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Oscillation criteria of second-order half-linear dynamic equations on time scales

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Abstract

In this paper, by using the Riccati transformation technique, chain rule and inequality

$$A^{\lambda} - \lambda A B^{\lambda-1} + (\lambda - 1) B^{\lambda} \geqslant 0, \quad \lambda > 1,$$

where A and B are positive constants, we will establish some oscillation criteria for the second-order half-linear dynamic equation

$$(p(t)(x^{\Delta}(t))^{\gamma})^{\Delta} + q(t)x^{\gamma}(t) = 0, \quad t \in [a, b]$$

on time scales, where $\gamma > 1$ is an odd positive integer. Our results not only unify the oscillation of half-linear differential and half-linear difference equations but can be applied on different types of time scales and improve some well-known results in the difference equation case. Some examples are considered here to illustrate our main results.

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Keywords: Oscillation; Second-order half-linear dynamic equations; Time scale

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1. Introduction

In this paper, we are concerned with oscillation of second-order half-linear dynamic equation

$$(p(t)(x^{\Delta}(t))^{\gamma})^{\Delta} + q(t)x^{\gamma}(t) = 0, \quad t \in [a,b] \tag{1.1}$$

on time scales, where

(H) p, q are positive, real-valued rd-continuous functions, and $\gamma > 1$ is an odd positive integer. We shall also consider the two cases

$$\int_{t_0}^{\infty} \left(\frac{1}{p(t)}\right)^{1/\gamma} \Delta t = \infty \tag{1.2}$$

and

$$\int_{t_0}^{\infty} \left(\frac{1}{p(t)}\right)^{1/\gamma} \Delta t < \infty. \tag{1.3}$$

By the solution of (1.1), we mean a nontrivial real-valued function $x(t) \in C^1_{\mathrm{rd}}[t_x, \infty)$, $t_x \geqslant t_0 \geqslant a$, which has the property $p(t)(x^{\Delta}(t))^{\gamma} \in C^1_{\mathrm{rd}}[t_x, \infty)$ and satisfying Eq. (1.1) for $t \geqslant t_x$. Our attention is restricted to those solutions of (1.1) which exist on some half-line $[t_x, \infty)$ and satisfy $\sup\{|x(t)| : t > t_1\} > 0$ for any $t_1 \geqslant t_x$. A solution $x(t)$ of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. Eq. (1.1) is said to be oscillatory if all its solutions are oscillatory. Half-linear dynamic equations derive their name from the fact that if $\{x(t)\}$ is a solution, then so is $\{cx(t)\}$ for any constant c.

Much recent attention has been given to dynamic equations on time scales (or measure chains), and we refer the reader to the landmark paper of Hilger [16] for a comprehensive treatment of the subject. Since then; several authors have expounded on various aspects of this new theory; see the survey paper by Agarwal et al. [1] and the references cited therein. A book on the subject of time scales, by Bohner and Peterson [4], summarizes and organizes much of time scale calculus; we refer also the last book by Bohner and Peterson [5] for advances in dynamic equations on time scales.

In recent years there has been much research activity concerning the oscillation and nonoscillation of solutions of dynamic equations on time scales. We refer the reader to the papers [2–4,6–15,18,19].

In this paper, we apply the Riccati transformation technique to obtain some oscillation criteria for (1.1) when (1.2) or (1.3) holds. Our results not only unify the oscillation of second-order half-linear differential and difference equations but can be also applied on different types of time scales. The paper is organized as follows. In Section 2, we present some basic definitions concerning the calculus on time scales. In Section 3, we apply the Riccati transformation technique, a simple consequence of Keller's chain rule, and the inequality

$$A^{\lambda} - \lambda A B^{\lambda-1} + (\lambda-1)B^{\lambda} \geqslant 0, \quad \lambda > 1, \tag{1.4}$$

where A and B are nonnegative constants, to obtain some oscillation criteria for Eq. (1.1). Our results when (1.2) holds are sufficient for oscillation of all solutions of (1.1) and when (1.3) holds our results ensure that all solutions either oscillate or converge to zero.

2. Some preliminaries on time scales

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. On any time scale $\mathbb{T}$ we define the forward and backward jump operators by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) := \sup\{s \in \mathbb{T}, s < t\}. \tag{2.1}$$

A point $t \in \mathbb{T}$, $t > \inf \mathbb{T}$, is said to be left-dense if $\rho(t) = t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, left-scattered if $\rho(t) < t$ and right-scattered if $\sigma(t) > t$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$.

For a function $f : \mathbb{T} \to \mathbb{R}$ (the range $\mathbb{R}$ of f may be actually replaced by any Banach space) the (delta) derivative is defined by

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}, \tag{2.2}$$

if *f* is continuous at *t* and *t* is right-scattered. If *t* is not right-scattered then the derivative is defined by

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}, \tag{2.3}$$

provided this limit exists. A function $f : [a, b] \to \mathbb{R}$ is said to be right-dense continuous if it is right continuous at each right-dense point and there exists a finite left limit at all left-dense points, and *f* is said to be differentiable if its derivative exists. A useful formula is

$$f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t). \tag{2.4}$$

We will make use of the following product and quotient rules for the derivative of the product *fg* and the quotient f/g (where $gg^{\sigma} \neq 0$) of two differentiable function *f* and *g*

$$(fg)^{\Delta} = f^{\Delta} g + f^{\sigma} g^{\Delta} = f g^{\Delta} + f^{\Delta} g^{\sigma}, \tag{2.5}$$

$$\left(\frac{f}{g}\right)^{\Delta} = \frac{f^{\Delta} g - f g^{\Delta}}{g g^{\sigma}}. \tag{2.6}$$

For $a, b \in \mathbb{T}$, and a differentiable function f, the Cauchy integral of f^{Δ} is defined by

$$\int_a^b f^{\Delta}(t) \Delta t = f(b) - f(a). \tag{2.7}$$

An integration by parts formula reads

$$\int_a^b f(t) g^{\Delta}(t) \Delta t = [f(t) g(t)]_a^b - \int_a^b f^{\Delta}(t) g(\sigma(t)) \Delta t, \tag{2.8}$$

and infinite integrals are defined as

$$\int_a^{\infty} f(t) \Delta t = \lim_{b \to \infty} \int_a^b f(t) \Delta t.$$

In case $\mathbb{T}=\mathbb{R}$, we have $\sigma(t)=\rho(t)=t,\ \ \mu(t)\equiv 0$,

$$f^{\Delta}=f' \quad \text{and} \quad \int_a^b f(t)\Delta t=\int_a^b f(t)\,\mathrm{d}t$$

and in case $\mathbb{T}=\mathbb{Z}$, we have $\sigma(t)=t+1,\ \ \mu(t)\equiv 1$,

$$f^{\Delta}=\Delta f \quad \text{and} \quad \int_a^b f(t)\Delta t=\sum_{t=a}^{b-1} f(t),$$

in the case $\mathbb{T}=h\mathbb{Z}$, $h>0$, we have $\sigma(t)=t+h$, $\mu(t)=h$,

$$f^{\Delta}=\Delta_h f=\frac{f(t+h)-f(t)}{h} \quad \text{and} \quad \int_a^b f(t)\Delta t=\sum_{i=a/h}^{b/h-1} f(i);$$

and in the case $\mathbb{T}=q^{\mathbb{N}}=\{t: t=q^k,\ \ k\in\mathbb{N},\ \ q>1\}$, we have $\sigma(t)=qt$, $\mu(t)=(q-1)t$,

$$x_q^{\Delta}(t)=\frac{x(qt)-x(t)}{(q-1)t} \quad \text{and} \quad \int_a^{\infty} f(t)\Delta t=\sum_{k=0}^{\infty}\mu(q^k)f(q^k).$$

3. Main results

In this section, we give some new oscillation criteria for (1.1). Since we are interested in oscillatory behavior, we suppose that the time scale under consideration is not bounded above, i.e., it is a time scale interval of the form $[a,\infty)$. Also, we will use the formula

$$(x^{\gamma}(t))^{\Delta}=\gamma\int_0^1 [hx^{\sigma}+(1-h)x]^{\gamma-1}\,\mathrm{d}h x^{\Delta}(t), \tag{3.1}$$

which is a simple consequence of Keller's chain rule [4, Theorem 1.90].

First, we consider the case when (1.2) holds

Theorem 3.1. *Assume that* (H) *and* (1.2) *hold. Furthermore, assume that there exists a positive Δ-differentiable function $\delta(t)$ such that*

$$\limsup_{t\to\infty}\int_a^t\left[\delta(s)q(s)-\frac{p(s)(\delta^{\Delta}(s))_+^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}(s)}\right]\Delta s=\infty, \tag{3.2}$$

where $(\delta^{\Delta}(t))_+=\max\{0,(\delta^{\Delta}(t))\}$. *Then every solution of Eq.* (1.1) *is oscillatory on* $[a,\infty)$.

Proof. Suppose to the contrary that $x(t)$ is a nonoscillatory solution of (1.1). Without loss of generality, we may assume that $x(t)$ is an eventually positive solution of (1.1) such that $x(t)>0$ for all $t\geqslant t_0>a$. We shall consider only this case, since the substitution $y(t)=-x(t)$ transforms Eq. (1.1) into an equation of the same form. In view of (1.1), we have

$$(p(t)(x^{\Delta}(t))^{\gamma})^{\Delta}=-q(t)x^{\gamma}(t)<0, \tag{3.3}$$

for all $t \geqslant t_0$, and so $\{p(t)(x^\Delta(t))^\gamma\}$ is an eventually decreasing function. We first show that $\{p(t)(x^\Delta(t))^\gamma\}$ is eventually nonnegative. Indeed, since $q(t)$ is a positive function, the decreasing function $p(t)(x^\Delta(t))^\gamma$ is either eventually positive or eventually negative. Suppose there exists an integer $t_1 \geqslant t_0$ such that $p(t_1)(x^\Delta(t_1))^\gamma = c < 0$, then from (3.3) we have $p(t)(x^\Delta(t))^\gamma < p(t_1)(x^\Delta(t_1))^\gamma = c$ for $t \geqslant t_1$, hence

$$x^\Delta(t) \leqslant c^{1/\gamma}\left(\frac{1}{p(t)}\right)^{1/\gamma},$$

which implies by (1.2) that

$$x(t) \leqslant x(t_1) + c^{1/\gamma}\int_{t_1}^{t}\left(\frac{1}{p(s)}\right)^{1/\gamma}\Delta s \to -\infty \quad \text{as } t \to \infty, \tag{3.4}$$

which contradicts the fact that $x(t) > 0$ for all $t \geqslant t_0$. Hence $p(t)(x^\Delta(t))^\gamma$ is eventually nonnegative. Therefore, we see that there is some t_0 such that

$$x(t) > 0,\ \ x^\Delta(t) \geqslant 0,\ \ (p(t)(x^\Delta(t))^\gamma)^\Delta < 0, \qquad t \geqslant t_0. \tag{3.5}$$

Define the function $w(t)$ by

$$w(t) = \delta(t)\frac{p(t)(x^\Delta(t))^\gamma}{x^\gamma(t)}, \qquad t \geqslant t_0. \tag{3.6}$$

Then $w(t) > 0$, and using (2.5) and (2.6) we obtain

$$w^\Delta(t) = \frac{\delta(t)}{x^\gamma(t)}(p(t)(x^\Delta(t))^\gamma)^\Delta + p(x^\Delta)^\gamma)^\sigma\left[\frac{x^\gamma(t)\delta^\Delta(t) - \delta(t)(x^\gamma(t))^\Delta}{x^\gamma(t)x^\gamma(\sigma(t))}\right]. \tag{3.7}$$

In view of (1.1) and (3.7), we get

$$w^\Delta(t) = -\delta(t)q(t) + \frac{\delta^\Delta(t)}{\delta^\sigma}w^\sigma - \frac{\delta(t)(p(x^\Delta)^\gamma)^\sigma(x^\gamma(t))^\Delta}{x^\gamma(t)x^\gamma(\sigma(t))}. \tag{3.8}$$

Using (3.4) we have $x^\sigma \geqslant x(t)$, and then from the chain rule (3.1) we obtain

$$\begin{aligned}(x^\gamma(t))^\Delta &= \gamma\int_0^1 [hx^\sigma + (1-h)x]^{\gamma-1}x^\Delta(t)\,\mathrm{d}h\\ &\geqslant \gamma\int_0^1 [hx + (1-h)x]^{\gamma-1}x^\Delta(t)\,\mathrm{d}h = \gamma(x(t))^{\gamma-1}x^\Delta(t).\end{aligned} \tag{3.9}$$

It follows from (3.8) and (3.9) that

$$\begin{aligned}w^\Delta(t) &\leqslant -\delta(t)q(t) + \frac{(\delta^\Delta(t))_+}{\delta^\sigma}w^\sigma - \frac{\delta(t)(p(x^\Delta)^\gamma)^\sigma\gamma(x(t))^{\gamma-1}x^\Delta(t)}{x^\gamma(t)x^\gamma(\sigma(t))}\\ &= -\delta(t)q(t) + \frac{(\delta^\Delta(t))_+}{\delta^\sigma}w^\sigma - \frac{\gamma\delta(t)(p(x^\Delta)^\gamma)^\sigma x^\Delta(t)}{x(t)x^\gamma(\sigma(t))}\\ &\leqslant -\delta(t)q(t) + \frac{(\delta^\Delta(t))_+}{\delta^\sigma}w^\sigma - \frac{\gamma\delta(t)(p(x^\Delta)^\gamma)^\sigma x^\Delta(t)}{x^{\gamma+1}(\sigma(t))}.\end{aligned} \tag{3.10}$$

From (3.5) since $(p(t)(x^{\Delta}(t))^{\gamma})^{\Delta}<0$ we have

$$x^{\Delta}(t)>\frac{(p^{\sigma})^{1/\gamma}}{p^{1/\gamma}}(x^{\Delta})^{\sigma}. \tag{3.11}$$

Substituting (3.11) in (3.10) we find that

$$\begin{aligned} w^{\Delta}(t)&<-\delta(t)q(t)+\frac{(\delta^{\Delta}(t))_{+}}{\delta^{\sigma}}w^{\sigma}-\frac{\gamma\delta(t)(p^{\sigma})^{(\gamma+1)/\gamma}(x^{\Delta})^{\gamma+1}(\sigma(t))}{p^{1/\gamma}x^{\gamma+1}(\sigma(t))}\\ &=-\delta(t)q(t)+\frac{(\delta^{\Delta}(t))_{+}}{\delta^{\sigma}}w^{\sigma}-\frac{\gamma\delta(t)}{(\delta^{\delta})^{\lambda}p^{\lambda-1}(t)}(w^{\sigma})^{\lambda}, \end{aligned} \tag{3.12}$$

where $\lambda=(\gamma+1)/\gamma$. Set

$$A=\left[\frac{\gamma\delta(t)}{(\delta^{\delta})^{\lambda}p^{\lambda-1}(t)}\right]^{1/\lambda}w^{\sigma}\quad\text{and}\quad B=\left[\frac{(\delta^{\Delta}(t))_{+}}{\lambda\delta^{\sigma}}\left(\frac{\gamma\delta(t)}{(\delta^{\delta})^{\lambda}p^{\lambda-1}(t)}\right)^{-1/\lambda}\right]^{1/(\lambda-1)}.$$

Using inequality (1.4), we have

$$\begin{aligned} &\frac{(\delta^{\Delta}(t))_{+}}{\delta^{\sigma}}w^{\sigma}-\frac{\gamma\delta(t)}{(\delta^{\delta})^{\lambda}p^{\lambda-1}(t)}(w^{\sigma})^{\lambda}\\ &\quad\leqslant(\lambda-1)\lambda^{\lambda/(\lambda-1)}\left(\frac{(\delta^{\Delta}(t))_{+}}{\delta^{\sigma}}\right)^{\lambda/(\lambda-1)}\left(\frac{\gamma\delta(t)}{(\delta^{\delta})^{\lambda}p^{\lambda-1}(t)}\right)^{-1/(\lambda-1)}\\ &\quad=C\frac{p(t)(\delta^{\Delta}(t))_{+}^{\lambda/(\lambda-1)}}{\delta^{1/(\lambda-1)}(t)}=C\frac{p(t)(\delta^{\Delta}(t))_{+}^{\gamma+1}}{\delta^{\gamma}(t)}, \end{aligned} \tag{3.13}$$

where $C=(\lambda-1)\lambda^{\lambda/(\lambda-1)}\gamma^{-1/(\lambda-1)}=1/(\gamma+1)^{\gamma+1}$. Thus, from (3.12) and (3.13) we obtain

$$w^{\Delta}(t)<-\left[\delta(t)q(t)-\frac{p(t)(\delta^{\Delta}(t))_{+}^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}(t)}\right]. \tag{3.14}$$

Integrating (3.14) from t_0 to t, we obtain

$$-w(t_0)<w(t)-w(t_0)<-\int_{t_0}^{t}\left[\delta(s)q(s)-\frac{p(s)(\delta^{\Delta}(s))_{+}^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}(s)}\right]\Delta s, \tag{3.15}$$

which yields

$$\int_{t_0}^{t}\left[\delta(s)q(s)-\frac{p(s)(\delta^{\Delta}(s))_{+}^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}(s)}\right]\Delta s<w(t_0),$$

for all large t. This is contrary to (3.2). The proof is complete. □

From Theorem 3.1, we can obtain different conditions for oscillation of all solutions of (1.1) by different choices of $\delta(t)$. For instance, let $\delta(t)=t$, $t\geqslant t_0$. By Theorem 3.1, we have the following result.

Corollary 3.1. *Assume that* (H) *and* (1.2) *hold. Furthermore, assume that*

$$\limsup_{t\to\infty}\int_a^t\left[sq(s)-\frac{p(s)}{(\gamma+1)^{\gamma+1}s^{\gamma}}\right]\Delta s=\infty. \tag{3.16}$$

Then, every solution of (1.1) *is oscillatory on* $[a,\infty)$.

Let $\delta(t)=1$, $t\geqslant t_0$. By Theorem 3.1, we have the following well-known result (Leighton–Wintner Theorem).

Corollary 3.2 (*Leighton–Wintner*). *Assume that* (H) *and* (1.2) *hold. If*

$$\int_a^\infty q(s)\Delta s=\infty. \tag{3.17}$$

Then every solution of (1.1) *is oscillatory on* $[a,\infty)$.

Remark 3.1. From Theorem 3.1, we can give sufficient conditions for oscillation of (1.1) on different types of time scales, for example we can deduce that

$$\int_{t_0}^\infty\frac{1}{(p(s))^{\gamma}}\,\mathrm{d}s=\infty\quad\text{and}\quad\limsup_{t\to\infty}\int_{t_0}^t\left[\delta(s)q(s)-\frac{p(s)(\delta'(s))_+^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}(s)}\right]\mathrm{d}s=\infty,$$

where $(\delta'(t))_+=\max\{0,\delta'(t)\}$, are sufficient conditions for oscillation of the second-order half-linear differential equation

$$(p(t)(x'(t))^{\gamma})'+q(t)x^{\gamma}(t)=0,\quad t\in[t_0,\infty], \tag{3.18}$$

$$\sum_{i=n_0}^{\infty}\left[\frac{1}{(p(i))^{\gamma}}\right]=\infty\quad\text{and}\quad\limsup_{n\to\infty}\sum_{i=n_0}^{n-1}\left[\delta(i)q(i)-\frac{p(i)(\Delta\delta(i))_+^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}(i)}\right]=\infty,$$

where $(\Delta\delta(i))_+=\max\{0,\Delta\delta(i)\}$, are sufficient conditions for oscillation of the second-order half-linear difference equation

$$\Delta(p(n)(\Delta x(n))^{\gamma})+q(n)x^{\gamma}(n)=0,\quad n\in[n_0,\infty], \tag{3.19}$$

$$\sum_{i=n_0/h}^{\infty}\left[\frac{1}{(p(i))^{\gamma}}\right]=\infty\quad\text{and}\quad\limsup_{n\to\infty}\sum_{i=n_0/h}^{n/h-1}\left[\delta(i)q(i)-\frac{p(i)(\Delta_h\delta(i))_+^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}(i)}\right]=\infty,$$

where $(\Delta_h\delta(i))_+=\max\{0,\Delta_h\delta(i)\}$, are sufficient conditions for oscillation of the general second-order half-linear difference equation

$$\Delta_h(p(n)(\Delta_h x(n))^{\gamma})+q(n)x^{\gamma}(n)=0,\quad n\in[n_0,\infty], \tag{3.20}$$

$$\sum_{k=0}^{\infty}\mu(\ell^k)\left[\frac{1}{(p(\ell^k))^{\gamma}}\right]=\infty\quad\text{and}\quad\sum_{k=0}^{\infty}\mu(\ell^k)\left[\delta(\ell^k)q(\ell^k)-\frac{p(\ell^k)(\Delta_\ell\delta(\ell^k))_+^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}(\ell^k)}\right]=\infty,$$

where $(\Delta_\ell \delta(i))_+ = \max\{0, \Delta_\ell \delta(i)\}$, are sufficient conditions for oscillation of the second-order half-linear ℓ-difference equation

$$\Delta_\ell(p(n)(\Delta_\ell x(n))^\gamma) + q(n)x^\gamma(n) = 0, \quad n \in [n_0, \infty]. \tag{3.21}$$

Remark 3.2. In [20], Thandpani et al., considered the second-order half-linear difference Eq. (3.19) when $p = 1$ and proved that every solution is oscillatory if

$$\sum_{n_0}^{\infty} q(n) = \infty. \tag{3.22}$$

But, one can easily see that this result cannot be applied in discrete half-linear Euler difference equation, so our results extend and improve the results in [20].

Theorem 3.2. *Assume that* (H) *and* (1.2) *hold. Let* $\delta(t)$ *be as defined in Theorem* 3.1. *If*

$$\limsup_{t\to\infty} \frac{1}{t^m} \int_a^t (t-s)^m \left[\delta(s)q(s) - \frac{p(s)(\delta^\Delta(s))_+^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^\gamma(s)} \right] \Delta s = \infty, \tag{3.23}$$

for an odd positive integer m. Then every solution of (1.1) *is oscillatory on* $[a, \infty)$.

The proof is similar to that of the proof of Theorem 3.2 in [19] by using inequality (3.14) and hence is omitted.

Note that when $\delta(t) = 1$, then (3.23) reduces to

$$\lim_{t\to\infty} \frac{1}{t^m} \int_a^t (t-s)^m q(s) \Delta s = \infty.$$

Then (2.23) can be considered as the extension of Kamenev-type oscillation criteria for second-order differential equations (see [17]).

From Theorem 3.2, we have the following oscillation criteria for Eqs. (3.18)–(3.20), and the oscillation conditions for Eq. (3.21) are left to the reader.

Corollary 3.3. *Assume that* (H) *holds and*

$$\int_{t_0}^{\infty} \frac{1}{(p(s))^\gamma} \, \mathrm{d}s = \infty.$$

Let $\delta(t)$ *be a positive real-valued differentiable function such that*

$$\limsup_{t\to\infty} \frac{1}{t^m} \int_{t_0}^t (t-s)^m \left[\delta(s)q(s) - \frac{p(s)\big(\delta'(s)\big)_+^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^\gamma(s)} \right] \mathrm{d}s = \infty,$$

for an odd positive integer m. Then every solution of (3.18) *is oscillatory.*

Corollary 3.4. *Assume that* (H) *holds and*

$$\sum_{i=n_0}^{\infty}\left[\frac{1}{(p(i))^{\gamma}}\right]=\infty.$$

Let $\{\delta(n)\}$ *be a positive sequence such that*

$$\limsup_{n\to\infty}\frac{1}{t^m}\sum_{s=n_0}^{n-1}(t-s)^m\left[\delta(s)q(s)-\frac{p(s)(\Delta\delta(s))_+^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}(s)}\right]=\infty,$$

for an odd positive integer m. Then every solution of (3.19) *is oscillatory.*

Corollary 3.5. *Assume that* (H) *holds and*

$$\sum_{i=n_0/h}^{\infty}\left[\frac{1}{(p(i))^{\gamma}}\right]=\infty.$$

Let $\{\delta(n)\}$ *be a positive sequence such that*

$$\limsup_{t\to\infty}\frac{1}{t^m}\sum_{s=n_0/h}^{n/h-1}(t-s)^m\left[\delta(s)q(s)-\frac{p(s)(\Delta_h\delta(s))_+^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}(s)}\right]=\infty,$$

for an odd positive integer m. Then every solution of (3.20) *is oscillatory.*

Next we consider the case when (1.3) holds.

Now, we give some sufficient conditions when (1.3) holds, which guarantee that every solution $x(t)$ of Eq. (1.1) oscillates or converges to zero.

Theorem 3.3. *Assume that* (H) *and* (1.3) *hold. Let* $\delta(t)$ *be as defined in Theorem* 3.1 *such that* (3.2) *holds. If*

$$\int_a^{\infty}\left[\frac{1}{p(t)}\int_a^t q(s)\Delta s\right]^{1/\gamma}\Delta t=\infty. \tag{3.24}$$

Then every solution of Eq. (1.1) *is oscillatory or converges to zero.*

Proof. We proceed as in the proof of Theorem 3.1. We assume that Eq. (1.1) has a nonoscillatory solution such that $x(t)>0$, for $t\geqslant t_0>a$. (We shall consider only this case, since the substitution $y(t)=-x(t)$ transforms Eq. (1.1) into an equation of the same form.) From the proof of Theorem 3.1 we see that there exist two possible cases for the sign of $x^{\Delta}(t)$. The proof when $x^{\Delta}(t)$ is eventually positive is similar to that of the proof of Theorem 3.1 and hence is omitted.

Next, suppose that $x^{\Delta}(t)<0$ for $t\geqslant t_1\geqslant t_0$. Then $x(t)$ is decreasing and $\lim_{t\to\infty}x(t)=b\geqslant 0$ exists. We assert that $b=0$. If not, then $x(t)>b>0$ for $t\geqslant t_2>t_1$. Define the function

$$u(t)=p(t)(x^{\Delta}(t))^{\gamma},$$

then from Eq. (1.1) for $t \geqslant t_2$, we obtain

$$u^{\Delta}(t) = -q(t)x^{\gamma}(t) \leqslant -b^{\gamma}q(t).$$

Hence, for $t \geqslant t_2$ we have

$$u(t) \leqslant u(t_2) - b^{\gamma}\int_{t_2}^{t} q(s)\Delta s < -b^{\gamma}\int_{t_2}^{t} q(s)\Delta s.$$

Since $u(t_2) = p(t_2)(x^{\Delta}(t_2))^{\gamma} < 0$, integrating the last inequality from t_2 to t, we have

$$\int_{t_2}^{t} x^{\Delta}(s)\Delta s \leqslant -b\int_{t_2}^{t}\left[\frac{1}{p(s)}\int_{t_2}^{s} q(\tau)\Delta\tau\right]^{1/\gamma}\Delta s.$$

By condition (3.24), we get $x(t) \to -\infty$ as $t \to \infty$, and this is a contradiction to the fact that $x(t) > 0$ for $t \geqslant t_0$. Thus $b = 0$ and $x(t) \to 0$ as $t \to \infty$. The proof is complete. □

Theorem 3.4. *Assume that* (H) *and* (1.3) *hold. Let* $\delta(t)$ *be as defined in Theorem* 3.1 *such that* (3.23) *and* (3.24) *hold. Then every solution of Eq.* (1.1) *is oscillatory or converges to zero.*

From Theorems 3.3 and 3.4 we can deduce some new sufficient conditions, which insure that the solutions of Eqs. (3.18)–(3.21) are oscillatory or converge to zero. For example, for Eq. (3.19) we have the following result which improves the result established in [20].

Corollary 3.6. *Assume that* (H) *hold,*

$$\sum_{i=n_0}^{\infty}\left[\frac{1}{(p(i))^{\gamma}}\right] < \infty$$

and

$$\sum_{n=n_0}^{\infty}\left[\frac{1}{p(n)}\sum_{i=n_0}^{n-1} q(i)\right]^{1/\gamma} = \infty.$$

Let $\{\delta(n)\}$ *be a positive sequence such that*

$$\limsup_{n\to\infty}\sum_{i=n_0}^{n-1}\left[\delta(i)q(i) - \frac{p(i)(\Delta\delta(i))_+^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}(i)}\right] = \infty,$$

where $(\Delta\delta(i))_+ = \max\{0, \Delta\delta(i)\}$. *Then every solution of Eq.* (3.19) *is oscillatory or converges to zero.*

4. Applications

In this section, we give some examples to illustrate our main results in this paper and show that the results improve the results in [20] for the difference equation case.

Example 4.1. Consider the half-linear dynamic equation

$$((x^{\Delta}(t))^3)^{\Delta} + \frac{1}{t^2}x^3(t) = 0, \tag{4.1}$$

for $t \in [1, \infty)$. Here $p(t) = 1$ and $q(t) = 1/t^2$. Then, by Corollary 3.1, we have

$$\begin{aligned}\limsup_{t\to\infty} \int_1^t \left[sq(s) - \frac{1}{(\gamma+1)^{\gamma+1}s^{\gamma}}\right]\Delta s &= \limsup_{t\to\infty} \int_1^t \left[sq(s) - \frac{1}{256s^3}\right]\Delta s \\ &= \limsup_{t\to\infty} \int_1^t \left[\frac{1}{s} - \frac{1}{256s^3}\right]\Delta s = \infty.\end{aligned}$$

Then, every solution of (4.1) is oscillatory on $[1, \infty)$.

Example 4.2. Consider the half-linear dynamic equation

$$((x^{\Delta}(t))^{\gamma})^{\Delta} + t^{\alpha-\gamma}x^{\gamma}(t) = 0, \tag{4.2}$$

for $t \in [1, \infty)$, where α is a positive constant and $\gamma > 1$ is a positive integer. In (4.2), $p(t) = 1$ and $q(t) = t^{\alpha-\gamma}$. Then, by Corollary 3.1, we have

$$\limsup_{t\to\infty} \int_1^t \left[sq(s) - \frac{1}{(\gamma+1)^{\gamma+1}s^{\gamma}}\right]\Delta s = \limsup_{t\to\infty} \int_1^t \left[s^{1+\alpha-\gamma} - \frac{1}{(\gamma+1)^{\gamma+1}s^{\gamma}}\right]\Delta s = \infty,$$

if $\alpha - \gamma \geqslant -2$. Then every solution of (4.2) is oscillatory when $\alpha - \gamma \geqslant -2$.

Example 4.3. Consider the half-linear dynamic equation

$$(t^{\gamma-1}(x^{\Delta}(t))^{\gamma})^{\Delta} + \frac{\beta}{t^2}x^{\gamma}(t) = 0, \tag{4.3}$$

for $t \in [1, \infty)$, where α is a positive constant and $\gamma > 1$ is a positive integer. In (4.3), $p(t) = t^{\gamma-1}$ which satisfies condition (1.2) since

$$\int_1^{\infty} \left(\frac{1}{t}\right)^{(\gamma-1)/\gamma} \Delta t = \infty, \quad \text{for } \gamma > 1,$$

and $q(t) = \beta/t^2$. Then, by Corollary 3.1, we have

$$\limsup_{t\to\infty} \int_1^t \left[sq(s) - \frac{1}{(\gamma+1)^{\gamma+1}s^{\gamma}}\right]\Delta s = \limsup_{t\to\infty} \int_1^t \left[\frac{\beta}{s} - \frac{1}{(\gamma+1)^{\gamma+1}s}\right]\Delta s = \infty,$$

if $\beta > 1/((\gamma+1)^{\gamma+1})$. Then every solution of (4.3) is oscillatory when $\beta > 1/((\gamma+1)^{\gamma+1})$.

Example 4.4. Consider the half-linear dynamic equation

$$(t^{\gamma+1}(x^{\Delta}(t))^{\gamma})^{\Delta} + \beta x^{\gamma}(t) = 0, \tag{4.4}$$

for $t \in [1, \infty)$, where α is a positive constant and $\gamma > 1$ is a positive integer. In (4.4), $p(t) = t^{\gamma+1}$ which satisfies condition (1.3) since

$$\int_1^{\infty} \left(\frac{1}{t^{\gamma+1}}\right)^{1/\gamma} \Delta t = \int_1^{\infty} \left(\frac{1}{t^{(\gamma+1)/\gamma}}\right) \Delta t < \infty \quad \text{for } \gamma > 1,$$

and $q(t)=\beta$. Then, by Corollary 3.1, we have

$$\begin{aligned}\limsup_{t\to\infty}\int_1^t\left[sq(s)-\frac{1}{(\gamma+1)^{\gamma+1}s^\gamma}\right]\Delta s&=\limsup_{t\to\infty}\int_1^t\left[\beta s-\frac{s^{\gamma+1}}{(\gamma+1)^{\gamma+1}s^\gamma}\right]\Delta s\\&=\limsup_{t\to\infty}\int_1^t\left[\beta-\frac{1}{(\gamma+1)^{\gamma+1}}\right]s\Delta s=\infty,\end{aligned}$$

if $\beta>1/(\gamma+1)^{\gamma+1}$. Then every solution of (4.4) is oscillatory or converges to zero when $\beta>1/(\gamma+1)^{\gamma+1}$.

Note that condition (3.22) in the difference equation case cannot be applied to Eq. (4.1) and also cannot be applied to Eqs. (4.2)–(4.4). So our results improve the results in [20] in the difference equation case.

5. Conclusion

In this paper, by using the chain rule and the Riccati transformation technique, we have established some new oscillation criteria of second-order half-linear dynamic equations on time scales. Our results not only unify the oscillation of differential and difference equations but also improve the results of second-order half-linear difference equations established in [20]. Also, we established some oscillation criteria for Eqs. (3.20) and (3.21) which are essentially new. Not only this, but also our results can be applied on different types of time scales. For examples, when $\mathbb{T}=\mathbb{N}_0^2=\{n^2 : n\in\mathbb{N}_0\}$, then $\sigma(t)=(\sqrt{t}+1)^2$, $\mu(t)=1+2\sqrt{t}$,

$$\Delta y_N(t)=\frac{y((\sqrt{t}+1)^2)-y(t)}{1+2\sqrt{t}}\quad\text{and}\quad\int_{t_0}^{\infty}f(t)\Delta t=\sum_{t=t_0^2}^{\infty}(1+2\sqrt{t})f(t^2),$$

and (1.1) becomes the second-order half-linear difference equation

$$\Delta_N(p(t)(\Delta_N x(t))^\gamma)+q(t)x^\gamma(t)=0,\quad t\in[t_0^2,\infty],\tag{5.1}$$

when $\mathbb{T}=\{t_n : n\in\mathbb{N}_0\}$ where t_n be the so-called harmonic numbers defined by

$$t_0=0,\quad t_n=\sum_{k=1}^{n}\frac{1}{k},\quad n\in\mathbb{N},$$

then $\sigma(t_n)=t_{n+1}$, $\mu(t_n)=1/(n+1)$,

$$x^\Delta(t_n)=(n+1)\Delta x(t_n)\quad\text{and}\quad\int_{t_0}^{\infty}f(t)\Delta t=\sum_{0}^{\infty}\frac{1}{n+1}f(t_n)$$

and Eq. (1.1) becomes the second-order half-linear difference equation

$$\Delta_{t_n}(a(t_n)(\Delta_{t_n}x(t_n))^\gamma)+x^\gamma(t_n)=0,\quad t_n\in[0,\infty].\tag{5.2}$$

By using the results in Section 3, we can obtain some sufficient conditions for oscillation of all solutions of Eqs. (5.1) and (5.2) which are essentially new. The details are left to the interested reader. The results are proved in the case $\gamma > 1$ and cannot be applied in the case when $\gamma = 1$ or if $0 < \gamma < 1$ is a quotient of odd positive integers. So it would be interesting to extend the above results to conclude the case when $\gamma = 1$ and find another method to study the case when γ is a quotient of odd positive integers.

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