Computers and Mathematics with Applications 62 (2011) 1531–1539

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Analysis of nonlinear dynamics and chaos in a fractional order financial system with time delay ☆

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ARTICLE INFO

Keywords:
Chaos
Delay
Fractional order financial system

ABSTRACT

In this paper, a delayed fractional order financial system is proposed and the complex dynamical behaviors of such a system are discussed by numerical simulations. A great variety of interesting dynamical behaviors of such a system including single-periodic, multiple-periodic, and chaotic motions are displayed. In particular, the effect of time delay on the chaotic behavior is investigated, it is found that an approximate time delay can enhance or suppress the emergence of chaos. Meanwhile, corresponding to different values of delay, the lowest orders for chaos to exist in the delayed fractional order financial systems are determined, respectively.

1. Introduction

In recent years, study on the complex dynamics of financial and economical systems has become a very prominent problem in both micro- and macroeconomics [1]. Researchers have been attempted to explain the central features of economic data by virtue of the dynamical behaviors exhibited in the systems. Several nonlinear continuous models have been proposed to study the complex economics dynamics, such as the forced van der Pol model [2,3], the IS-LM model [4–6], the Kaldorian model [7] and Goodwin's accelerate model [8]. As we all know, it is natural for nonlinear systems to exhibit periodic or chaotic behaviors, so is economical models. However, if there exist chaotic phenomena in economic systems, that means the systems will have inherent indefiniteness in itself, and this makes it hard to give a reasonable or effective economic prediction. Therefore, it is indispensable to investigate the complex dynamical behaviors, especially chaos, in economical and financial systems.

Fractional calculus, as an old mathematical topic, is now attracting intensive research in nearly all kinds of fields [9–16]. The major merit of fractional calculus, different from integer calculus, lies in the fact that it has memory, and has proven to be a very suitable tool for the description of memory and hereditary properties of various materials and processes [15,17,16,18]. As is well known, financial variables possess long memories, which make it appropriate to use fractional modeling to describe dynamic behaviors in financial systems. Chen [19] proposed a fractional order financial system, studied its dynamic behaviors, such as fixed points, periodic motions, chaotic motions, and identified period doubling and intermittency routes to chaos, Meanwhile, it has been shown that chaos exists in the system with orders less than 3 by numerical simulations. Refs. [20,21] studied the chaos control method of such a kind of system by slide mode and feedback control, respectively.

☆ This work was supported by the National Natural Science Foundation of China under grant 61004078, the Natural Science Foundation of Shandong Province under grant ZR2009GQ009 and the Spring Buds Program of Shandong University of Science and Technology under grant 2010AZZ182.

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doi:10.1016/j.camwa.2011.04.057

Time delay (time lag) in a financial system means that one policy from being made to taking effect will have to go through a period of time, and its existence and influence have been known to be not negligible. Though it is still difficult to determine the delay in a financial system accurately, but as an objective existence, it dominates the financial policy and makes it produce effect on the economy to a great extent. Modern research on economic dynamics has incorporated time delay into economic models. It was the first time that Kalecki introduced a time delay into the dynamical economic processes [22]. After the pioneering work of Kalecki, the research on a delayed financial system has become a hot issue in recent years [23–28]. However, to the best of our knowledge, there is no effort being made in the literature to study the dynamics of the delayed financial Chen system so far. The main aim of this paper is to study the complex dynamics of the delayed financial system, such as period, chaos, etc., and the effect of the time delay.

The rest of this paper is organized as follows. In Section 2, the analytical conditions which ensure the existence of chaos are derived, and chaos is simultaneously verified by numerical simulations. In Section 3, a delayed fractional order financial system is proposed, and its complex dynamics are investigated, with varying fractional order and time delay. Meanwhile, the lowest order for chaos to exist is determined. Finally, some concluding remarks are given in Section 4.

2. Definitions and lemmas

Till now, there are three different definitions for fractional derivatives [15], that is, Riemann–Liouville, Grünwald–Letnikov and Caputo definitions. This paper is based on Caputo definition, and the Adams–Bashforth–Moulton method [29] is used in the subsequent simulations.

Definition 1 (*[15]*)**.** The Caputo fractional derivative of order α of a continuous function $f : R^+ \to R$ is defined as follows

$$D_t^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t \dfrac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}} \mathrm{d}\tau, & m-1 < \alpha < m, \\ \dfrac{\mathrm{d}^m}{\mathrm{d}t^m} f(t), & \alpha = m, \end{cases}$$

where Γ is Γ-function, and

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \mathrm{d}t, \qquad \Gamma(z+1) = z\Gamma(z).$$

In a 3D nonlinear integer order system, saddle point and its index are defined as follows.

Definition 2. A saddle point is an equilibrium point at which the equivalent linearized model has at least one eigenvalue in the stable region and one in the unstable region.

Definition 3. A saddle point is called **a saddle point of index 1** if one of the eigenvalues is unstable and the others are stable. **A saddle point of index 2** is a saddle point with one stable eigenvalue and two unstable ones.

It should be noted that, for chaotic systems, scrolls are generated only around the saddle points of index 2. The saddle points of index 1 are responsible only for connecting the scrolls. Likewise, in the 3D commensurate fractional order systems, the saddle points of index 2 play a key role in the generation of scrolls. Similar to its integer order counterpart, a necessary condition for a fractional order system $D_t^\alpha x = f(x)$ to exhibit chaotic attractor is the instability of the equilibrium points. Otherwise, one of these equilibrium points becomes asymptotically stable and attracts the nearby trajectories [30].

Lemma 1 (*[31]*)**.** *Autonomous system* $D_t^\alpha x(t) = Ax$, $x(0) = x_0$ *is asymptotically stable if and only if*

$$|arg(eig(A))| > \frac{\alpha\pi}{2}. \tag{1}$$

In this case, each component of the states decays toward 0 like $t^{-\alpha}$. *Also, this system is stable if and only if* $|arg(eig(A))| \geq \frac{\alpha\pi}{2}$ *and those critical eigenvalues that satisfy* $|arg(eig(A))| = \frac{\alpha\pi}{2}$ *have geometric multiplicity one.*

Lemma 2 (*[32]*)**.** *Consider the following n-dimensional linear fractional order system:*

$$\begin{aligned} D_t^{\alpha_1} x_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n, \\ D_t^{\alpha_2} x_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n, \\ &\cdots\cdots \\ D_t^{\alpha_n} x_n &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{aligned} \tag{2}$$

Table 1
Equilibrium points and corresponding eigenvalues.

Equilibrium point	Eigenvalues	Nature	Index
$p_1(0, 10, 0)$	$6.8730, -0.8730, -0.1000$	Saddle point	1
$p_2(-\frac{\sqrt{15}}{5}, 4, \frac{\sqrt{15}}{5})$	$0.3128 \pm 1.2474i, -0.7256$	Saddle point	2
$p_3(\frac{\sqrt{15}}{5}, 4, -\frac{\sqrt{15}}{5})$	$0.3128 \pm 1.2474i, -0.7256$	Saddle point	2

where all α_i are rational numbers between 0 and 1. Assume M to be the lowest common multiple of the denominators u_i's of α_i's, where $\alpha_i = v_i/u_i$, $(u_i, v_i) = 1$, $u_i, v_i \in Z^+$, $i = 1, \dots n$. Then the zero solution of system (2) is Lyapunov globally asymptotically stable if all the roots of equation

$$det(\Delta(\lambda)) = det(diag([\lambda^{M\alpha_1}, \lambda^{M\alpha_2}, \dots, \lambda^{M\alpha_n}]) - (a_{ij})_{n\times n}) = 0 \tag{3}$$

satisfy $|arg(\lambda)| > \frac{\pi}{2M}$.

From Lemma 2, we can see that an equilibrium point is asymptotically stable if the condition $\frac{\pi}{2M} - \min_i |arg(\lambda_i)| < 0$ is satisfied. The term $\frac{\pi}{2M} - \min_i |arg(\lambda_i)|$ is called the instability measure for equilibrium points in fractional order systems (IMFOSs). It should be noticed that some authors have proved by numerical simulations that the condition IMFOS ≥ 0 is only a necessary condition and not the sufficient one for the system to show chaos [33].

3. Dynamics in a fractional order financial system

The authors [34,35] reported a model composed of three first order differential equations to describe the running of financial system:

$$\begin{aligned} \dot{x} &= z + (y-a)x, \\ \dot{y} &= 1 - by - x^2, \\ \dot{z} &= -x - cz. \end{aligned} \tag{4}$$

The three state variables x, y, z stand for the interest rate, the investment demand, the price index, respectively; a, b, c are nonnegative constants, and a is the saving amount (sometimes may be considered as a control parameter), b is the cost per investment, c is the elasticity of demand of the commercial markets. In this paper, the constants are chosen as $a = 3$, $b = 0.1$, and $c = 1$.

In [19], Chen generalized system (4) to fractional order case with the form

$$\begin{aligned} D_t^{\alpha_1} x(t) &= z + (y-a)x, \\ D_t^{\alpha_2} y(t) &= 1 - by - x^2, \\ D_t^{\alpha_3} z(t) &= -x - cz. \end{aligned} \tag{5}$$

It is not difficult to see that the Jacobian matrix of system (5) is

$$J(x, y, z) = \begin{bmatrix} y-a & x & 1 \\ -2x & -b & 0 \\ -1 & 0 & -c \end{bmatrix}.$$

3.1. Commensurate order case ($\alpha_1 = \alpha_2 = \alpha_3 = \alpha$)

The equilibrium points of system (5) and the eigenvalues of its corresponding Jacobian matrix are given in Table 1. From Table 1, it can be seen that p_2, p_3 are saddle points of index 2. Thus, there exists a 2-scroll attractor in system (5) [30].

On the other hand, system (5) will exhibit regular behavior if it satisfies (1). For the equilibrium points p_2, p_3, $\min_i |arg(p_i)| = 1.3251$, and hence, from Lemma 1, (1) is equivalent to

$$\alpha < \frac{2}{\pi} \min_i |arg(p_i)| \approx 0.8436. \tag{6}$$

For the initial values $x_0 = 0.1$, $y_0 = 4$, $z_0 = 0.5$, the numerical method proposed by Diethelm et al. [29] is applied to the simulation of system (5). Fig. 1 indicates that system (5) will converge to a fixed point when $\alpha = 0.83$, and Fig. 2 shows that system (5) will exhibit chaotic attractor when $\alpha = 0.85$.

3.2. Incommensurate order case

As examples, in this section we consider the following cases.

Case I: $\alpha = (0.66, 1, 1)$. In this case, since $v_1 = 33$, $u_1 = 50$, $v_2 = v_3 = u_2 = u_3 = 1$, we get $M = 50$, and

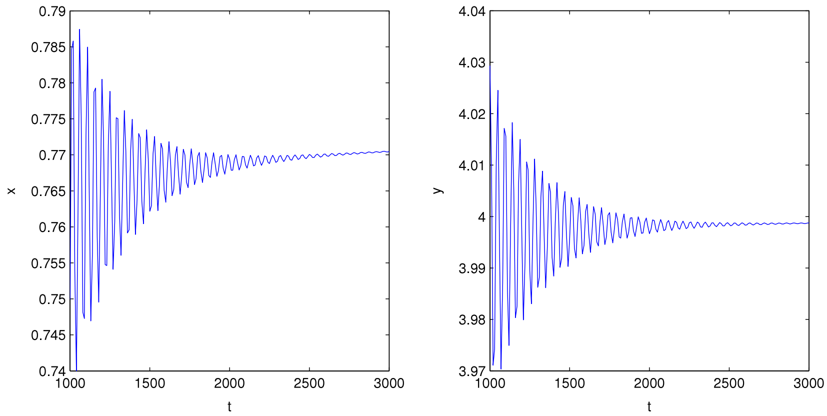

Fig. 1. Time trajectories of system (5) with $\alpha = 0.83$.

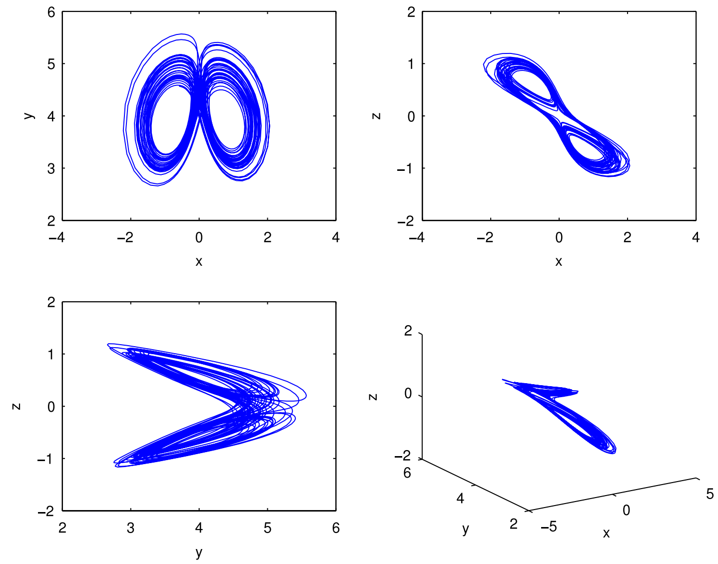

Fig. 2. Phase diagram of system (5) with fractional order $\alpha = 0.85$.

$$\Delta(\lambda) = diag(\lambda^{M\alpha_1}, \lambda^{M\alpha_2}, \lambda^{M\alpha_3}) - J(p_{2,3}) \tag{7}$$

in which $J(\cdot)$ denote the Jacobian matrix and p_2 or p_3 can be searched from Table 1.

By Lemma 2, Eq. (3) is simplified as

$$\lambda^{133} - \lambda^{100} + 1.1\lambda^{83} + 1.1\lambda^{50} + 0.1\lambda^{33} + 1.2 = 0. \tag{8}$$

Solving the above equation we have

$$\frac{\pi}{2M} - \min_i |arg(\lambda_i)| = \frac{\pi}{100} - 0.0317 = -0.0003 < 0. \tag{9}$$

And therefore, for the given derivative orders $\alpha = (0.66, 1, 1)$, system (5) does not satisfy the necessary condition to exhibit chaos. Numerical simulation in Fig. 3(a) confirms this conclusion.

Case II: $\alpha = (0.67, 1, 1)$. By the same procedure, we obtain $M = 100$, and

$$det(\Delta(\lambda)) = \lambda^{269} - \lambda^{200} + 1.1\lambda^{167} + 1.1\lambda^{100} + 0.1\lambda^{67} + 1.2 = 0. \tag{10}$$

Solving the above equation we have

$$\frac{\pi}{2M} - \min_i |arg(\lambda_i)| = \frac{\pi}{200} - 0.0156 = 0.0001 > 0. \tag{11}$$

Therefore, for the given derivative orders $\alpha = (0.67, 1, 1)$, system (5) satisfies the necessary condition to exhibit a 2-scroll chaotic attractor. Numerical simulation in Fig. 3(b) illustrates the existence of 2-scroll chaotic attractor.

It should be noted that all the other values of derivative orders can be checked by the same ways.

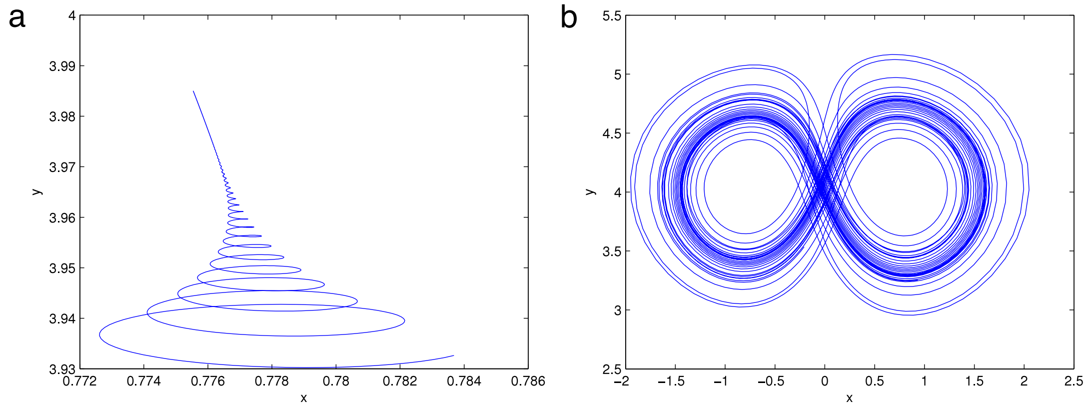

Fig. 3. Phase diagram of system (5) with fractional order (a) $\alpha = (0.65, 1, 1)$ and (b) $\alpha = (0.67, 1, 1)$.

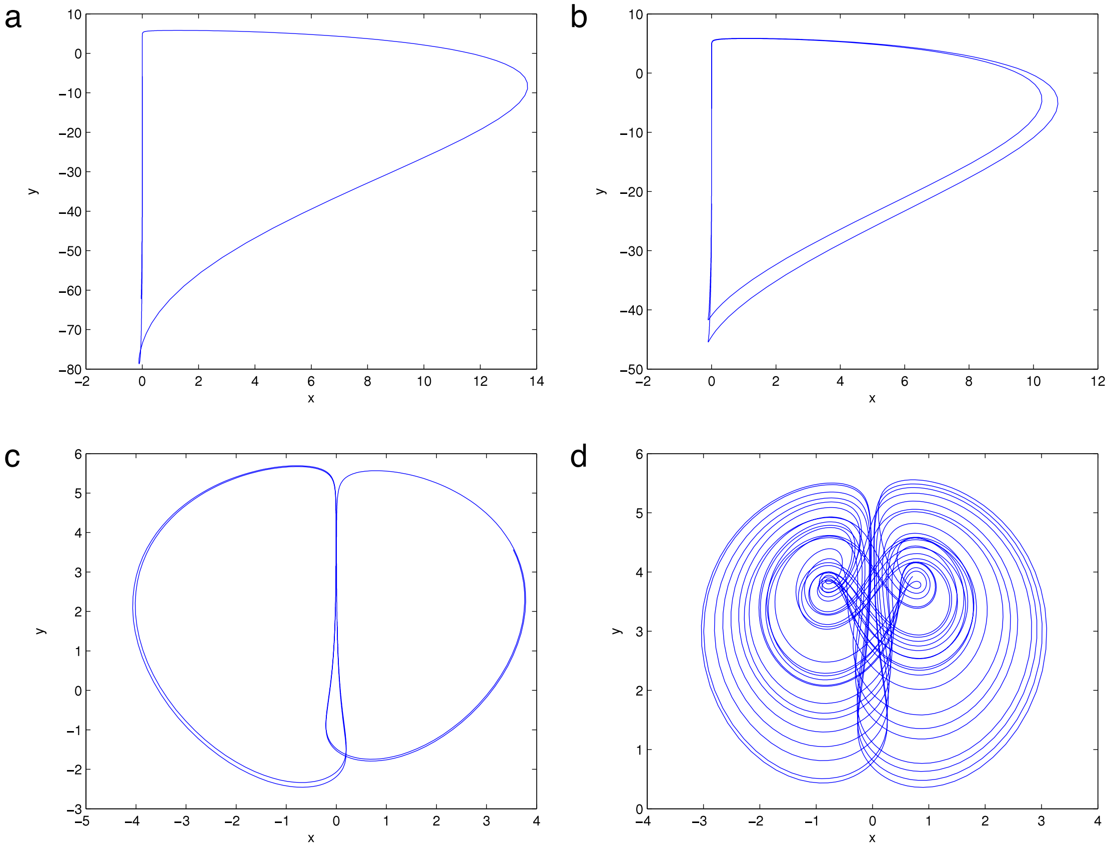

Fig. 4. Phase diagram of system (12) with fractional order $\alpha = 0.97$ and (a) $\tau = 0.3$, (b) $\tau = 0.25$, (c) $\tau = 0.1$, (d) $\tau = 0.06$.

4. Dynamics and chaos in a delayed fractional order financial system

In this section, we study a generalization of system (5) which takes time delay into consideration and is described as

$$\begin{aligned} D_t^{\alpha_1}x(t) &= z + (y(t-\tau) - a)x, \\ D_t^{\alpha_2}y(t) &= 1 - by - x^2(t-\tau), \\ D_t^{\alpha_3}z(t) &= -x(t-\tau) - cz, \end{aligned} \tag{12}$$

where $\tau > 0$ represents the delay term of the system and the constants a, b, c have the same values as those in system (5). We identify the dynamics of the system by using the time histories, phase diagrams and the largest Lyapunov exponents

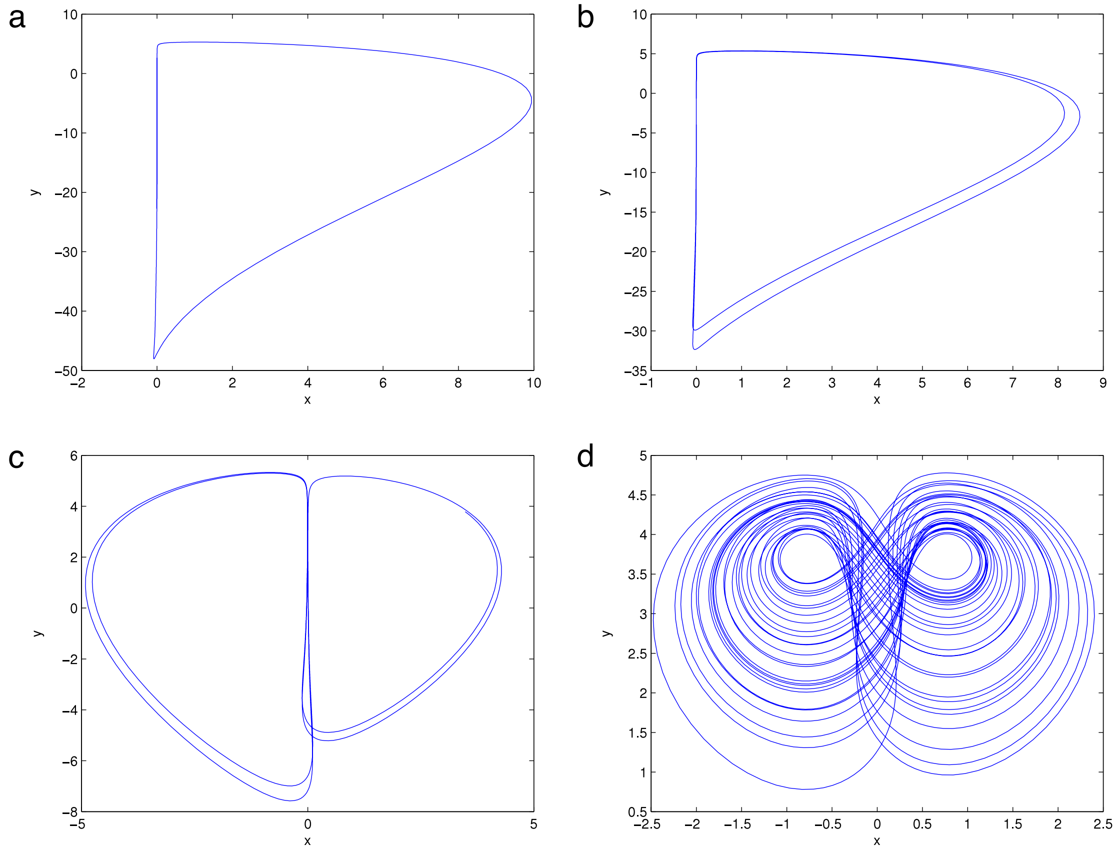

Fig. 5. Phase diagram of system (12) with fractional order $\alpha = 0.90$ and (a) $\tau = 0.35$, (b) $\tau = 0.30$, (c) $\tau = 0.20$, (d) $\tau = 0.15$.

Table 2
Dynamics of system (12) for fixed α and different τ.

α	τ	Observation	Figure
0.97	0.30	Period-1	Fig. 4(a)
	0.25	Period-2	Fig. 4(b)
	0.10	Period-4	Fig. 4(c)
	$0 < \tau \leq 0.06$	Chaos	Fig. 4(d)
0.9	0.35	Period-1	Fig. 5(a)
	0.30	Period-2	Fig. 5(b)
	0.20	Period-4	Fig. 5(c)
	$0 < \tau \leq 0.15$	Chaos	Fig. 5(d)
0.85	0.40	Period-1	Fig. 6(a)
	0.35	Period-2	Fig. 6(b)
	0.22	Period-4	Fig. 6(c)
	$0 < \tau \leq 0.05$	Chaos	Fig. 6(d)

(LEs). The largest LEs are calculated by using Wolf scheme [36], in which the time delay and the embedding windows are calculated by the C–C method [37].

4.1. *Dynamics for commensurate fixed α and different τ*

For fixed α, we study the dynamics of the delayed financial system by numerical simulation. In the simulations, the initial values are taken as $x(t) = 0.1, y(t) = 4, z(t) = 0.5,\ t \leq 0$. For $\alpha = 0.97$, Fig. 4(a)–(d) shows the phase diagram in the *xy* plane for different values of τ, it can be seen that with the decreasing of τ the system will exhibit period-1, period-2, period-4 and chaotic behaviors. When $\tau = 0.06$, the largest LE is calculated as $\lambda_{max} = 0.038$, which simultaneously confirms the existence of the chaotic behavior. At the same time, we find that system (12) always maintains chaotic behavior with $\tau \in [0, 0.06]$. Fig. 5 with $\alpha = 0.9$ and Fig. 6 with $\alpha = 0.85$ show nearly the same dynamical behavior as the case $\alpha = 0.97$.

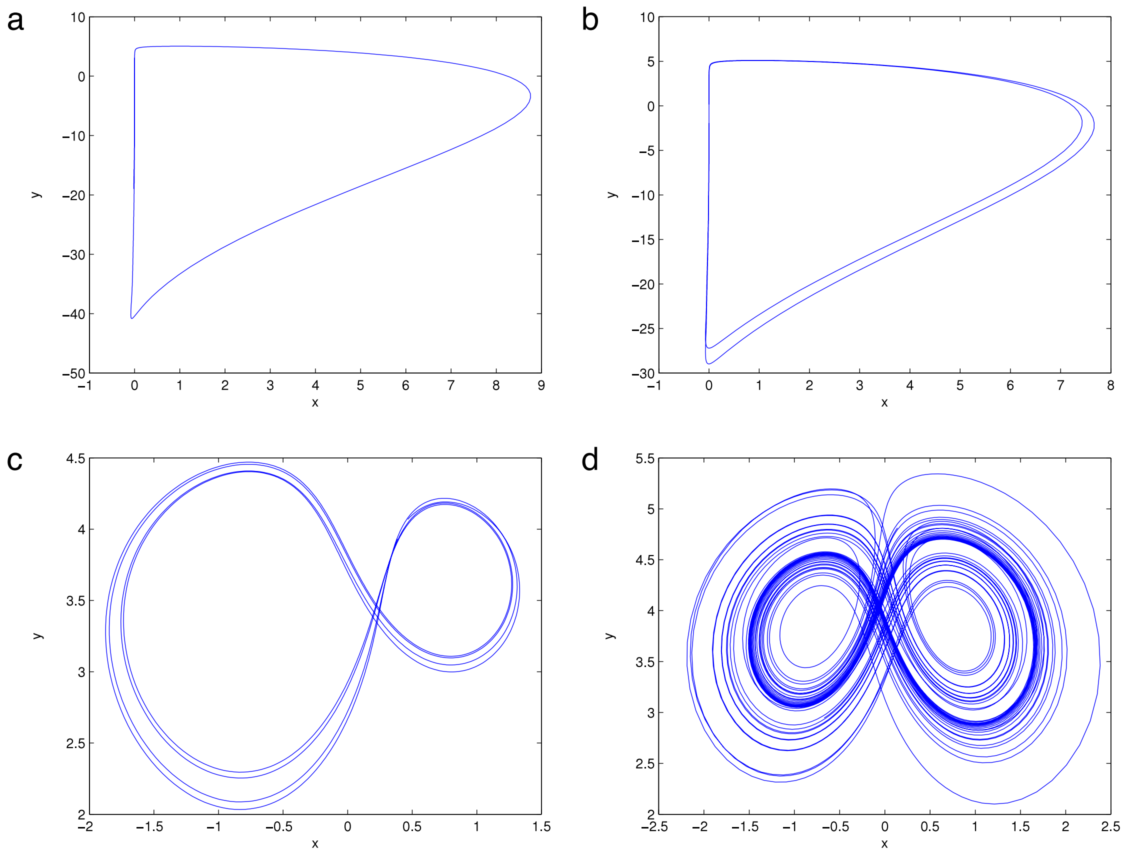

Fig. 6. Phase diagram of system (12) with fractional order $\alpha = 0.85$ and (a) $\tau = 0.40$, (b) $\tau = 0.35$, (c) $\tau = 0.17$, (d) $\tau = 0.05$.

All these simulations indicate that time delay is a sensitive factor for the financial system, and chaos can be suppressed by choosing a proper time delay. Observations for different values of α and τ are summarized in Table 2.

4.2. Dynamics for incommensurate order financial system with fixed τ

Case I: when $\alpha_2 = \alpha_3 = 1$, and let α_1 reduce to less than 1

System (12) is calculated numerically against $\alpha_1 \in [0.7, 1]$, and α_1 has an incremental value of 0.01. Numerical results show that system (12) will maintain chaotic motion for $\alpha_1 \in [0.76, 0.86]$, and it exhibits periodic motion in the other interval range. The largest LEs when $\alpha_1 = 0.76, 0.86$ are 0.0114 and 0.0429, respectively. Fig. 7(a)–(d) showed the phase diagram with $\alpha = 0.87, 0.86, 0.76, 0.75$, respectively. From the figures, we can see that the lowest order for chaos to exist is 2.76 in this case. The authors in [19] studied the chaotic dynamics of system (12) with $\tau = 0$. For the same parameters, the lowest order obtained in [19] for chaos to exist is 2.35. This shows that a suitable delay can suppress the emergence of chaos indeed.

Case II: when $\alpha_1 = \alpha_3 = 1$, and let α_2 reduce to less than 1

System (12) is calculated numerically against $\alpha_2 \in [0.75, 1]$, and α_1 has an incremental value of 0.01. Numerical results show that system (12) will maintain chaotic motion for $\alpha_2 \in [0.80, 1]$, and it exhibits periodic motion in the other interval range. The largest LE when $\alpha_2 = 0.8$ is 0.0092. Fig. 7(e)–(f) showed the phase diagrams with $\alpha_2 = 0.80, 0.79$, respectively. From the diagrams, we can see that the lowest order for chaos to exist is 2.80 in this case, and it is smaller than the result 2.90 obtained in [19], and this shows that a suitable delay can enhance the emergence of chaos indeed.

Case III: when $\alpha_1 = \alpha_2 = 1$, and let α_3 reduce to less than 1

System (12) is calculated numerically against $\alpha_3 \in [0.8, 1]$, and α_1 has an incremental value of 0.01. Numerical results show that system (12) will maintain chaotic motion for $\alpha_3 \in [0.83, 1]$, and it exhibits periodic motion in the other interval range. The largest LE when $\alpha_3 = 0.83$ is 0.0135. Fig. 7(g)–(h) showed the phase diagrams with $\alpha_3 = 0.83, 0.82$, respectively. From the diagrams, we can see that the lowest order for chaos to exist is 2.83 in this case, and it is bigger than the result 2.35 obtained in [19]. This shows that a suitable delay can suppress the emergence of chaos indeed.

All the observations of the above three cases are summarized in Table 3.

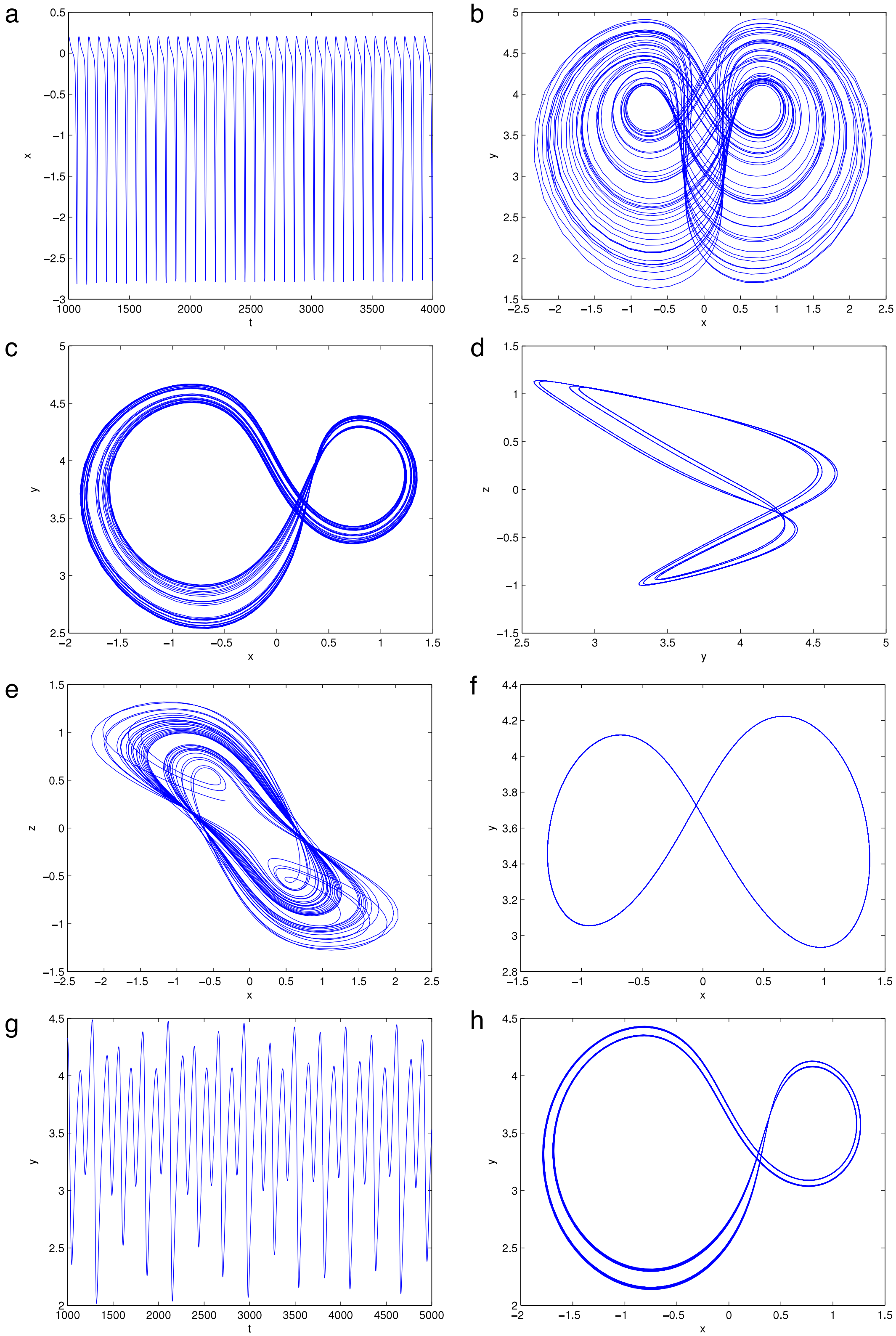

Fig. 7. Phase diagram for system (12) with $\tau = 0.08$, and fractional order (a) $\alpha = (0.87, 1, 1)$, (b) $\alpha = (0.86, 1, 1)$, (c) $\alpha = (0.76, 1, 1)$, (d) $\alpha = (0.75, 1, 1)$, (e) $\alpha = (1, 0.8, 1)$, (f) $\alpha = (1, 0.79, 1)$, (g) $\alpha = (1, 1, 0.83)$, (h) $\alpha = (1, 1, 0.82)$.

Table 3
Dynamics of system (12) for fixed $\tau = 0.08$ and different $\alpha = (\alpha_1, \alpha_2, \alpha_3)$.

α	Observation	Figure	Description
(0.87, 1, 1)	Periodic orbit	Fig. 7(a)	x trajectory
(0.86, 1, 1)	Chaos	Fig. 7(b)	x–y phase diagram
(0.76, 1, 1)	Chaos	Fig. 7(c)	x–y phase diagram
(0.75, 1, 1)	Periodic orbit	Fig. 7(d)	y–z phase diagram
(1, 0.8, 1)	Chaos	Fig. 7(e)	x–z phase diagram
(1, 0.79, 1)	Periodic orbit	Fig. 7(f)	x–y phase diagram
(1, 1, 0.83)	Chaos	Fig. 7(g)	y trajectory
(1, 1, 0.82)	Periodic orbit	Fig. 7(h)	x–y phase diagram

5. Conclusions

In this paper, the complex dynamics of a fractional order financial system with time delay are discussed in detail via numerical simulations. The influence of delay on the dynamics of such a system is explored, and it has been found that an appropriate time delay will enhance or suppress the emergence of chaotic or periodic motions. The complex dynamics of the system are analyzed with the change of fractional order or delay, respectively. Period doubling route to chaos was found, and meanwhile, the lowest order for chaos to exist is determined for each cases.

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