Q-Matrices and Spherical Geometry

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ABSTRACT

A real $n\times n$ matrix M is a Q-matrix if the linear complementarity problem $w-Mz=q$, $w\geqslant 0$, $z\geqslant 0$, $w^tz=0$ has a solution for all real n-vectors q. M is nondegenerate if all its principal minors are nonzero. Spherical geometry is applied to the problem of characterizing nondegenerate Q-matrices. The stability of 3×3 nondegenerate Q-matrices and a generalization of the partitioning property of P-matrices are rather easily proved using spherical geometry. It is also proved that the set of 4×4 nondegenerate Q-matrices is not open.

0. NOTATION

Let E^n be n-dimensional Euclidean space, and $E^{n\times n}$ the set of all real $n\times n$ matrices. The standard basis of E^n is denoted by e_i, $i=1,\dots,n$. For $I,J\subset\{1,\dots,n\}$ and $M\in E^{n\times n}$, M_{ij} is the (i,j) entry in M, $M_{I\cdot}$ is the submatrix of M consisting of the rows indexed by I, and $M_{\cdot J}$ consists of the columns indexed by J. The *cone* generated by vectors $V_{\cdot 1},\dots,V_{\cdot k}$ is

$$\mathcal{C}(V)=\mathcal{C}(V_{\cdot 1},\dots,V_{\cdot k})=\left\{\sum_{i=1}^{k}\alpha_i V_{\cdot i}\,\middle|\,\alpha_i\geqslant 0,\ i=1,\dots,k\right\}.$$

For $v_1,\dots,v_n\in E^n$, $\mathcal{C}(v_1,\dots,v_n)$ is *nondegenerate* if $v_1,\dots,v_n$ are independent. $M\in E^{n\times n}$ is *nondegenerate* if all its principal minors are nonzero. $y\in E^n$ is said to be *nondegenerate with respect to* $M\in E^{n\times n}$ if y is not a linear combination of $n-1$ columns from the matrix $(I,-M)$, where I is the identity matrix. For $x\in E^n$, $x\geqslant 0$ means each component $x_i\geqslant 0$.

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1. INTRODUCTION

Given a real $n\times n$ matrix M and a vector $q\in E^n$, the linear complementarity problem, denoted by (M/q), is to find vectors w and z such that

$$w-Mz=q, \qquad w\geqslant 0, \quad z\geqslant 0, \quad w^t z=0.$$

This problem arises in such areas as economics, game theory, geometry, linear and quadratic programming, mechanics, and numerical analysis [2–7, 9, 10, 13]. Typically the matrix M will have a very special structure depending on the source of the problem, and this structure will be extensively exploited. The usual questions of existence and uniqueness of solutions to (M/q) are answered, and often whether or not (M/q) has a solution for all q is considered. Answers to this latter question usually use special properties of M provided by the source of the problem.

In this paper we are concerned with the fundamental question: What conditions on M in the class of $n\times n$ real matrices are necessary and sufficient for (M/q) to have a solution for all q? A matrix M such that (M/q) has a solution for all $q\in E^n$ is called a Q-matrix. Such matrices were studied in considerable detail in [11] and [15], but a concise, nontrivial characterization of them has remained elusive, as predicted by Ingleton [6]. There are a large number of conditions *sufficient* for M to be a Q-matrix (most of which are mentioned in [15]), and some *necessary* conditions were proved in [11] and [15]. It is known, for example, that there is little connection between Q-matrices and principal minor signs, and that eigenvalues (without symmetry) are irrelevant [15].

Most of the work on the linear complementarity problem has been algebraic in nature, with a few exceptions such as [8], [11], [12], [13], [14], [15]. The geometric concept of a complementary cone has proved useful. $\mathcal{C}(A)$, where $A_{\cdot i}\in\{e_i, -M_{\cdot i}\}$, $i=1,\ldots,n$, is called a *complementary cone* formed from M. (M/q) has a solution if and only if q is in some complementary cone, and M is a Q-matrix if and only if the union of all the complementary cones $\mathcal{C}(A)$ is E^n. A similar geometric view of the linear complementarity problem using spheres is presented here, and used to produce a 4×4 nondegenerate Q-matrix which is on the boundary of the set of 4×4 Q-matrices. Besides producing this 4×4 example, the spherical geometry approach has certain conceptual advantages worth pursuing.

A counterexample to a reasonable characterization conjecture is presented in Sec. 2. Section 3 develops the spherical geometry, a characterization of nondegenerate Q-matrices, and a generalization of the P-matrix partition theorem [11, 13]. Section 4 discusses 3×3 Q-matrices, and Sec. 5 derives the 4×4 counterexample of [8].

2. A CONJECTURE

For simplicity, only nondegenerate matrices will be considered; this is no serious loss of generality, since every Q-matrix is in the closure of the set of nondegenerate Q-matrices [14]. A practical characterization of Q-matrices means that given a nondegenerate matrix M, it is possible to determine in a finite number of (algebraic) steps whether or not M is a Q-matrix. Given q, whether or not (M/q) has a solution can be verified. Thus a nice characterization would say "(M/q) has a solution for all q iff (M/q) has a solution for all $q \in S$, S a finite set." This is particularly desirable because (M/q) can often be solved very efficiently [15]. Murty [11] has a similar finite set characterization for P-matrices.

PROPOSITION. *Let M be a real 2×2 nondegenerate matrix. Then M is a Q-matrix if and only if $(M/-e_1)$ and $(M/-e_2)$ have solutions.*

Proof. Necessity: Trivial.

Sufficiency: Suppose M is not a Q-matrix, and assume that $(M/-e_1)$ has a solution. Then $-e_1 \in \mathcal{C}(-M_{.1}, v)$, where $v \in \{e_2, -M_{.2}\}$. Suppose $v = e_2$, in which case $-M_{.1}$ must lie in the third quadrant. Since M is not a Q-matrix, $-M_{.2}$ must lie in the first or second quadrants with $\det M < 0$. But then $(M/-e_2)$ has no solution. Suppose $v = -M_{.2}$. Then $-e_1 \in \mathcal{C}(-M_{.1}, -M_{.2})$ and M not a Q-matrix imply $-M_{.1}$ must lie in the third quadrant, which was the previous case. ∎

This leads to the very reasonable

CONJECTURE. Let M be a real $n\times n$ nondegenerate matrix. Then M is a Q-matrix if and only if $(M/-e_i)$ has a solution for $i=1,\ldots,n$.

Unfortunately, a counterexample is provided by

$$M(\epsilon)=\begin{bmatrix} 1 & -1 & 4 \\ 4 & -3 & 1 \\ 1 & 4\epsilon & -\epsilon \end{bmatrix}, \quad q(\epsilon)=\begin{pmatrix} 0 \\ 1 \\ -\epsilon \end{pmatrix}, \qquad 0<\epsilon<0.1.$$

The complementary cones cover everything except a small polyhedral "wedge" bounded by $\mathcal{C}(e_1,e_2)$, $\mathcal{C}(-M_{.2},e_3)$, $\mathcal{C}(-M_{.2},-M_{.3})$, $\mathcal{C}(-M_{.1},e_2)$. As ϵ goes to zero, this open wedge becomes arbitrarily small (in measure), and $M(0)$ is a Q-matrix. This shows that it is *impossible* to have a finite "test" set S *independent* of M. Furthermore, the example in Sec. 5 shows that any such finite test set S, if it exists, must depend on *every* column of M. This is also a counterexample for Murty's P-matrix test set $\{-e_1,\ldots,-e_n,M_{\cdot 1},\ldots,M_{\cdot n}\}$. A simple test set depending on M still remains a possibility, though, and warrants further investigation.

3. PRELIMINARIES ON SPHERES

The question posed in the first paragraph of the introduction is clearly subject to the following reformulation. Let $M\equiv\{M_i\}$, $N\equiv\{N_i\}$, $i=1,2,\ldots,n$, be two n-tuples of linearly independent points on the unit sphere S^{n-1} in E^n such that every set $\{P_1,\ldots,P_n\}$ with $P_i\in\{M_i,N_i\}$ is independent, and call a spherical $(n-1)$-simplex with vertices P_i *complementary* relative to M and N if $P_i\in\{M_i,N_i\}$. Under what circumstances will a set of 2^n complementary $(n-1)$-simplices cover the sphere?

Our first criterion will be in terms of visibility sets, which we now define.

DEFINITION. If in $S^{n-1},[E^n]$ U is a nonempty set and P a point, then $\mathrm{Vis}(P,U)$ is the union of the half-open segments $\overline{PX}^{\bullet}$ in $S^{n-1},[E^n]$ which lie entirely in the complement of U.

DEFINITION. If in $S^{n-1},[E^n]$ U is a nonempty set and P a point, then $\mathrm{St}(P,U)$ is the union of closed segments $\overline{PX}$, $X\in U$.

$\mathrm{St}(P,U)$ is called the *star of P relative to U*. If for some point P of U, $\mathrm{St}(P,U)=U$, then U is said to be *starlike* (*from* P). The set of points from which a set is starlike is its *nucleus*.

Segments in these two definitions refer, of course, to spherical segments in S^{n-1}, that is, great circle arcs of length less than or equal to π and Euclidean segments in E^n.

LEMMA 1. *For nonempty closed* $U\subset S^{n-1}$, $\mathrm{St}(P,U)$ *and* $\mathrm{Vis}(-P,U)$ *are set theoretic complements.*

Proof. Clear. ∎

LEMMA 2. *For nonempty closed* $U\subset S^{n-1}$, $\mathrm{St}(P,U)\cup\mathrm{St}(Q,U)=S^{n-1}$ *iff* $\mathrm{Vis}(-P,U)\cap\mathrm{Vis}(-Q,U)=\varnothing$.

Proof. Let $*$ denote complementation. Then $[\mathrm{St}(P,U)\cup\mathrm{St}(Q,U)]^*=[\mathrm{St}(P,U)]^*\cap[\mathrm{St}(Q,U)]^*=\mathrm{Vis}(-P,U)\cap\mathrm{Vis}(-Q,U)$. ∎

It is convenient to extend the definition of complementary simplices for $k\neq n$.

Let $M \equiv \{M_i\}$, $N \equiv \{N_i\}$, $i=1,2,\dots,k$, be two k-tuples of linearly independent points in $S^{n-1},[E^n]$ such that every set $\{P_1,\dots,P_k\}$ with $P_i \in \{M_i,N_i\}$ is independent. A $(k-1)$-simplex with vertices $P_i \in \{M_i,N_i\}$ is a *complementary* $(k-1)$*-simplex relative to M and N*. The set of such simplices is denoted $C^k(M,N)$. By $C_j^k(M,N)$ we will mean the $C^{k-1}(\hat{M},\hat{N})$, where $\hat{M}=M\setminus\{M_j\}$ and $\hat{N}=N\setminus\{N_j\}$.

Observation. $C^{k+1}(M,N)$ is a combinatorial k-cycle. That is, it is a combinatorial k-complex having 2^{k+1} k-simplices and a null $(k-1)$-boundary. Conceivably the well-developed theory concerning such complexes could be useful in these considerations, although in this paper we make no explicit use of it aside from the fact that the complement of a nonsingular [as a combinatorial $(n-2)$-cycle] $C^{n-1}(M,N)$ on an S^{n-1} has at least two components.

Definition. $C^n(M,N)$ is a Q-arrangement on S^{n-1} if the $(n-1)$-simplices in $C^n(M,N)$ cover S^{n-1}.

Observe that $C^n(M,N)$ is a Q-arrangement iff for some $j=1,\dots,n$ the stars $\mathrm{St}(M_j,C_j^n(M,N))$ and $\mathrm{St}(N_j,C_j^n(M,N))$ together cover S^{n-1}.

The above lemmas will be used in the form of:

Theorem 3. *$C^n(M,N)$ is a Q-arrangement on S^{n-1} iff $\mathrm{Vis}[-M_j,C_j^n(M,N)]\cap \mathrm{Vis}[-N_j,C_j^n(M,N)]=\varnothing$ for some $j=1,2,\dots,n$.*

Note that if this intersection is not empty, then it is precisely the set of points not covered by $C^n(M,N)$.

Theorem 4. *$C^n(M,N)$ is a Q-arrangement on S^{n-1} if $-M_j$ and $-N_j$ are in different components of the complement of $C_j^n(M,N)$ for some $j=1,\dots,n$.*

Proof. This is an immediate consequence of Theorem 3. ∎

Theorem 5. *If $C^n(M,N)$ is a Q-arrangement on S^{n-1}, then the interior of $\{\mathrm{Vis}[-M_j,C_j^n(M,N)]\}^*\cap\{\mathrm{Vis}[-N_j,C_j^n(M,N)]\}^*=\{\mathrm{Vis}[-M_j,C_j^n(M,N)]\cup \mathrm{Vis}[-N_j,C_j^n(M,N)]\}^*$ is in the (interior of the) set of multiply covered points.*

Proof. This is an immediate consequence of Theorem 3 and the subsequent observation. ∎

Definition. A Q-arrangement in which no point of S^{n-1} not on an $(n-2)$-simplex is multiply covered is a P-arrangement. That is to say, each two of the covering simplices have disjoint interiors.

THEOREM 6. *$C^n(M,N)$ is a P-arrangement on S^{n-1} iff for all $j=1,\dots,n$ the complement of $C_j^n(M,N)$ is the union of two starlike components with $-M_j$ and $-N_j$ in their respective nuclei.*

Proof. Necessity: If some component of the complement of $C_j^n(M,N)$ contains neither $-M_j$ or $-N_j$, then that component does not intersect $\mathrm{Vis}[-M_j, C_j^n(M,N)]$ or $\mathrm{Vis}[-N_j, C_j^n(M,N)]$. Thus it is in the complement of their union. According to Theorem 5 this means that each interior point of this component is multiply covered. Hence the complement of $C_j^n(M,N)$ consists of at most two components. Since no two $(n-2)$-cells of the $(n-2)$-cycle $C_j^n(M,N)$ can have interior points in common, it follows from topological considerations that $C_j^n(M,N)$ separates S^{n-1}. Thus the complement of $C_j^n(M,N)$ consists of precisely two components, one of which contains $-M_j$ and the other $-N_j$.

If $-M_j$ were not in the nucleus of its component, then some point, p, of that component would not be in $\mathrm{Vis}[-M_j, C_j^n(M,N)]$. Since that point is certainly not in $\mathrm{Vis}[-N_j, C_j^n(M,N)]$, it would be in the interior of the complement of the union of these two sets. According to Theorem 5, p would be multiply covered. Hence $-M_j$ must be in the nucleus of the component of $C_j^n(M,N)$ in which it lies, and similarly for $-N_j$.

The sufficiency follows from Theorem 4 and $\mathrm{Vis}(-M_j, C_j^n(M,N)) \cup \mathrm{Vis}(-N_j, C_j^n(M,N)) \cup C_j^n(M,N) = S^{n-1}$, whence there are no multiply covered (interior) points by Theorem 5. ■

COROLLARY [11], [13]. *$C^n(M,N)$ is a P-arrangement on S^{n-1} iff M_i and N_i are on opposite sides of each $(n-2)$-simplex of $C_i^n(M,N)$ for each i; i.e., M_i and N_i are on opposite sides of the hyperplane determined by the vertices of that face and the center 0 of S^{n-1}.*

Proof. Let $-M_j$ and $-N_j$ be in the nuclei of their respective starlike components, and let Δ be any face in $C_j^n(M,N)$. Note that neither M_j nor N_j are in Δ or the hyper-great-sphere σ containing Δ. Now let $P \neq M_j$ be a point in the relative interior of Δ. Join $-M_j$ to P, and let Q be a point such that P is between $-M_j$ and Q, length $(\mathrm{arc}(-M_j,Q)) < \pi$, and $Q \notin C_j^n(M,N)$. Q is not visible from $-M_j$ and hence must be in the component containing $-N_j$. If $-N_j = M_j$, then certainly M_j and N_j are on opposite sides of σ. So $-M_j$, $-N_j$ may be assumed to be end points of a unique segment $\mathrm{arc}(-M_j, -N_j)$. Now σ cuts one side of triangle $-M_j, -N_j, Q$ and hence must cut precisely one other side. Let Q_i be a sequence of points on the segment $\mathrm{arc}(QP)$ approaching P. If $\mathrm{arc}(-N_j, Q_i) \cap \sigma = T_i$ for each i, then T_i must approach P. But $T_i \notin \Delta$ for any i, since Q_i is visible from $-N_j$ for each i. Hence for some i

$\mathrm{arc}(-N_j, Q_i) \cap \sigma = \varnothing$ and therefore $\mathrm{arc}(-M_j, -N_j) \cap \sigma \neq \varnothing$, i.e., $-M_j$ and $-N_j$ are on opposite sides of σ.

Conversely, suppose that for each j, $-M_j$ and $-N_j$ are on opposite sides of each face of $C_j^n(M,N)$. This says that there can be no multiply covered (interior) points, i.e., $C^n(M,N)$ is a P-arrangement. ∎

In algebraic terms this says that if two complementary simplices in a P-arrangement $C^n(M,N)$ differ in exactly one vertex, then the two nth order determinants of their defining vectors have opposite signs.

A matrix $M=(M_1,\dots,M_n)$ is said to be a P-matrix if all its principal minors are positive [11, 14]. We then have

COROLLARY [11, 14]. *$C^n(-M,I)$ is a P-arrangement iff $M=(M_1,\dots,M_n)$ is a P-matrix.*

The following theorem has some general interest and is specifically needed in Sec. 5.

THEOREM 7. *If $\sigma = S^{n-1}$ is a hypersphere with center 0, $\pi = E^{n-1}$ a hyperplane, not through 0, both in E^n, and if $T:\pi\to\sigma$ is a map such that for each $X\in\pi$, $X, T(X), 0$ are collinear with 0 not between X and $T(X)$, then for any set U and point P of π for which* $\mathrm{Vis}(P,U)$ *is bounded,* $T[\mathrm{Vis}(P,U)] = \mathrm{Vis}[T(P),T(U)]$.

Proof. T, of course, is a central projection of π into σ from the center 0. The image $T(\pi)$ is confined to an open hemisphere σ^+ of σ, and the map is 1-1. Straight lines in π map into great semicircles, and segments in π map into spherical segments in σ. It follows at once that if $\overline{PX}$ is in the complement of U, then $T(\overline{PX}) = \mathrm{arc}(T(P)T(X))$ is in the complement of $T(U)$. That is to say, $T[\mathrm{Vis}(P,U)] \subset \mathrm{Vis}[T(P),T(U)]$.

Suppose now that $Y\in\sigma$ is in $\mathrm{Vis}[T(P),T(U)]$. If Y is not on σ^+, then there is a sequence of points $\{Y_i\}$ on σ^+ and the segment $\overline{YT(P)}$ converging to a point on the boundary of σ^+. $T^{-1}(Y_i)$ is then an unbounded sequence of points in $\mathrm{Vis}(P,U)$, and this is impossible, since $\mathrm{Vis}(P,U)$ is bounded. Thus $\mathrm{Vis}[T(P),T(U)]$ is confined to σ^+.

Since both T and T^{-1} preserve segments, it follows that $\mathrm{Vis}[T(P),T(U)] = T[\mathrm{Vis}(P,U)]$. ∎

4. A CHARACTERIZATION OF Q-ARRANGEMENTS ON S^2

Let $M\equiv\{M_1,M_2,M_3\}$, $N\equiv\{N_1,N_2,N_3\}$ be two triples of points of S^2, $M\cap N=\varnothing$. Then $C_1^3(M,N)\equiv \overline{M_2M_3}\cup\overline{N_2N_3}\cup\overline{M_2N_3}\cup\overline{M_3N_2}$, and an easy analysis shows that the complement of this spherical 1-cycle consists of 1, 2, or 3 components depending on whether $C_1^3(M,N)$ is a simple closed curve or not. In the third case exactly one pair of opposite edges of the 1-cycle intersect. In all cases the components are easily seen to be starlike (i.e., each is the star of some point in the set) and thus if P and Q are any two points in such a component, U, then $\mathrm{Vis}(P, C_1^3(M,N) \cap \mathrm{Vis}(Q, C_1^3(M,N) \neq \varnothing$. It follows from Theorem 3 that $\mathrm{St}[M_1, C_1^3(M,N)] \cup \mathrm{St}[N_1, C_1^3(M,N)] = S^2$ iff $-M_1$ and $-N_1$ are in different components of the complement of $C_1^3(M,N)$. Therefore

THEOREM 8. *M and N form a Q-arrangement on S^2 iff $-M_1$ and $-N_1$ are in different components of the complement of $C_1^3(M,N)$.*

Note that Theorem 8 and Fig. 1 make the stability of Q-arrangements on S^2 obvious.

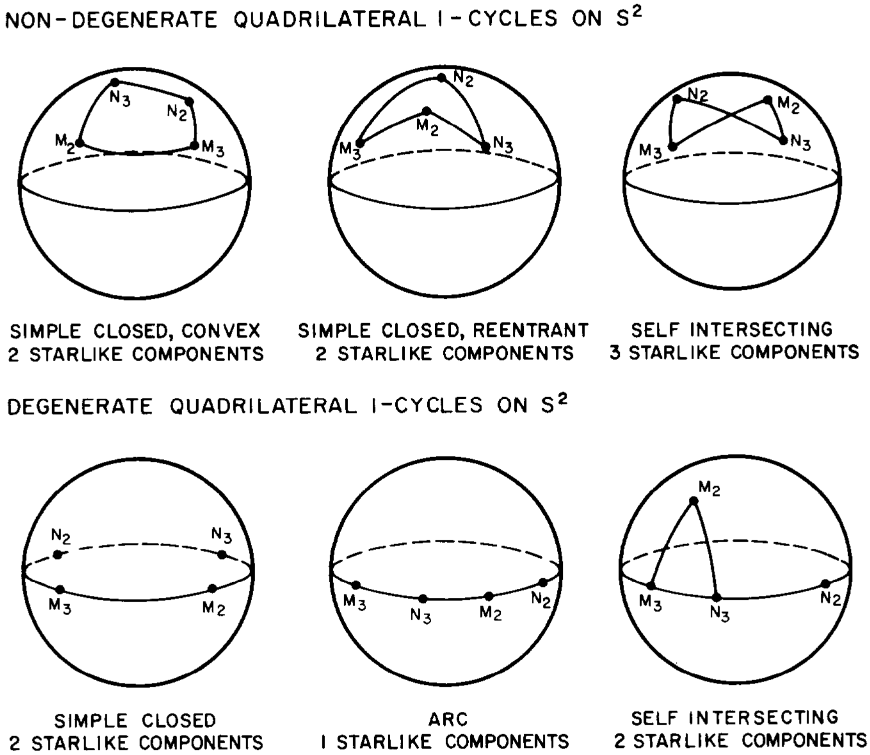


FIG. 1. 1-cycles $C_1^3(M,N)$ on S^2.

5. THE STRUCTURE OF Q-ARRANGEMENTS IN S^3

If the components of the complement of $C_j^4(M,N)$ on S^3 were all starlike, then the S^2 characterization theorem would extend to S^3. However, we will construct a $C^4(M,N)$ in S^3 such that one of the components of the complement of $C_4^4(M,N)$ is not starlike, and this will lead to an unstable nondegenerate Q-arrangement. This implies that the set of 4×4 nondegenerate Q-matrices is not open in the usual matrix topologies.

We will first informally describe the configuration and then present the appropriate coordinate description together with the associated "unstable" 4×4 nondegenerate Q-matrix.

We wish to construct a "small" complementary 2-cycle with 8 triangular 2-cells on S^3 in the neighborhood of the north pole $(0,0,0,1)$, such that one of the components of its complement is rather complicated, i.e., not starlike. To do this we construct such a configuration in the tangent hyperplane, τ, to S^3 at the north pole, and from 0 centrally project the configuration onto the sphere. Such a projection, we have seen, preserves bounded visibility sets, and so various relevant properties of the spherical configuration can be "read off" of the E^3 configuration. We could, in fact, bypass the spherical projection altogether and go directly to the Murty [11] cone interpretation, though this would entail restating some of our earlier spherical observations into cone language, which, in some cases, would be awkward.

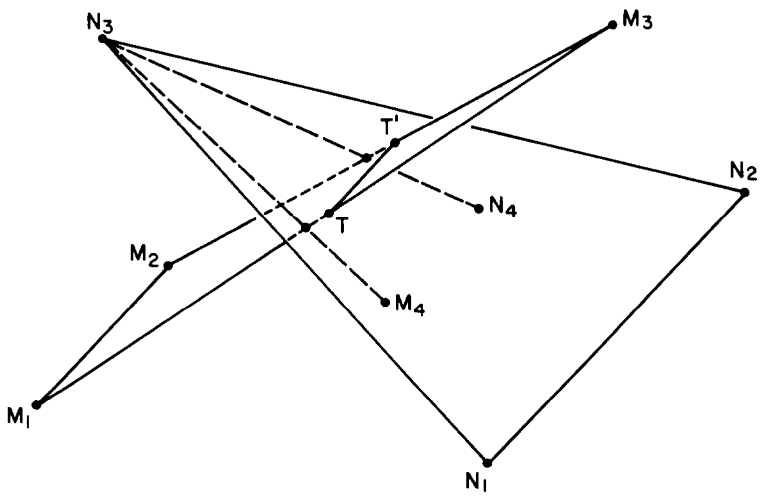


FIG. 2.

The E^3 configuration consists of two triangles, $\triangle M_1M_2M_3$ and $\triangle N_1N_2N_3$ (see Fig. 2), with the first piercing the interior of the second in a short segment parallel to both bases $\overline{M_1M_2}$ and $\overline{N_1N_2}$. $\overline{M_1M_2}$ and $\overline{N_1N_2}$ are two bases of an isosceles trapezoid in a "horizontal" plane. N_3 may be taken directly above the $\overline{M_1M_2}$ midpoint and M_3 directly above the $\overline{N_1N_2}$ midpoint.

More specifically, consider the following six points in E^3: $M_1(1,-2,0)$, $M_2(-1,-2,0)$, $M_3(0,2,2)$, $N_1(2,2,0)$, $N_2(-2,2,0)$, $N_3(0,-2,2)$. As observed earlier, the complementary triangles in $C^3(M,N)$ form a topological 2-cell cycle of 8 triangular cells whose complement consists of more than one component. Our only concern is to see that one of these components is not starlike and to use this component to appropriately define M_4 and N_4 so that their respective visibility sets relative to $C^3(M,N)$ "just barely" fail to intersect. The associated projected points in S^3 under the mapping T described in Theorem 7 are denoted $\hat{M}_i=T(M_i)$ and $\hat{N}_i=T(N_i)$, $i=1,2,3,4$. Since T is a homeomorphism of the $E^3=\tau$ onto σ^+ which also preserves segments and bounded visibility sets, it will follow that if $\hat{M}\equiv\{\hat{M}_1,\hat{M}_2,\hat{M}_3,-\hat{M}_4\}$ and $\hat{N}\equiv\{\hat{N}_1,\hat{N}_2,\hat{N}_3,-\hat{N}_4\}$, then $C^4(\hat{M},\hat{N})$ is an unstable nondegenerate Q-arrangement in S^3.

The defining vectors for the associated complementary cones can then be obtained by adjoining 1 as a fourth component to the M_i and N_i $(i=1,2,3)$, and -1 to $-M_4$ and $-N_4$. This gives rise to a nondegenerate unstable Q-matrix $\mathfrak{M}$.

Returning to specifics, we claim that if $M=\{M_1,M_2,M_3,M_4\}$ and $N=\{N_1,N_2,N_3,N_4\}$ where $M_4=(\frac{3}{4},\frac{1}{3},\frac{1}{2})$, $N_4=(-\frac{3}{4},\frac{1}{3},\frac{1}{2})$, then $\mathrm{Vis}[M_4,C_4^4(M,N)]\cap\mathrm{Vis}[N_4,C_4^4(M,N)]=\varnothing$, while for a suitably chosen sequence $\{P_i\}$ converging to N_4 we have $\mathrm{Vis}[P_i,C_4^4(M,N)]\cap\mathrm{Vis}[M_4,C_4^4(M,N)\neq\varnothing$, $i=1,2,\ldots$. Furthermore $C^4(\hat{M},\hat{N})$ is not degenerate. This will imply that the set of nondegenerate Q-arrangements on S^3 is not open and hence that there are unstable nondegenerate 4×4 Q-matrices.

We now present a geometric argument showing that the matrix $\mathfrak{M}$, referred to above, is a nondegenerate Q-matrix on the boundary of the set of 4×4 Q-matrices.

Geometric argument. The tangent hyperplane τ to the unit sphere S^3 in E^4 has the equation $x_4=1$. We will operate exclusively in this hyperplane in this argument, and in describing points in this plane we will omit the x_4 coordinate.

Consider $C^4(M,N)$ in τ defined by the points $M_1(1,-2,0)$, $M_2(-1,-2,0)$, $M_3(0,2,2)$, $M_4(\frac{3}{4},\frac{1}{3},\frac{1}{2})$, $N_1(2,2,0)$, $N_2(-2,2,0)$, $N_3(0,-2,2)$, $N_4(-\frac{3}{4},\frac{1}{3},\frac{1}{2})$.

Referring to Fig. 3 (and Fig. 4 for easier visualization), let

$$\overline{M_1M_3} \cap \triangle M_2N_3N_1 = S\left(\tfrac{3}{5}, -\tfrac{2}{5}, \tfrac{4}{5}\right),$$

$$\overline{M_1M_3} \cap \triangle N_1N_2N_3 = T\left(\tfrac{1}{2}, 0, 1\right),$$

$$\overline{N_1N_3} \cap \text{plane}\, M_1M_3N_2 = K\left(\tfrac{6}{7}, -\tfrac{2}{7}, \tfrac{8}{7}\right),$$

$$\overline{M_2M_3} \cap \triangle M_1N_2N_3 = S'\left(-\tfrac{3}{5}, -\tfrac{2}{5}, \tfrac{4}{5}\right),$$

$$\overline{M_2M_3} \cap \triangle N_1N_2N_3 = T'\left(-\tfrac{1}{2}, 0, 1\right),$$

$$\overline{N_2N_3} \cap \text{plane}\, N_1M_2M_3 = K'\left(-\tfrac{6}{7}, -\tfrac{2}{7}, \tfrac{8}{7}\right),$$

$$\overline{M_1N_2} \cap \overline{M_2N_1} = L\left(0, -\tfrac{2}{3}, 0\right),$$

$$\overline{LM_3} \cap \text{plane}\, N_1N_2N_3 = J\left(0, \tfrac{2}{5}, \tfrac{4}{5}\right),$$

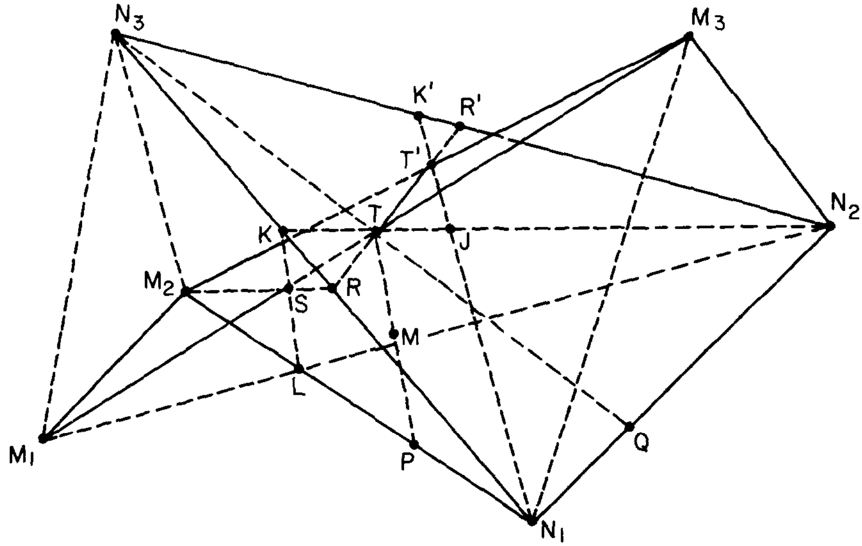
$$\overline{N_3T} \cap \overline{N_1N_2} = Q(1, 2, 0),$$

and

$$\overline{M_1Q} \cap \overline{M_2N_1} = P\left(1, \tfrac{2}{3}, 0\right).$$

Observe the following: M_4 is on $\overline{TP}$ and is interior to tetrahedron KN_1JL. $\triangle KN_1L \subset \triangle M_2N_3N_1$, $\triangle N_1JL \subseteq \triangle M_2M_3N_1$, $\triangle KN_1J \subset \triangle N_1N_2N_3$, and $\triangle KJL$ is in the plane of $\triangle M_1M_3N_2$ but is not a subset of this triangle. However, the convex quadrilateral $LSTJ$ is a subset of $\triangle KLJ$. $\triangle KST$ is the only subset of the surface of the tetrahedron KN_1JL not a subset of $C_4^4(M,N)$.

Now M_4 is also clearly in the interior of $\Sigma = \text{tet}\, KN_1JL \cup \text{tet}\, N_3KST$. ($N_3$ and N_1 are on opposite sides of plane $KST =$ plane $M_1M_3N_2$.) But $\triangle KSN_3 \subset \triangle N_1M_2N_3$, $\triangle KTN_3 \subset \triangle N_1N_2N_3$. Thus $\text{Vis}[M_4, C_4^4(M,N)] \subset \text{int tet}\, JKLN_1 \cup \text{int tet}\, KSTN_3 \cup \text{rel int}\, \triangle KST \cup \text{cone}[M_4, \triangle TSN_3]$. But since M_4 is in the plane of $\triangle TSN_3$ and $\overline{ST} \subset \overline{M_1M_3}$, we conclude that $\text{Vis}[M_4, C_4^4(M,N)] \subset \text{int tet}\, KJLN_1 \cup \text{int tet}\, KSTN_3 \cup \text{rel int}\, \triangle KST \subset \{x_1 > 0\}$. Similarly $\text{Vis}[N_4, C_4^4(M,N)] \subset \{x_1 < 0\}$, and hence $\text{Vis}[M_4, C_4^4(M,N)] \cap \text{Vis}[N_4, C_4^4(M,N)] = \varnothing$. This concludes the proof that the two visibility sets are disjoint and hence that $C^4(\hat{M}, \hat{N})$ on S^3 is a Q-arrangement. Actually, with a little more argument we could show that these last two set inclusions



$M_1(1,-2,0)$	$T(\frac{1}{2},0,1)$ $T'(-\frac{1}{2},0,1)$
$M_2(-1,-2,0)$	$S(\frac{3}{5},-\frac{2}{5},\frac{4}{5})$
$M_3(0,2,2)$	$K(\frac{6}{7},-\frac{2}{7},\frac{8}{7})$ $K'(-\frac{6}{7},-\frac{2}{7},\frac{8}{7})$
$M_4(\frac{3}{4},\frac{1}{3},\frac{1}{2})$	$J(0,\frac{2}{5},\frac{4}{5})$
$N_1(2,2,0)$	$L(0,-\frac{2}{3},0)$
$N_2(-2,2,0)$	$Q(1,2,0)$
$N_3(0,-2,2)$	$P(1,-\frac{2}{3},0)$
$N_4(-\frac{3}{4},\frac{1}{3},\frac{1}{2})$	$R(1,0,1)$ $R'(-1,0,1)$

FIG. 3. $C^4(M,N)$ in the hyperplane τ.

are set identities, but the weaker conclusion will serve our present purpose.

We now wish to make clear that the arrangement is unstable. Let $\overline{N_4N_3} \cap \overline{S'T'} = F$, and consider a sequence of points $\{X_i\}$ in $\mathrm{Vis}[M_4, C_4^4(M,N)]$ converging to N_3. Define Y_i and P_i so that F is between X_i and Y_i, $\mathrm{dist}(N_4,F) = \mathrm{dist}(Y_i,F)$, and Y_i is the midpoint between P_i and N_4. It should now be clear that as $X_i \to N_3$, $P_i \to N_4$, and that for i sufficiently large the segment $\overline{X_iP_i}$ is in a component of the complement of $C_4^4(M,N)$ (which contains both M_4 and N_4). This means that $\mathrm{Vis}[P_i, C_4^4(M,N)] \cap \mathrm{Vis}[M_4, C_4^4(M,N)] \neq \varnothing$, or in other words that $C^4(\hat{M},\hat{N})$ is an unstable nondegenerate Q-arrangement on S^3, from which the desired properties of $\mathfrak{M}$ follow. ∎

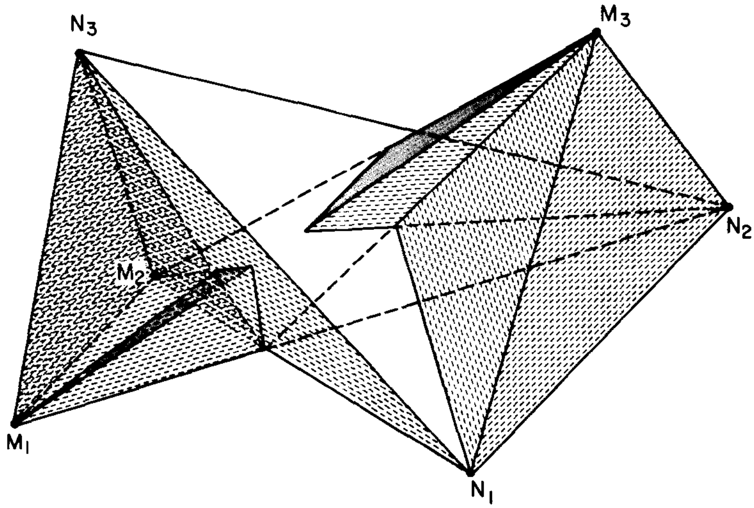


FIG. 4.

The points in $\hat{M}$ and $\hat{N}$ lead to the algebraic problem

$$Rw - Sz = q,$$

$$w \geq 0, \quad z \geq 0, \quad w^t z = 0,$$

where

$$R = \begin{pmatrix} N_1 & N_2 & N_3 & -M_4 \\ 1 & 1 & 1 & -1 \end{pmatrix} = \begin{bmatrix} 2 & -2 & 0 & -\frac{3}{4} \\ 2 & 2 & -2 & -\frac{1}{3} \\ 0 & 0 & 2 & -\frac{1}{2} \\ 1 & 1 & 1 & -1 \end{bmatrix},$$

$$S = \begin{pmatrix} -M_1 & -M_2 & -M_3 & N_4 \\ -1 & -1 & -1 & 1 \end{pmatrix} = \begin{bmatrix} -1 & 1 & 0 & -\frac{3}{4} \\ 2 & 2 & -2 & \frac{1}{3} \\ 0 & 0 & -2 & \frac{1}{2} \\ -1 & -1 & -1 & 1 \end{bmatrix},$$

and

$$q = \begin{bmatrix} 0.04 \\ -1.90 \\ 1.94 \\ 1.00 \end{bmatrix}$$

is a nondegenerate point in $\operatorname{int}\operatorname{tet} TN_1M_2N_3$. With $\mathcal{M} = \operatorname{diag}(8,8,8,\frac{2}{3})R^{-1}S\operatorname{diag}(1,1,1,12)$ this reduces to the problem (as stated in [8])

$$\tilde{w} - \mathcal{M}\tilde{z} = \tilde{q},$$

$$\tilde{w} \geqslant 0, \quad \tilde{z} \geqslant 0, \quad \tilde{w}^t\tilde{z} = 0,$$

where

$$\mathcal{M} = \begin{bmatrix} 21 & 25 & -27 & -36 \\ 7 & 3 & -9 & 36 \\ 12 & 12 & -20 & 0 \\ 4 & 4 & -4 & -8 \end{bmatrix}, \qquad \tilde{q} = \begin{bmatrix} 0.26 \\ -0.02 \\ 30.8 \\ -0.08 \end{bmatrix}.$$

The points X_i, Y_i, and P_i mentioned above lead [after the transformation which took R, S, q, and $(N_3, 1)$ to $\mathcal{M}$, $\tilde{q}$, and $32e_3$] to the perturbations

$$\delta\mathcal{M} = \begin{bmatrix} 0 & 0 & 0 & -\epsilon \\ 0 & 0 & 0 & \epsilon \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$q = (1-\epsilon)32e_3 + \epsilon\tilde{q},$$

$0 < \epsilon < 1$. In algebraic terms, we have proved

LEMMA. *$\mathcal{M}$ is a nondegenerate Q-matrix, but $\mathcal{M} + \delta\mathcal{M}$ is nondegenerate and not a Q-matrix, and $(\mathcal{M} + \delta\mathcal{M}/q)$ has no solution.*

THEOREM 9. *The set of real 4×4 nondegenerate Q-matrices is not open.*

The authors have also proved Theorem 9 (which was stated in [8] without proof) using only the concept of complementary cones. That approach is exceedingly tedious and cumbersome compared to the preceding spherical geometry, and in general verifying that the complementary cones cover E^4 is extremely difficult.

6. CONCLUSION

Is there any reasonably broad subclass of Q-matrices (other than the one given in [8]) which is open? Describing such a class in terms of linear independence and nonzero minors seems unlikely, since slight modifications of our choice of M_4 and N_4 in the construction of the 4×4 matrix produce unstable Q-arrangements of great variety. Results such as the stability of nondegenerate Q-arrangements in E^3 [14] and the partitioning property of P-matrices [11] are transparent using spherical geometry. It is hoped that this approach will aid further investigations.

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