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The Copeland measure of Condorcet choice functions

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Abstract

We propose a measure to compare an arbitrary choice function with the Copeland choice function. We compute this measure for the familiar Condorcet choice functions.

1. Introduction

A tournament $T = (X, U)$ is a nonempty finite set X and a binary relation U on X (read xUy as x dominates y) that is complete ($x \neq y \Rightarrow xUy$ or yUx) and asymmetric (see Moon [10]). A choice function F on the class $\mathscr{T}$ of all tournaments is a map that assigns a nonempty choice set $F(T) \subseteq X$ to each $T = (X, U)$ in $\mathscr{T}$.

Several specific choice functions have been proposed to reflect $F(T)$ as the set of most dominant members of X with respect to U. A *Condorcet choice function* is a choice function F such that for all $T = (X, U)$ with xUy $\forall y \neq x$ for some $x \in X$, $F(T) = \{x\}$[1]. All the choice functions considered in this paper fall in this class. One of the most popular is Copeland's function C which defines $C(T)$ as the subset of X whose members maximize $s(T, x)$, the number of $y \in X$ for which xUy (Copeland [4]). Our purpose is to compare C with other choice functions by the *Copeland measure* C_F for choice function F as defined by

$$C_F = \inf_{T \in \mathscr{T}} \left[\frac{\max_{x \in F(T)} s(T, x)}{\max_{x \in X} s(T, x)} \right].$$

This measure provides an evaluation of the poorest conceivable scores of the outcomes chosen by the choice function F. By definition $C_F \leqslant 1$ with equality iff $F(T) \cap C(T) \neq \emptyset$ for all T in $\mathscr{T}$. It represents a first estimation of the magnitude of the disagreement between the Copeland function and an arbitrary choice function.

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[1] Such an outcome is called a Condorcet winner.

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The familiar choice functions we consider are organized in three groups according to C_F values, namely $C_F = 1$, $C_F = \frac{1}{2}$ and $C_F \leqslant \frac{1}{3}$. Proofs of our results appear in Section 3.

2. Copeland measure of the familiar choice functions

Let $T = (X, U)$ be an arbitrary tournament. We denote by $V(T, x)$ the set $\{y \in X: yUx\}$. Given $A \subseteq X$, $T|A$ denotes the tournament $(A, U|A)$ where $U|A$ is the restriction of U to A. $L(T)$ denotes the family of subsets $A \subseteq X$ such that $T|A$ is a maximal (with respect to inclusion) transitive tournament, and $l(T)$ denotes the largest cardinality of the elements of $L(T)$.

Let $T' = (X, U')$ be another tournament on X. We denote by $\Delta(T, T')$ the cardinality of the set $\{(x, y) \in X^2: x \neq y, xUy \text{ and } yU'x\}$. It is easy to verify that Δ is a distance on the set of tournaments defined over X.

We define the composition $\Pi(T; T_1, \ldots, T_n)$ of a tournament T on $\{y_1, \ldots, y_n\}$ and n tournaments T_i on $\{x_{i1}, \ldots, x_{in_i}\}$, $i = 1, \ldots, n$ as the tournament on $\bigcup_{i=1}^{n} \{x_{i1}, \ldots, x_{in_i}\}$ by: x_{ij} dominates x_{kl} iff $i = k$ and x_{ij} dominates x_{il} in T_i, or $i \neq k$ and y_i dominates y_k in T.

Given two choice functions F and F' we define a new choice function $F(F')$ by $F(F')(T) = F(T|F'(T))$. In particular, given a choice function F, we define inductively F^k *by* $F^1 = F$ and $F^{k+1} = F(F^k)$; $F^\infty(T)$ is defined by $\bigcap_k F^k(T)$.

2.1. *Choice functions F with* $C_F = 1$

The *top cycle* of T, discussed by Schwartz [12], is the subset $TC(T)$ of X whose members x satisfy: for all $y \in X$ there exists an integer n and a sequence $x = x_0, x_1, \ldots, x_n = y$ such that $x_i U x_{i+1}$, $i = 0, \ldots, n-1$.

The *uncovered set* of T proposed by Fishburn [6] and Miller [9] is the subset $UC(T)$ of X whose members x are the maximal elements of the covering relation defined by: z covers y iff zUy and wUz implies wUy. It is easy to see that the covering relation is transitive and so $UC(T)$ is nonempty.

It is well known that $C(T) \subseteq UC(T) \subseteq TC(T)$. We deduce that $C_{TC} = C_{UC} = 1$.

2.2. *Choice functions F with* $C_F = \frac{1}{2}$

Moulin [11] presents a tournament T such that: $TC(UC)(T) \cap C(T) = \emptyset$, implying $C_{TC(UC)} < 1$. The same observation applies to the iterates UC^k of UC. Dutta [5] proposed a choice function in the continuation of UC. Define a covering set of T as a subset A of X satisfying: $UC(T|A) = A$ and $x \notin UC(T|A \cup \{x\})$ for all $x \notin A$. Dutta [5] proved that $UC^\infty(T)$ is a covering set of T and that the family of covering sets of T has a minimal element with respect to inclusion, called the *minimal covering set* of T and denoted by $MC(T)$. Since $MC(T) \subseteq UC^\infty(T)$ we deduce that $C_{MC} < 1$.

Slater [14] proposed the following choice function. Define a slater order of T as an order O over X which is a solution of the problem Min $\Delta(O, T)$ over the set of orders on X. The *Slater set* of T is the subset $SL(T)$ of X whose members are top elements of at least one Slater order of T. Bermond [3] gives a tournament T such that $SL(T) \cap C(T) = \emptyset$, implying $C_{SL} < 1$.

All the choice function defined in this subsection have a Copeland measure strictly smaller than 1. Our first proposition states that they all have a Copeland measure equal to $\frac{1}{2}$. Precisely:

Proposition 1. *If F is a choice function such that*
(i) $F(T) \subseteq TC(UC)(T)\ \forall T \in \mathscr{T}$ *then* $C_F \leqslant \frac{1}{2}$,
(ii) $F(T) \supseteq MC(T)\ \forall T \in \mathscr{T}$ *then* $C_F \geqslant \frac{1}{2}$,
(iii) $F(T) \supseteq SL(T)\ \forall T \in \mathscr{T}$ *then* $C_F \geqslant \frac{1}{2}$.

Indeed from (i) and (ii) if follows that $C_{TC(UC)} = C_{UC^k} = C_{MC} = \frac{1}{2}$. From (i), (iii) and the inclusion $SL(T) \subseteq TC(UC(T))$ proved in Banks et al. [2], it follows that $C_{SL} = \frac{1}{2}$.

2.3. Choice functions F with $C_F \leqslant \frac{1}{3}$

The two choice functions of this subsection arise in descriptive models of the majority voting process.

The *Banks set* (Banks [1]) of T is the subset $B(T)$ of X whose members are top elements of at least one maximal transitive subtournament of T. Banks proved that $B(T)$ is a subset of $TC(UC)(T)$, implying $CB < 1$.

The *Tournament Equilibrium set* of T denoted by $TEQ(T)$ has been proposed by Schwartz [13]. It is defined inductively on the order of the tournament. Suppose it is defined for all tournaments of order smaller or equal to n and consider T of order $n+1$. Define the binary relation $D(T)$ on X by: $xD(T)y$ iff $x \in TEQ(T \mid V(T, y))$. The tournament equilibrium set of T is the subset of X whose members belong to a maximal component of the transitive closure of $D(T)$. Schwartz proved that $TEQ(T)$ is a subset of $B(T)$ implying $C_{TEQ} < 1$.

These two choice functions have a Copeland measure smaller than $\frac{1}{3}$. This is equivalent to the following proposition.

Proposition 2. *If F is a choice function such that $F(T) \subseteq B(T)\ \forall T \in \mathscr{T}$ then $C_F \leqslant \frac{1}{3}$.*

3. Proofs

Proof of Proposition 1. (i) Let $T = (X, U)$ be a cyclical tournament[2] of order $2n+1$ and consider the tournament T'_n defined as

$$T'_n = \Pi(T; T^1, \ldots, T^{2n+1})$$

[2] A tournament (X, U) is cyclical if there is labelling of X, $X = \{x_1, \ldots, x_{2n+1}\}$ such that $x_i U x_j$ iff $j - i$ $\{1, \ldots, n\}$ (addition mode $2n+1$).

where $T^i = (\{x_1^i\} \cup X_2^i, U^i)$ with $|X_2^i| = n$, $x_1^i U^i x_2^i$ for all x_2^i in X_2^i and $T^i | X_2^i$ arbitrary.

Denote by X^1 and X^2 the sets $\bigcup_{i=1}^{2n+1} \{x_1^i\}$ and $\bigcup_{i=1}^{2n+1} X_2^i$.

To define T_n, we consider an additional point a and we define U_n over $X_1 \cup X_2 \cup \{a\}$ by:

$$X_1 U_n a,$$
$$a U_n X^2$$

and

$$T_n | X_1 \cup X_2 = T_n'.$$

It is easy to check that $UC(T_n) = X_1 \cup \{a\}$ and thus $TC(UC(T_n)) = X_1$; further:

$$s(T_n, a) = 2n^2 + n$$

and

$$s(T_n, x_1) = n^2 + 2n + 1 \text{ for all } x_1 \in X^1.$$

Consequently, $C(T) = \{a\}$ implying $CTC(UC) \leqslant (n^2 + 2n + 1)/(2n^2 + n)$.

Since the right-hand side of the above inequality tends to 1/2 as $n \to \infty$, the proof of (i) is complete.

(ii) The proof proceeds in a sequence of claims.

Claim 1. *Let $T = (X, U)$ and consider $x \in MC(T)$, $y \in X - MC(T)$ such that xUy. Let $T_{xy} = (X, U')$ be defined as:*

- *$yU'x$,*
- *$uU'v$ iff uUv if $\{u, v\} \neq \{x, y\}$.*

Then if $MC(T_{xy}) = MC(T)$,

$$\frac{\max_{z \in MC(T_{xy})} s(T_{xy}, z)}{\max_{z \in X} s(T_{xy}, z)} \leqslant \frac{\max_{z \in MC(T)} s(T, z)}{\max_{z \in X} s(T, z)}.$$

The proof of this claim is obvious. We denote by T' the family of tournaments $T = (X, U)$ such that $\forall x \in MC(T)$, $\forall y \in MC(T)$, xUy implies $MC(Txy) \neq MC(T)$.

Claim 2. *Let $T = (X, U) \in T'$. Then for all $x \notin MC(T)$, there exists a unique $y \in MC(T)$ such that yUx and for all $z \in MC(T)$, xUz iff yUz.*

Let $x \notin MC(T)$. From the definition of $MC(T)$, x is covered by y in $T | MC(T) \cup \{x\}$. We assert that this y verifies the conditions of the claim. Assume on the contrary that $\exists z \in MC(T)$ with yUz and zUx. Consider Tx, z.

It is easy to see that $v \notin MC(T)$, we have $v \notin UC(Tx, z \,|\, MC(T) \cup \{v\})$. This implies that $MC(Tx, z) \subset MC(T)$. Since MC satisfies the strong superset property[3], we

[3] A solution S satisfies the strong superset property if for all $T \in T'$ and $S(T) \subseteq A \subseteq X$, then $S(T | A) = S(T)$. Dutta [5] proves that MC satisfies the strong superset property.

deduce that $MC(Tx, z \mid MC(T)) = MC(Tx, z)$. But, from the construction, $Tx, z \mid MC(T) = T \mid MC(T)$. Since $MC(T \mid MC(T)) = MC(T)$ from the strong superset property, we deduce that $MC(Txz) = MC(T)$, contradicting $T \in T'$.

Given $T \in T'$, let us introduce the following notations. For all $x \notin MC(T)$, we denote by $f(x)$ the unique vertex whose existence is asserted in Claim 2. For all $y \in MC(T)$, denote respectively by $\zeta(y)$ and $\Psi(y)$ the sets $\{x \in X - MC(T)\colon f(x) = y\}$ and $\{y' \in MC(T)\colon yUy'\}$.

Claim 3. $\forall y \in MC(T)$, $\{z \in X\colon yUz\} = \Psi(y) \cup \zeta(y) \cup \bigcup_{y' \in \Psi(y)} \zeta(y')$.

Of course, $\Psi(y) \cup \zeta(y)$ is included in $\{z \in X\colon yUz\}$.

If $y' \notin \Psi(y)$ then $y'Uy$. Since every x in $\zeta(y')$ dominates every z in $MC(T)$ such that $y'Uz$, we deduce that xUy.

If $y' \in \Psi(y)$ and $z \in \zeta(y')$ then as above, since yUy', we deduce yUz.

Let us now conclude the proof. From Claim 1 it is enough to prove that $C_{MC}(T) \geqslant \frac{1}{2}$ for T in T'.

Let $T = (X, U)$ in T' and define $T' = (X, U')$ as:

$$T \mid MC(T) = T' \mid MC(T) \quad \text{for all } y \in MC(T),\ T' \mid \zeta\{y\}(y) = T \mid \{y\} \cup \zeta(y)$$

and

$$\text{for all } x, x' \notin MC(T) \text{ such that } f(x) \neq f(x')\colon\ xU'x' \text{ iff } f(x)Uf(x').$$

From Claim 3 it comes that T and T' only differ on the complement of $MC(T)$. Since MC is monotonic[4] and satisfies the strong superset property, we deduce that $MC(T) = MC(T')$. Further, from Claim 3 it comes also that $s(T, y) = s(T', y)$ for all $y \in MC(T)$. Finally, from the construction of T' it is easy to check that $UC(T') = MC(T')$. Combining these informations, we get $C(T') \subseteq MC(T)$.

This concludes the proof since for all y in $C(T')$, $s(T, y) \geqslant n/2$.

(iii) This follows from the property $x \in SL(T)$ implies $s(T, x) \geqslant [n/2]$ proved by Bermond [3] and thus $C_{SL} \geqslant \frac{1}{2}$. □

Proof of Proposition 2. We prove that $C_B \leqslant \frac{1}{3}$. The following simple claim will be needed.

Claim 4. For all $\varepsilon > 0$ there exists a regular[5] tournament $T = (X, U)$ such that:

$$\frac{l(T)}{o(T)} \leqslant \varepsilon.$$

Proof. Let T^1 be a cyclical tournament over a set X^1 of 5 vertices. We have $l(T^1) = 3$. We define inductively a sequence (T^k) $k \geqslant 1$ as follows:

$$T^{k+1} = \Pi(T^k; T^k, \ldots, T^k).$$

[4] A solution S is monotonic if for all $T = (X, U)$, $T' = (X, U')$ in T such that $(x, y) \in U$, $(y, x) \in U'$ and $T = T'$ otherwise, $x \in S(T')$. It is easy to see that if S is monotonic and satisfies the strong superset property then $S(T)$ is independent from the arcs within the complement of $S(T)$ in X. Dutta [5] proves that MC is monotonic.

[5] A tournament is regular if $C(T) = X$.

It is easy to check that for all k, T^k is regular, with:

$$o(T^k) = 5^{2k-1}$$

and

$$l(T^k) = 3^{2k-1}.$$

Since the sequence $(\frac{3}{5})^{2k-1}$ tends to 0 as k goes to ∞, T^k will satisfy the conditions of the lemma for k large enough.

Let $\varepsilon > 0$ and consider (according to Claim 4) a regular tournament $T = (X, U)$ with $l(T)/o(T) \leqslant \varepsilon$. Denote by γ the set of subsets of X belonging to $L(T)$. Given two integers n and m, we define three tournaments T_n^1, T^2 and T_m^3 as follows.

$T_n^1 = (X_n^1, U_n^1)$ is the tournament $\Pi(T; R_n, \ldots, R_n)$ where R_n is cyclical tournament of order $2n + 1$.

$T^2 = (X^2, U^2)$ is an arbitrary tournament over $\mathrm{X}^2 = \gamma$, $T_m^3 = (X_m^3, U_m^3)$ is an arbitrary transitive tournament of order $m + 1$.

We construct a tournament $T_{n,m}$ over $X_{n,m} = X_n^1 X^2 X_m^3$ as follows: $T_{n,m} = (X_{n,m}, U')$,

$$T_{n,m} | X_n^1 = T_1^n, \qquad T_{n,m} | X^2 = T^2, \qquad T_{n,m} | X_m^3 = T_m^3,$$

$$x^3 U' x^1 \ \ \forall x^3 \in X_m^3,\ \forall x^1 \in X_n^1,$$

$$x^2 U' x^3 \ \ \forall x^3 \in X_m^3,\ \forall x^2 \in X^2,$$

and $x^2 U' x^1$ iff the component of T_n^1 containing x^1 belongs to x^2.

It is easy to check that $B(T_{n,m})$ is a subset of $X_n^1 \cup X^2$. Furthermore, we obtain,

$$s(T_{n,m}, x^1) = \left(\frac{o(T) - 1}{2}\right)(2n + 1) + n + K^1 \quad \forall x^1 \in X_n^1,$$

$$s(T_{n,m}, x^2) \leqslant m + l(T)(2n + 1) + K^2, \quad x^2 \in X^2$$

and

$$s(T_{n,m}, x^3) = m + o(T)(2n + 1)$$

where x^3 is the top element in T_m^3 and K^1 and K^2 are constants (not depending on n or m).

It follows that, if m and n are large enough, $C(T_{n,m}) = \{x^3\}$.

Further, if $m = (o(T) - 2l(T) - \rho)n$ where ρ is a small positive number, then we have:

$$s(T_{n,m}, x^1) > s(T_{n,m}, x^2) \ \forall x^1 \in X^1,\ \forall x^2 \in X^2 \text{ for } n \text{ large enough.}$$

We deduce

$$C_B \leqslant \frac{((o(T) - 1)/2)(2n + 1) + n + K^1}{o(T)(2n + 1) + (o(T) + 1 - 2l(T) - \rho)n}$$

for n large enough.

If we take the limit of the right hand side of the inequality we obtain:

$$C_B \leqslant \frac{1}{3 + 2/o(T) - 2(l(T)/o(T)) - \rho/o(T)} \leqslant \frac{1}{3 - 2\varepsilon - \rho/o(T)}.$$

Since ε and ρ can be chosen arbitrary small the proof is complete. □

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