the generating function for the number of $(\mathbf{2}+\mathbf{2})$-free posets of size n: $P(t)=\sum_{n\geq 0} p_n t^n = \sum_{n\geq 0}\prod_{i=1}^{n}\left(1-(1-t)^i\right)$. We extend this result in two ways. First, we find the generating function for $(\mathbf{2}+\mathbf{2})$-free posets when four statistics are taken into account, one of which is the number of minimal elements in a poset. Second, we show that if $p_{n,k}$ equals the number of $(\mathbf{2}+\mathbf{2})$-free posets of size n with k minimal elements, then $P(t,z)=\sum_{n,k\geq 0} p_{n,k}t^n z^k = 1+\sum_{n\geq 0}\frac{zt}{(1-zt)^{n+1}}\prod_{i=1}^{n}(1-(1-t)^i)$. The second result cannot be derived from the first one by a substitution. Our enumeration results are extended to certain restricted permutations and to regular linearized chord diagrams through bijections in [1,2]. Finally, we define a subset of ascent sequences counted by the Catalan numbers and we discuss its relations with $(\mathbf{2}+\mathbf{2})$- and $(\mathbf{3}+\mathbf{1})$-free posets.

1. Introduction

An unlabeled poset is said to be $(\mathbf{2}+\mathbf{2})$-*free* if it does not contain an induced subposet that is isomorphic to $\mathbf{2}+\mathbf{2}$, the union of two disjoint 2-element chains. We let $\mathcal{P}$ (resp. $\mathcal{P}_n$) denote the set of $(\mathbf{2}+\mathbf{2})$-free posets (resp. on n elements). Fishburn [5] showed that a poset is $(\mathbf{2}+\mathbf{2})$-free precisely when it is isomorphic to an *interval order*. Another important characterization of $(\mathbf{2}+\mathbf{2})$-free posets (see [6,4,9]) is that a poset is $(\mathbf{2}+\mathbf{2})$-free if and only if the collection of strict principal down-sets can be linearly ordered by inclusion. Here for any poset $\mathbf{P}=(P,<_p)$ and $x\in P$, the strict principal down set of x, $D(x)$, in $\mathbf{P}$ is the set of all $y\in P$ such that $y<_p x$. The *trivial down-set* is the empty set. Thus if $\mathbf{P}$ is a $(\mathbf{2}+\mathbf{2})$-free poset, we can write $D(\mathbf{P})=\{D(x):x\in P\}$ as

$$D(\mathbf{P})=\{D_0,D_1,\ldots,D_k\}$$

where $\emptyset=D_0\subset D_1\subset\cdots\subset D_k$. In such a situation, we say that $x\in P$ has level i if $D(x)=D_i$.

Let p_n be the number of $(\mathbf{2}+\mathbf{2})$-free posets on n elements. Bousquet-Mélou et al. [1] showed that the generating function for the numbers p_n is

$$P(t)=\sum_{n\geq 0}p_n t^n=\sum_{n\geq 0}\prod_{i=1}^{n}\left(1-(1-t)^i\right). \tag{1}$$

Note that the term corresponding to $n=0$ in the last sum is 1.

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Zagier [13] proved that (1) is also the generating function which counts certain involutions introduced by Stoimenow [11]. Bousquet-Mélou et al. [1] gave bijections between $(\mathbf{2}+\mathbf{2})$-free posets and such involutions, between $(\mathbf{2}+\mathbf{2})$-free posets and a certain restricted class of permutations, and between $(\mathbf{2}+\mathbf{2})$-free posets and *ascent sequences*. A sequence $x_1x_2\cdots x_n \in \mathbb{N}^n$ is an *ascent sequence of length n* if and only if it satisfies $x_1 = 0$ and $x_i \in [0, 1 + \mathrm{asc}(x_1x_2\cdots x_{i-1})]$ for all $2 \le i \le n$. Here, for any integer sequence $x_1x_2\cdots x_i$, the number of *ascents* of this sequence is

$$\mathrm{asc}(x_1x_2\cdots x_i) = |\{1 \le j < i : x_j < x_{j+1}\}|.$$

For instance, 010231002 is an ascent sequence. We let $\mathcal{A}$ denote the set of all ascent sequences where we assume that the empty word is also an ascent sequence.

To define the bijection between $(\mathbf{2}+\mathbf{2})$-free posets and ascent sequences, Bousquet-Mélou et al. [1] used a step by step decomposition of a $(\mathbf{2}+\mathbf{2})$-free poset P where at each step one removes a maximal element located on the lowest level together with certain relations. If one records the levels from which one removed such maximal elements and then reads the resulting sequence backwards, one obtains an ascent sequence associated to the poset. We shall briefly review this bijection in Section 2. In the process of decomposing the $(\mathbf{2}+\mathbf{2})$-free poset P, one will reach a point where the remaining poset consists of an antichain, possibly having one element. We define $\mathrm{lds}(P)$ to be the maximum size of such an antichain, which is also equal to the size of the down-set of the last removed element that has a non-trivial down-set ("lds" stands for "last down-set"). By definition, the value of lds on an antichain is 0 as there are no non-trivial down-sets for such a poset.

Bousquet-Mélou et al. [1] studied a more general generating function $F(t, u, v)$ of $(\mathbf{2}+\mathbf{2})$-free posets according to size="number of elements" (variable t), levels="number of levels" (variable u), and minmax="level of minimum maximal element" (variable v). Via their bijection between $(\mathbf{2}+\mathbf{2})$-free posets and ascent sequences, Bousquet-Mélou et al. [1] showed that $F(t, u, v)$ is also the generating function of the ascent sequence according to length (variable t), the number of ascents (variable u), and the last entry (variable v). Using the interpretation of $F(t, u, v)$ in terms of ascent sequences, they showed that $F(t, u, v)$ satisfies the functional equation

$$(v - 1 - tv(1-u))F(t, u, v) = (v-1)(1-tuv) - tF(t, u, 1) + tuv^2F(t, uv, 1). \tag{2}$$

This functional equation then allowed them to show that

$$F(t, u, 1) = \sum_{k\ge1} \frac{(1-u)u^{k-1}(1-t)^k}{(u-(u-1)(1-t)^k)\prod_{i=1}^{k}(u-(u-1)(1-t)^i)}. \tag{3}$$

One cannot substitute $u = 1$ directly into this series to obtain the generating function for $P(t)$ given in (1). Instead, Bousquet-Mélou et al. [1] showed how one can rewrite this series as

$$F(t, u, 1) = \sum_{n\ge0} F_n(t, u) \tag{4}$$

where $(F_n(t, u))_{n\ge0}$ is a certain sequence of polynomials. Then one can set $u = 1$ to obtain the fact that $P(t) = F(t, 1, 1)$ is given by (1).

The main result of this paper, Theorem 4, is an explicit form of the generating function $G(t, u, v, z, x)$ for a generalization of $F(t, u, v)$, when two more statistics are taken into account, i.e., min = "number of minimal elements" in a poset (variable z) and lds = "size of non-trivial last down-set" (variable x). That is, we shall find an explicit formula for

$$G(t, u, v, z, x) = \sum_{P\in\mathcal{P}} t^{\mathrm{size}(P)}u^{\mathrm{levels}(P)}v^{\mathrm{minmax}(P)}z^{\mathrm{min}(P)}x^{\mathrm{lds}(P)}$$

where, as above, $\mathcal{P}$ is the set of all $(\mathbf{2}+\mathbf{2})$-free posets. As in [1], to find $G(t, u, v, z, x)$, we translate our problem on $(\mathbf{2}+\mathbf{2})$-free posets to an equivalent problem on ascent sequences. That is, we define the following statistics on an ascent sequence: length = "the number of elements in the sequence", last = "the rightmost element of the sequence", zeros = "the number of 0's in the sequence", run = "the number of elements in the leftmost run of 0's" = "the number of 0's to the left of the leftmost non-zero element". By definition, if there are no non-zero elements in an ascent sequence, the value of run is 0. Then we shall prove the following.

Lemma 1. *The function $G(t, u, v, z, x)$ defined above can alternatively be defined on ascent sequences as*

$$G(t, u, v, z, x) = \sum_{w\in\mathcal{A}} t^{\mathrm{length}(w)}u^{\mathrm{asc}(w)}v^{\mathrm{last}(w)}z^{\mathrm{zeros}(w)}x^{\mathrm{run}(w)} = \sum_{n,a,\ell,m,r\ge0} G_{n,a,\ell,m,r}t^nu^av^\ell z^m x^r. \tag{5}$$

We can then follow the basic strategy of Bousquet-Mélou et al. [1] to find an explicit formula for a specialization of $G(t, u, v, z, x)$, namely $G(t, 1, 1, z, 1)$. That is, one can find a functional equation for $G(t, u, v, z, x)$ by using the interpretation of $G(t, u, v, z, x)$ in terms of ascent sequences. This functional equation then allows us to give an explicit expression of $G(t, u, v, z, x)$ which has a similar flavor of (3) but is necessarily much more complicated. One can set $v = x = 1$ in our formula for $G(t, u, v, z, x)$ to obtain a series for $G(t, u, 1, z, 1)$ which has a similar flavor to the series expression for $F(t, u, 1)$ given in (3). As was the case for (3), one cannot substitute $u = 1$ directly in the series for $G(t, u, 1, z, 1)$ so that one has to rewrite the series in a form where one can make such a substitution. However, in our case, the required rewriting is not just a sum of a sequence of polynomials. Thus while the outline of our derivation of an explicit formula $G(t, 1, 1, z, 1)$ follows similar steps as the derivation of (1) in [1], the details of each of the steps are considerably different. Nevertheless, we are able to derive an explicit formula for $G(t, 1, 1, z, 1) = P(t, z) = \sum_{n,k\geq 0} p_{n,k}t^n z^k$ where $p_{n,k}$ denotes the number of $(\mathbf{2}+\mathbf{2})$-free posets of size n with k minimal elements or, equivalently, the number of ascent sequences of length n with k zeros. That is, we shall show that

$$P(t,z) = \sum_{n,k\geq 0} p_{n,k}t^n z^k = 1 + \sum_{n\geq 0} \frac{zt}{(1-zt)^{n+1}} \prod_{i=1}^{n} (1-(1-t)^i). \tag{6}$$

In fact, we conjectured another form of writing $P(t, z)$:

$$P(t,z) = \sum_{n,k\geq 0} p_{n,k}t^n z^k = \sum_{n\geq 0} \prod_{i=1}^{n} (1-(1-t)^{i-1}(1-zt)).$$

Three different proofs were presented of the last formula [8,12].

A poset P is $(\mathbf{3}+\mathbf{1})$-free if it does not contain, as an induced subposet, a 3-element chain and an element which incomparable to the elements in the 3-element chain. It is known that the number of posets avoiding $(\mathbf{2}+\mathbf{2})$ and $(\mathbf{3}+\mathbf{1})$ is given by the Catalan numbers (see [10,9]). Define a *restricted ascent sequence* as follows. A sequence $x_1x_2\cdots x_n \in \mathbb{N}^n$ is a restricted ascent sequence of length n if it satisfies $x_1 = 0$ and $x_i \in [m-1, 1+\mathrm{asc}(x_1x_2\cdots x_{i-1})]$ for all $2 \leq i \leq n$, where m is the maximum element in $x_1x_2\cdots x_{i-1}$. For instance, 010232232 is a restricted ascent sequence, whereas 010201 is not. We shall show that restricted ascent sequences are counted by the Catalan numbers. For $n \leq 6$, the bijection in [1] sends $(\mathbf{2}+\mathbf{2})$- and $(\mathbf{3}+\mathbf{1})$-free posets to restricted ascent sequences which lead us to initially conjecture that it always the case that the bijection in [1] sends $(\mathbf{2}+\mathbf{2})$- and $(\mathbf{3}+\mathbf{1})$-free posets to restricted ascent sequences. However, this is not true as we shall produce counterexamples when $n = 7$.

This paper is organized as follows. In Section 2, we briefly describe the bijection between $(\mathbf{2}+\mathbf{2})$-free posets and ascent sequences given in [1, Section 3] so that we can establish the claims about the properties of the bijection used in Lemma 1. This bijection then allows us to reduce the enumerative problem on posets to an equivalent enumerative problem on ascent sequences. In Section 3 we find explicitly the function $G(t, u, v, z, x)$ using the ascent sequences (see Theorem 4). In Section 4, we shall derive our formula for $P(t, z)$ and show how to get $P(t)$ from $P(t, z)$. Finally, in Section 5 we define a subset of ascent sequences counted by the Catalan numbers and discuss its relations to $(\mathbf{2}+\mathbf{2})$- and $(\mathbf{3}+\mathbf{1})$-free posets.

2. $(\mathbf{2}+\mathbf{2})$-free posets and ascent sequences

In this section, we shall review the bijection between $(\mathbf{2}+\mathbf{2})$-free posets and ascent sequences given in [1, Section 3]. In order to do this, Bousquet-Mélou et al. [1] introduced an operation on posets called the subtraction operation which removes a maximal element m_P from $P \in \mathcal{P}_n$ and results in $Q \in \mathcal{P}_{n-1}$. Before giving this operation, we need to define some terminology.

Let $D(x)$ be the set of predecessors of x (the strict down-set of x): $D(x) = \{y : y < x\}$. Clearly, any poset is uniquely specified by listing the sets of predecessors. It is well-known (see for example [7]) that a poset is $(\mathbf{2}+\mathbf{2})$-free if and only if its sets of predecessors, $\{D(x) : x \in P\}$, can be linearly ordered by inclusion. Let

$$D(P) = (D_0, D_1, \ldots, D_{k-1})$$

with $\emptyset = D_0 \subset D_1 \subset \cdots \subset D_{k-1}$ be the chain for P. In this context we define $D_i(P) = D_i$ and $\ell(P) = k$. We say that an element x is at *level* i in P if $D(x) = D_i$ and we write $\ell(x) = i$. The set of all elements at level i is denoted by $L_i(P) = \{x \in P : \ell(x) = i\}$ and we let

$$L(P) = \big(L_0(P), L_1(P), \ldots, L_{k-1}(P)\big).$$

For instance, $L_0(P)$ is the set of minimal elements and $L_{k-1}(P)$ is the set of maximal elements whose set of predecessors is also maximal. Let m_P be a maximal element of P whose set of predecessors is smallest. This element may not be unique but the level on which it resides is. Let us write $\ell^\star(P) = \ell(m_P)$.

Clearly, any $(\mathbf{2}+\mathbf{2})$-free poset P is determined by the pair $\big(D(P), L(P)\big)$. Thus when defining the subtraction operation below it suffices to specify how $D(P)$ and $L(P)$ change.

For non-empty $P \in \mathcal{P}_n$, let $\Psi(P) = (Q, i)$ where $i = \ell^\star(P)$ and Q is the poset that results from applying:

(Sub1) If m_P is not alone on level i, then remove m_P. In terms of predecessors and levels, $D(Q) = D(P)$ and

$$L_j(Q) = \begin{cases} L_j(P) & \text{if } j \neq i, \\ L_i(P) - \{m_P\} & \text{if } j = i. \end{cases}$$

(Sub2) If m_P is alone on level $i = \ell(P)$, then remove the unique element of level i.

(Sub3) If m_P is alone on level $i \le \ell(P) - 1$, then set $\mathbb{N} = D_{i+1}(P) - D_i(P)$. Make each element in $\mathbb{N}$ a maximal element of the poset by removing any covers. Finally, remove the element m_P. In terms of predecessors and levels,

$$D_j(Q) = \begin{cases} D_j(P) & \text{if } 0 \le j < i, \\ D_{j+1}(P) - \mathbb{N} & \text{if } i \le j < \ell(P) - 1, \end{cases}$$

and

$$L_j(Q) = \begin{cases} L_j(P) & \text{if } 0 \le j < i, \\ L_{j+1}(P) & \text{if } i \le j < \ell(P) - 1. \end{cases}$$

Thus if we start with a $(\mathbf{2}+\mathbf{2})$-free poset P of size n, we can produce a sequence of posets $P = P_1, P_2, \ldots, P_n$ such that P_{i+1} is obtained from P_i by applying the subtraction operation. The bijection of [1, Section 3] maps P to the ascent sequence $(\ell^\star(P_n), \ell^\star(P_{n-1}), \ldots, \ell^\star(P_1))$. The details of the fact that $(\ell^\star(P_n), \ell^\star(P_{n-1}), \ldots, \ell^\star(P_1))$ is always an ascent sequence and how to define the inverse of this map via an operation called addition can be found in [1, Section 3].

Proof of Lemma 1. To prove the statement we need to show equidistribution of the statistics involved. All but one case follows from Theorem 11 of [1] where Bousquet-Mélou et al. proved that the bijection from $(\mathbf{2}+\mathbf{2})$-free posets to ascent sequences sends size → length, levels → asc, minmax → last, and min → zeros. The fact that lds goes to run also follows from the bijection. That is, in the process of decomposing the poset, there will be a point where we remove the element, say e, whose down-set gives lds. At that point, we will be left with incomparable elements located on level 0, which gives the initial run of 0's followed by the 1 (corresponding to element e on level 1) in the corresponding ascent sequence. □

3. Main results

For $r \ge 1$, let $G_r(t, u, v, z)$ denote the coefficient of x^r in $G(t, u, v, z, x)$. Thus $G_r(t, u, v, z)$ is the generating function of those ascent sequences that begin with $r \ge 1$ 0's followed by 1. We let $G^r_{n,a,\ell,m}$ denote the number of ascent sequences of length n which begin with r 0's followed by 1, have a ascents, last letter ℓ, and a total of m zeros. We then let

$$G_r := G_r(t, u, v, z) = \sum_{a,\ell,m\ge 0, n\ge r+1} G^r_{n,a,\ell,m} t^n u^a v^\ell z^m. \tag{7}$$

Clearly, since the sequence $0 \cdots 0$ has no ascents and no initial run of 0's (by definition), we have that the generating function for such sequences is

$$1 + tz + (tz)^2 + \cdots = \frac{1}{1 - tz}$$

where 1 corresponds to the empty word. Thus, we have the following relation between G and G_r:

$$G = \frac{1}{1 - tz} + \sum_{r\ge 1} G_r x^r. \tag{8}$$

Lemma 2. *For $r \ge 1$, the generating function $G_r(t, u, v, z)$ satisfies*

$$(v - 1 - tv(1 - u))G_r = (v - 1)t^{r+1}uvz^r + t((v - 1)z - v)G_r(t, u, 1, z) + tuv^2 G_r(t, uv, 1, z). \tag{9}$$

Proof. Our proof follows the same steps as in Lemma 13 in [1]. Fix $r \ge 1$. Let $x' = x_1x_2 \cdots x_{n-1}$ be an ascent sequence beginning with r 0's followed by 1, with a ascents and m zeros where $x_{n-1} = \ell$. Then $x = x_1x_2 \cdots x_{n-1}i$ is an ascent sequence if and only if $i \in [0, a + 1]$. Clearly x also begins with r 0's followed by 1. Now, if $i = 0$, the sequence x has a ascents and $m + 1$ zeros. If $1 \le i \le \ell$, x has a ascents and m zeros. Finally if $i \in [\ell + 1, a + 1]$, then x has $a + 1$ ascents and m zeros. Counting the sequence $00 \cdots 01$ with r 0's separately, we have

$$\begin{aligned} G_r &= t^{r+1}u^1v^1z^r + \sum_{\substack{a,\ell,m\ge 0\\ n\ge r+1}} G^r_{n,a,\ell,m} t^{n+1} \left(u^a v^0 z^{m+1} + \sum_{i=1}^{\ell} u^a v^i z^m + \sum_{i=\ell+1}^{a+1} u^{a+1} v^i z^m \right) \\ &= t^{r+1}uvz^r + t \sum_{\substack{a,\ell,m\ge 0\\ n\ge r+1}} G_{n,a,\ell,m} t^n u^a z^m \left(z + \frac{v^{\ell+1} - v}{v - 1} + u\frac{v^{a+2} - v^{\ell+1}}{v - 1} \right) \\ &= t^{r+1}uvz^r + tzG_r(t, u, 1, z) + tv\frac{G_r - G_r(t, u, 1, z)}{v - 1} + tuv\frac{vG_r(t, uv, 1, z) - G_r}{v - 1}. \end{aligned}$$

The result follows. □

Next just like in Section 6.2 of [1], we use the kernel method to proceed. Setting $(v-1-tv(1-u))=0$ and solving for v, we obtain that the substitution $v=1/(1+t(u-1))$ will kill the left-hand side of (9). We can then solve for $G_r(t,u,1,z)$ to obtain that

$$G_r(t,u,1,z)=\frac{(1-u)t^{r+1}uz^r+uG_r\left(t,\frac{u}{1+t(u-1)},1,z\right)}{(1+zt(u-1))(1+t(u-1))}. \tag{10}$$

Next we define

$$\delta_k=u-(1-t)^k(u-1) \quad \text{and} \tag{11}$$

$$\gamma_k=u-(1-zt)(1-t)^{k-1}(u-1) \tag{12}$$

for $k\geq 1$. We also set $\delta_0=\gamma_0=1$. Observe that $\delta_1=u-(1-t)(u-1)=1+t(u-1)$ and $\gamma_1=u-(1-zt)(u-1)=1+zt(u-1)$. Thus we can rewrite (10) as

$$G_r(t,u,1,z)=\frac{t^{r+1}z^ru(1-u)}{\delta_1\gamma_1}+\frac{u}{\delta_1\gamma_1}G_r\left(t,\frac{u}{\delta_1},1,z\right). \tag{13}$$

For any function of $f(u)$, we shall write $f(u)|_{u=\frac{u}{\delta_k}}$ for $f(u/\delta_k)$. It is then easy to check that

1. $(u-1)|_{u=\frac{u}{\delta_k}}=\frac{(1-t)^k(u-1)}{\delta_k}$,
2. $\delta_s|_{u=\frac{u}{\delta_k}}=\frac{\delta_{s+k}}{\delta_k}$,
3. $\gamma_s|_{u=\frac{u}{\delta_k}}=\frac{\gamma_{s+k}}{\delta_k}$, and
4. $\frac{u}{\delta_s}|_{u=\frac{u}{\delta_k}}=\frac{u}{\delta_{s+k}}$.

Using these relations, one can iterate the recursion (13) to prove by induction that for all $n\geq 1$,

$$G_r(t,u,1,z)=\frac{t^{r+1}z^ru(1-u)}{\delta_1\gamma_1}+\left(t^{r+1}z^ru(1-u)\sum_{s=1}^{2^n-1}\frac{u^s(1-t)^s}{\delta_s\delta_{s+1}\prod_{i=1}^{s+1}\gamma_i}\right)+\frac{u^{2^n}}{\delta_{2^n}\prod_{i=1}^{2^n}\gamma_i}G_r\left(t,\frac{u}{\delta_{2^n}},1,z\right). \tag{14}$$

Since $\delta_0=1$, it follows that as a power series in u,

$$G_r(t,u,1,z)=t^{r+1}z^ru(1-u)\sum_{s\geq 0}\frac{u^s(1-t)^s}{\delta_s\delta_{s+1}\prod_{i=1}^{s+1}\gamma_i}. \tag{15}$$

Note that we can rewrite (9) as

$$G_r(t,u,v,z)=\frac{t^{r+1}z^ruv(1-v)}{v\delta_1-1}+\frac{t(z(v-1)-v)}{v\delta_1-1}G_r(t,u,1,z)+\frac{uv^2t}{v\delta_1-1}G_r(t,uv,1,z). \tag{16}$$

For $s\geq 1$, we let

$$\bar{\delta}_s=\delta_s|_{u=uv}=uv-(1-t)^s(uv-1) \quad \text{and}$$

$$\bar{\gamma}_s=\gamma_s|_{u=uv}=uv-(1-zt)(1-t)^{s-1}(uv-1)$$

and set $\bar{\delta}_0=\bar{\gamma}_0=1$. Then using (16) and (15), we have the following theorem.

Theorem 3. *For all $r\geq 1$,*

$$G_r(t,u,v,z)=\frac{t^{r+1}z^ru}{v\delta_1-1}\left(v(v-1)+t(1-u)(z(v-1)-v)\sum_{s\geq 0}\frac{u^s(1-t)^s}{\delta_s\delta_{s+1}\prod_{i=1}^{s+1}\gamma_i}\right.$$
$$\left.+uv^3t(1-uv)\sum_{s\geq 0}\frac{(uv)^s(1-t)^s}{\bar{\delta}_s\bar{\delta}_{s+1}\prod_{i=1}^{s+1}\bar{\gamma}_i}\right). \tag{17}$$

It is easy to see from Theorem 3 that

$$G_r(t,u,v,z)=t^{r-1}z^{r-1}G_1(t,u,v,z). \tag{18}$$

This is also easy to see combinatorially since every ascent sequence counted by $G_r(t,u,v,z)$ is of the form $0^{r-1}a$ where a is an ascent sequence a counted by $G_1(t,u,v,z)$.

Note that

$$\begin{aligned} G(t,u,v,z,x) &= \frac{1}{(1-tz)} + \sum_{r\geq 1} G_r(t,u,v,z)x^r \\ &= \frac{1}{(1-tz)} + \sum_{r\geq 1} t^{r-1}z^{r-1}G_1(t,u,v,z)x^r \\ &= \frac{1}{(1-tz)} + \frac{1}{1-tzx}xG_1(t,u,v,z). \end{aligned}$$

Thus we have the following theorem.

Theorem 4.

$$\begin{aligned} G(t,u,v,z,x) &= \sum_{P\in\mathcal{P}} t^{\mathrm{size}(P)}u^{\mathrm{levels}(P)}v^{\mathrm{minmax}(P)}z^{\min(P)}x^{\mathrm{lds}(P)} \\ &= \sum_{w\in\mathcal{A}} t^{\mathrm{length}(w)}u^{\mathrm{asc}(w)}v^{\mathrm{last}(w)}z^{\mathrm{zeros}(w)}x^{\mathrm{run}(w)} \\ &= \frac{1}{(1-tz)} + \frac{t^2zxu}{(1-tzx)(v\delta_1-1)} + \left(v(v-1) + t(1-u)(z(v-1)-v)\sum_{s\geq 0}\frac{u^s(1-t)^s}{\delta_s\delta_{s+1}\prod_{i=1}^{s+1}\gamma_i} \right. \\ &\quad \left. + \, uv^3t(1-uv)\sum_{s\geq 0}\frac{(uv)^s(1-t)^s}{\bar{\delta}_s\bar{\delta}_{s+1}\prod_{i=1}^{s+1}\bar{\gamma}_i} \right). \end{aligned} \tag{19}$$

4. Counting (2 + 2)-free posets by size and number of minimal elements

In this section, we shall compute the generating function of (**2** + **2**)-free posets by size and the number of minimal elements which is equivalent to finding the generating function for ascent sequences by length and the number of zeros.

For $n \geq 1$, let $H_{n,a,\ell,m}$ denote the number of ascent sequences of length n with a ascents and m zeros which have last letter ℓ. Then we first wish to compute

$$H(t,u,v,z) = \sum_{n\geq 1, a,\ell,m\geq 0} H_{n,a,\ell,m}t^n u^a v^\ell z^m. \tag{20}$$

Using the same reasoning as in the previous section, we see that

$$\begin{aligned} H(t,u,v,z) &= tz + \sum_{\substack{a,\ell,m\geq 0\\ n\geq 1}} H_{n,a,\ell,m}t^{n+1}\left(u^a v^0 z^{b+1} + \sum_{i=1}^{\ell} u^a v^i z^b + \sum_{i=\ell+1}^{a+1} u^{a+1}v^i z^b\right) \\ &= tz + t\sum_{\substack{a,\ell,m\geq 0\\ n\geq r+1}} H_{n,a,\ell,m}t^n u^a z^b\left(z + \frac{v^{\ell+1}-v}{v-1} + u\frac{v^{a+2}-v^{\ell+1}}{v-1}\right) \\ &= tz + \frac{tv(1-u)}{v-1}H(t,u,v,z) + \frac{t(z(v-1)-v)}{v-1}H(t,u,1,z) + \frac{tuv^2}{v-1}H(t,uv,1,z). \end{aligned}$$

Thus we have the following lemma.

Lemma 5.

$$(v-1-tv(1-u))H(t,u,v,z) = tz(v-1) + t(z(v-1)-v)H(t,u,1,z) + tuv^2H(t,uv,1,z). \tag{21}$$

Setting $(v-1-tv(1-u)) = 0$, we see that the substitution $v = 1/(1+t(u-1)) = 1/\delta_1$ kills the left-hand side of (21). We can then solve for $H(t,u,1,z)$ to obtain the recursion

$$H(t,u,1,z) = \frac{zt(1-u)}{\gamma_1} + \frac{u}{\delta_1\gamma_1}H(t,uv,1,z). \tag{22}$$

By iterating (22), we can prove by induction that for all $n \geq 1$,

$$H(t,u,1,z) = \frac{zt(1-u)}{\gamma_1} + \left(\sum_{s=1}^{2^n-1} \frac{zt(1-u)u^s(1-t)^s}{\delta_s \prod_{i=1}^{s+1} \gamma_i} \right) + \frac{u^{2^n}}{\delta_{2^n} \prod_{i=1}^{2^n} \gamma_i} H\left(t, \frac{u}{\delta_{2^n}}, 1, z\right). \tag{23}$$

Since $\delta_0 = 1$, we can rewrite (23) as

$$H(t,u,1,z) = \left(\sum_{s=0}^{2^n-1} \frac{zt(1-u)u^s(1-t)^s}{\delta_s \prod_{i=1}^{s+1} \gamma_i} \right) + \frac{u^{2^n}}{\delta_{2^n} \prod_{i=1}^{2^n} \gamma_i} H\left(t, \frac{u}{\delta_{2^n}}, 1, z\right). \tag{24}$$

Thus as a power series in u, we can conclude the following.

Theorem 6.

$$H(t,u,1,z) = \sum_{s=0}^{\infty} \frac{zt(1-u)u^s(1-t)^s}{\delta_s \prod_{i=1}^{s+1} \gamma_i}. \tag{25}$$

We would like to set $u = 1$ in the power series $\sum_{s=0}^{\infty} \frac{zt(1-u)u^s(1-t)^s}{\delta_s \prod_{i=1}^{s+1} \gamma_i}$, but the factor $(1-u)$ in the series does not allow us to do that in this form. Thus our next step is to rewrite the series in a form where it is obvious that we can set $u = 1$ in the series. To that end, observe that for $k \geq 1$,

$$\delta_k = u - (1-t)^k(u-1) = 1 + u - 1 - (1-t)^k(u-1) = 1 - ((1-t)^k - 1)(u-1)$$

so that

$$\frac{1}{\delta_k} = \sum_{n\geq 0} ((1-t)^k - 1)^n (u-1)^n = \sum_{n\geq 0} (u-1)^n \sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m} (1-t)^{km}. \tag{26}$$

Substituting (26) into (25), we see that

$$\begin{aligned}
H(t,u,1,z) &= \frac{zt(1-u)}{\gamma_1} + \sum_{k\geq 1} \frac{zt(1-u)u^k(1-t)^k}{\prod_{i=1}^{k+1} \gamma_i} \sum_{n\geq 0} (u-1)^n \sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m} (1-t)^{km} \\
&= \frac{zt(1-u)}{\gamma_1} + \sum_{n\geq 0} \sum_{m=0}^{n} (-1)^{n-m-1} \binom{n}{m} (u-1)^{n-m} zt \sum_{k\geq 1} \frac{(u-1)^{m+1} u^k (1-t)^{k(m+1)}}{\prod_{i=1}^{k+1} \gamma_i} \\
&= \frac{zt(1-u)}{\gamma_1} + \sum_{n\geq 0} \sum_{m=0}^{n} (-1)^{n-m-1} \binom{n}{m} (u-1)^{n-m} \frac{zt}{(1-zt)^{m+1}} \\
&\quad \times \sum_{k\geq 1} \frac{(u-1)^{m+1}(1-zt)^{m+1} u^k (1-t)^{k(m+1)}}{\prod_{i=1}^{k+1} \gamma_i}.
\end{aligned}$$

Next we need to study the series

$$\sum_{k\geq 1} \frac{(u-1)^{m+1}(1-zt)^{m+1} u^k (1-t)^{k(m+1)}}{\prod_{i=1}^{k+1} \gamma_i}$$

where $m \geq 0$. We can rewrite this series in the form

$$-\frac{(u-1)^{m+1}(1-zt)^{m+1}}{\gamma_1} + \sum_{k\geq 0} \frac{(u-1)^{m+1}(1-zt)^{m+1} u^k (1-t)^{k(m+1)}}{\prod_{i=1}^{k+1} \gamma_i}.$$

We let

$$\psi_{m+1}(u) = \sum_{k\geq 0} \frac{(u-1)^{m+1}(1-zt)^{m+1}u^k(1-t)^{k(m+1)}}{\prod_{i=1}^{k+1}\gamma_i}. \tag{27}$$

We shall show that $\psi_{m+1}(u)$ is in fact a polynomial for all $m \geq 0$. First, we claim that $\psi_{m+1}(u)$ satisfies the following recursion:

$$\psi_{m+1}(u) = \frac{(u-1)^{m+1}(1-zt)^{m+1}}{\gamma_1} + \frac{u\delta_1^m}{\gamma_1}\psi_{m+1}\left(\frac{u}{\delta_1}\right). \tag{28}$$

That is, one can easily iterate (28) to prove by induction that for all $n \geq 1$,

$$\psi_{m+1}(u) = \left(\sum_{s=0}^{2^n-1} \frac{(u-1)^{m+1}(1-zt)^{m+1}u^s(1-t)^{s(m+1)}}{\prod_{i=1}^{s+1}\gamma_i}\right) + \frac{u^{2^n}(\delta_{2^n})^m}{\prod_{i=1}^{2^n}\gamma_i}\psi_{m+1}\left(\frac{u}{\delta_{2^n}}\right). \tag{29}$$

By taking the limit as $n \to \infty$ it follows that if $\psi_{m+1}(u)$ satisfies the recursion (28), then $\psi_{m+1}(u)$ is given by the power series in (27). However, it is routine to check that the polynomial

$$\phi_{m+1}(u) = -\sum_{j=0}^{m}(u-1)^j(1-zt)^j u^{m-j}\prod_{i=j+1}^{m}(1-(1-t)^i) \tag{30}$$

satisfies the recursion that

$$\gamma_1\phi_{m+1}(u) = (u-1)^{m+1}(1-zt)^{m+1} + u\delta_1^m\phi_{m+1}\left(\frac{u}{\delta_1}\right). \tag{31}$$

Thus we have proved the following lemma.

Lemma 7.

$$\begin{aligned}\psi_{m+1}(u) &= \sum_{k\geq 0} \frac{(u-1)^{m+1}(1-zt)^{m+1}u^k(1-t)^{k(m+1)}}{\prod_{i=1}^{k+1}\gamma_i} \\ &= -\sum_{j=0}^{m}(u-1)^j(1-zt)^j u^{m-j}\prod_{i=j+1}^{m}(1-(1-t)^i).\end{aligned} \tag{32}$$

It thus follows that

$$\begin{aligned}H(t,u,1,z) = {}& \frac{zt(1-u)}{\gamma_1} + \sum_{n\geq 0}\sum_{m=0}^{n}(-1)^{n-m-1}\binom{n}{m}(u-1)^{n-m}\frac{zt}{(1-zt)^{m+1}} \\ & -\frac{(u-1)^{m+1}(1-zt)^{m+1}}{\gamma_1} - \sum_{j=0}^{m}(u-1)^j(1-zt)^j u^{m-j}\prod_{i=j+1}^{m}(1-(1-t)^i).\end{aligned}$$

There is no problem in setting $u = 1$ in this expression to obtain that

$$H(t,1,1,z) = \sum_{n\geq 0}\frac{zt}{(1-zt)^{n+1}}\prod_{i=1}^{n}(1-(1-t)^i). \tag{33}$$

Clearly our definitions ensure that $1+H(t,1,1,z) = P(t,z)$ as defined in the introduction so that we have the following theorem.

Theorem 8.

$$P(t,z) = \sum_{n,k\geq 0} p_{n,k}t^n z^k = 1 + \sum_{n\geq 0}\frac{zt}{(1-zt)^{n+1}}\prod_{i=1}^{n}(1-(1-t)^i). \tag{34}$$

Next we observe that one can easily derive the ordinary generating function for the number of (**2** + **2**)-free posets or, equivalently, for the number of ascent sequences proved by Bousquet-Mélou et al. [1] from Theorem 8. That is, for any sequence of natural numbers $a = a_1 \cdots a_n$, let $a^+ = (a_1 + 1) \cdots (a_n + 1)$ be the result of adding one from each element of the sequence. Moreover, if all the elements of $a = a_1 \cdots a_n$ are positive, then we let $a^- = (a_1 - 1) \cdots (a_n - 1)$ be the result of subtracting one to each element of the sequence. It is easy to see that if $a = a_1 \cdots a_n$ is an ascent sequence, then $0a^+$ is also an ascent sequence. Vice versa, if $b = 0a$ is an ascent sequence with only one zero where $a = a_1 \cdots a_n$, then a^- is an ascent sequence. It follows that the number of ascent sequences of length n is equal to the number of ascent sequences of length $n + 1$ which have only one zero. Hence

$$\begin{aligned} P(t) &= \sum_{n\geq 0} p_n t^n = \frac{1}{t}\frac{\partial P(t,z)}{\partial z}\bigg|_{z=0} \\ &= \sum_{n\geq 0}\prod_{i=1}^{n}(1-(1-t)^i). \end{aligned}$$

Results in [1–3] show that (**2** + **2**)-free posets of size n with k minimal elements are in bijection with the following objects. (See [1–3] for the precise definitions.)

- ascent sequences of length n with k zeros;
- permutations of length n *avoiding*  whose *leftmost-decreasing run* is of size k;
- *regular linearized chord diagrams* on $2n$ points with *initial run of openers* of size k;
- *upper triangular matrices* whose non-negative integer entries sum up to n, each row and column contains a non-zero element, and the sum of entries in the first row is k.

Thus (34) provides generating functions for  -avoiding permutations by the size of the leftmost-decreasing run, for regular linearized chord diagrams by the size of the initial run of openers, and for the upper triangular matrices by the sum of entries in the first row. Moreover, Theorem 4, together with bijections in [1–3] can be used to enumerate the permutations, diagrams, and matrices subject to four statistics.

5. Restricted ascent sequences and the Catalan numbers

Recall that a sequence $x_1x_2\cdots x_n \in \mathbb{N}^n$ is a restricted ascent sequence of length n if it satisfies $x_1 = 0$ and $x_i \in [m-1, 1+\mathrm{asc}(x_1x_2\cdots x_{i-1})]$ for all $2 \leq i \leq n$, where m is the maximum element in $x_1x_2\cdots x_{i-1}$.

Theorem 9. *The number of restricted ascent sequences of length n is given by the n-th Catalan number.*

Proof. Lets R_n denote the number of restricted ascent sequences of length n. The Catalan numbers C_n can be defined by the recursion

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}$$

with the initial condition that $C_0 = 1$. It is easy to see that $R_0 = 1$ since the empty sequence is a restricted ascent sequence. We must show that

$$R_{n+1} = \sum_{k=0}^{n} R_k R_{n-k}. \tag{35}$$

Thus we need a procedure to take a restricted ascent sequence D_1 of length k and a restricted ascent sequence D_2 of length $n-k$ and produce a restricted ascent sequence D of length $n+1$. We shall describe a procedure "gluing" two ascent sequences, D_1 and D_2 which is equivalent to gluing two Dyck paths together. To define our gluing procedure we first need the concept of the "rightmost maximum" in an ascent sequence, defined as a left-to-right maximum x such that x is one more than the number of ascents to the left of x, and none of the left-to-right maxima to the right of x has this property (in other words, this is the last time we use the maximum option in the interval $[m-1, 1+\mathrm{asc}]$ among the left-to-right maxima). The sequence $00\cdots 0$ is the only one that does not have the rightmost maximum. For example, 0010101003 has the rightmost maximum (the leftmost) 1, whereas 0010103323234 has the rightmost maximum (the leftmost) 3. Then procedure of "gluing" two ascent sequences, D_1 and D_2 together can be described as follows.

1. For $D_1 \neq \emptyset$, define $D_1 + D_2 := D_1(1+\mathrm{asc}(D_1))(D_2 + +(\mathrm{asc}(D_1)))$ where "++" means increasing *each* element of D_2 by the number $\mathrm{asc}(D_1)$. For example, if $D_1 = 01021$ and $D_2 = 01212$, then $D_1 + D_2 = 01021323434$.
2. For $D_1 = \emptyset$ define $D_1 + D_2 := D_2$ with the rightmost maximum element duplicated (add extra 0 if $D_2 = 00\cdots 0$). For example, $\epsilon + 01212 = 012212$.

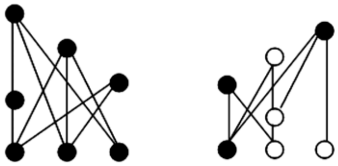

Fig. 1. Counterexamples to the statement that restricted ascent sequences correspond to (**2** + **2**)- and (**3** + **1**)-free posets under the bijection in [1].

It is easy to see that in Case 1, the element $(1 + \mathrm{asc}(D_1))$ is the rightmost maximum element of $D_1(1 + \mathrm{asc}(D_1))(D_2 + +(\mathrm{asc}(D_1)))$ which is either the rightmost element if $D_2 = \epsilon$ or is followed by $\mathrm{asc}(D_1)$ if $D_2 \neq \epsilon$ since D_2 must start with 0 in that case. It follows that the rightmost maximal element is not duplicated in $D_1 + D_2$ in Case 1 and, hence, it is easy to recover D_1 and D_2 from $D_1 + D_2$. Clearly, in Case 2, the rightmost maximal element of $D_1 + D_2$ is duplicated so that we can distinguish Case 1 from Case 2. Moreover, it is easy to see that we can recover D_2 from $D_1 + D_2$ in Case 2. This proves that (35) holds and hence $R_n = C_n$ for all n.

Here are examples of decompositions for $n = 3$ and $n = 4$ (ϵ stays for the empty word):

$000 = \epsilon + 00$	$0000 = \epsilon + 000$	$0100 = 0 + 00$	$0112 = 011 + \epsilon$
$001 = 00 + \epsilon$	$0001 = 000 + \epsilon$	$0101 = 0 + 01$	$0121 = 01 + 0$
$010 = 0 + 0$	$0010 = 00 + 0$	$0102 = 010 + \epsilon$	$0122 = \epsilon + 012$
$011 = \epsilon + 01$	$0011 = \epsilon + 001$	$0110 = \epsilon + 010$	$0123 = 012 + \epsilon$
$012 = 01 + \epsilon$	$0012 = 001 + \epsilon$	$0111 = \epsilon + 011$	. □

Recall that posets avoiding (**3**+**1**) are those that do not contain, as an induced subposet, a 3-element chain together with another element which is incomparable to all elements in the 3-element chain. As we mentioned in the introduction, the number of posets avoiding (**2**+**2**) and (**3**+**1**) is given by the Catalan numbers (see [10,9]). Using the bijection in [1] applied to small restricted ascent sequences, one would be tempted to conjecture that restricted ascent sequences are bijectively mapped to (**2** + **2**)- and (**3** + **1**)-free posets as both of the objects are counted by the Catalan numbers. Indeed, this is true for posets of size less than or equal to six.

Moreover, we can show that, for the first time one violates the restricted ascent sequence condition, the corresponding (**2** + **2**)-free poset contains an induced copy of (**3** + **1**). That is, suppose that $a = a_1a_2 \cdots a_n$ is a restricted ascent sequence, $m = \max\{a_1, a_2, \ldots, a_n\} \geq 2$, and $x < m - 1$. Then we claim that the poset corresponding to ax, must contain an induced copy of (**3** + **1**). That is, let r be the element on level x that corresponds to x under the bijection of Section 2. Now in ax, x is preceded by a larger element, and thus r has a neighbor, say s, on its level, level x. Because the first time we encounter m in a, its corresponding element z in the poset covers all maximal elements, it follows that there must be at least one non-maximal element, say u, on level $m - 1$. Next, since $x < m - 1$, there exists an element e in the poset such that $e < c$ and $e \not< b$. That is, c is on a higher level than b and the down-sets are linearly ordered by inclusion according to their levels. Since r copies relations of s, $e \not< r$. Since r is a maximal element, also $r \not< e$ and $r \not< u$. Finally, u is a non-maximal element, thus there exists $v > u$. Finally $v \not< r$ since r is maximal so that the four elements $e < u < v$ and r form a (**3** + **1**)configuration.

If it was the case that our addition operations preserved the property of containing (**3**+**1**) configuration, then it would be the case that the bijection in Section 2 would send (**2**+**2**)- and (**3**+**1**)-free posets to restricted ascent sequences. However, this is not the case. For example, consider the poset on the left in Fig. 1 which corresponds to (**2**+**2**)-free poset corresponding to the ascent sequence 0101202. One can check that there is an induced (**3** + **1**) in the poset corresponding to the non-restricted ascent sequence 010120, but clearly there is no induced (**3** + **1**) in the (**2** + **2**)-free poset corresponding to the ascent sequence 0101202. This means that there must be a restricted ascent sequence of length seven whose corresponding (**2** + **2**)-free poset does contain an induced copy of (**3** + **1**). Such a sequence and its corresponding (**2** + **2**)-free poset is shown on the right of Fig. 1.

We leave it as an open problem to characterize (**2** + **2**)-free posets corresponding to restricted ascent sequences under the bijection in [1] and to characterize ascent sequences corresponding to (**2** + **2**)- and (**3** + **1**)-free posets under the same bijection.

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