Orthogonal cubic spline collocation method for the extended Fisher–Kolmogorov equation

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Abstract

A second-order splitting combined with orthogonal cubic spline collocation method is formulated and analysed for the extended Fisher–Kolmogorov equation. With the help of Lyapunov functional, a bound in maximum norm is derived for the semidiscrete solution. Optimal error estimates are established for the semidiscrete case. Finally, using the monomial basis functions we present the numerical results in which the integration in time is performed using RADAU 5 software library. © 2004 Elsevier B.V. All rights reserved.

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1. Introduction

In this paper, we discuss a second-order splitting procedure combined with orthogonal cubic spline collocation method for the following extended Fisher–Kolmogorov (EFK) equation. For $\gamma > 0$, find a real valued function $u(x,t)$ on $I \times [0,T)$ satisfying

$$u_t + \gamma u_{xxxx} - u_{xx} + f(u) = 0, \quad x \in I, \quad t \in (0,T)$$

with initial condition

$$u(x,0) = u_0(x), \quad x \in I$$

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and the boundary conditions
\[ u(0,t) = u(1,t) = 0, \quad t \in (0,T), \]
\[ u_{xx}(0,t) = u_{xx}(1,t) = 0, \quad t \in (0,T). \] (1.3)

Here, \( f(u) = u^3 - u \), \( 0 < T < \infty \) and \( I = (0,1) \).

When \( \gamma = 0 \) in (1.1), we obtain the standard Fisher–Kolmogorov (FK) equation. However, by adding a stabilizing fourth-order derivative term to the Fisher–Kolmogorov equation, Coullet et al. [5] and Dee and van Saarloos [7, 21–23] proposed (1.1) and called the model described by (1.1) as the EFK equation.

Eq. (1.1) arises in a variety of applications such as pattern formation in bi-stable systems [7], propagation of domain walls in liquid crystals [24], travelling waves in reaction diffusion system [1,2] and mesoscopic model of a phase transition in a binary system near Lipschitz point [12]. In particular, in phase transitions near critical points (Lipschitz points), the fourth-order derivative becomes important.

In recent years, attention has been focused on the steady-state equation of (1.1) [19]. Typically, depending on the parameter \( \gamma \), the stationary problem displays a multitude of periodic, homoclinic and heteroclinic solutions. The steady-state equation of (1.1) has been analysed by shooting methods in [18,19] and by variational methods in [13].

Regarding computational studies, there are some numerical experiments conducted in [7], without any convergence analysis. Therefore, the objective of the present study is to discuss the convergence of the numerical solution of (1.1) using the second-order splitting combined with orthogonal cubic spline collocation method.

The theoretical results like existence, uniqueness and regularity of (1.1) have been discussed in [6]. Earlier, second-order splitting scheme combined with orthogonal spline collocation method has been discussed in [16,17] and in [14]. The popularity of orthogonal cubic spline collocation method is due to its conceptual simplicity, wide applicability and ease of implementation. It is much superior to B-splines in terms of stability, efficiency and conditioning of the resulting matrix [3]. Compared to finite element method (FEM) the calculation of the coefficients of the mass and stiffness matrices determining the approximate solution is very fast since no integrals need to be evaluated or approximated. Another advantage of this method is that it systematically incorporates boundary conditions and interface conditions. The resulting semidiscrete system is solved by RADAU 5 software library [10,11] which is suitable for solving differential algebraic equations (DAEs). In this paper, we use second-order splitting procedure combined with orthogonal cubic spline collocation method for Eq. (1.1) and approximate the discrete solution using monomial basis function. The semidiscrete system is then integrated in time using RADAU 5 time integrator.

Recently, the orthogonal spline collocation methods have been applied to Fisher equation with a parameter and reaction diffusion equation with cubic nonlinearity, i.e., Eq. (1.1) with \( \gamma = 0 \). We note that only computational experiments are discussed there using the latest version of a software called AUTO developed in [8]. However, in the present paper, we have analysed the splitting method combined with orthogonal cubic spline collocation techniques with numerical experiments for the EFK equation (1.1), i.e., when \( \gamma \neq 0 \) and also discussed the convergence analysis.
To motivate the present study, we first split Eq. (1.1) by setting $v = -u_{xx}$ to obtain
\[
\begin{align*}
    u_t - \gamma v_{xx} + v + f(u) &= 0, \quad (x,t) \in I \times (0,T], \quad (1.4) \\
    v + u_{xx} &= 0 \quad (1.5)
\end{align*}
\]
with initial condition
\[
    u(x,0) = u_0(x), \quad x \in I \quad (1.6)
\]
and the boundary conditions
\[
    u(0,t) = u(1,t) = 0, \quad t \in (0,T], \quad v(0,t) = v(1,t) = 0, \quad t \in (0,T]. \quad (1.7)
\]
Then, an orthogonal cubic spline collocation method in the spatial direction using monomial basis functions are applied to approximate the solutions of system (1.4)–(1.7).

For the orthogonal cubic spline collocation method, let $\mathcal{A} = \{x_j\}_{j=1}^{N+1}$ denote a partition of $I = [0,1]$ and be such that
\[
    A : 0 = x_1 < x_2 < \cdots < x_{N+1} = 1.
\]
\[
    I_j = (x_{j+1}, x_j), \quad h_j = x_{j+1} - x_j, \quad j = 1,2,3,\ldots,N
\]
and
\[
    h = \max_{1 \leq j \leq N} h_j.
\]
Assume that the partition is quasi-uniform, i.e., there exists a finite positive constant $\delta$ such that
\[
    \max_{1 \leq j \leq N} \left( \frac{h}{h_j} \right) \leq \delta.
\]
We define a finite-dimensional subspace $\mathcal{S}_3(\mathcal{A})$ as
\[
    \mathcal{S}_3(\mathcal{A}) = \{ \varphi \in C^1(I) : \varphi|_{I_j} \in P_3, \ j = 1,2,\ldots,N \text{ and } \varphi(0) = \varphi(1) = 0 \},
\]
where $P_3$ denotes the set of all cubic polynomials. Let $\{\lambda_k\}_{k=1}^2$ denote the roots of the Legendre Polynomial of degree 2 (i.e., $\lambda_1 = \frac{1}{2}(1 - 1/\sqrt{3}), \lambda_2 = \frac{1}{2}(1 + 1/\sqrt{3})$). These are the nodes of the 2-point Gaussian quadrature rule on the interval $I$ with corresponding weights $w_k = \frac{1}{2}, \ k = 1,2$. Now setting
\[
    \lambda_{jk} = x_j + h_j \lambda_k, \quad j = 1,2,\ldots,N, \quad k = 1,2.
\]
For $\varphi, \psi \in C^0(I)$, we define a discrete innerproduct and its induced norm by
\[
    \langle \varphi, \psi \rangle = \sum_{j=1}^{N} \langle \varphi, \psi \rangle_j, \quad |\varphi|_D = \langle \varphi, \varphi \rangle^{1/2},
\]
where
\[
    \langle \varphi, \psi \rangle_j = \frac{h_j}{2} \sum_{k=1}^{2} \varphi(\lambda_{jk}) \psi(\lambda_{jk}).
\]
Lemma 1.1. For \( f, g \in \mathcal{F}_3(\Delta) \),
\[
-\langle f'', g \rangle = (f', g') + \frac{1}{1080} \sum_{j=1}^{N} f_j^{(3)} g_j^{(3)} h_j^5 = -\langle g'', f \rangle ,
\]
where \( f_j^{(3)} \) (respectively, \( g_j^{(3)} \)), the third derivative is constant on \( \bar{I}_j \).
Note that when \( g = f \) with \( f \in \mathcal{F}_3(\Delta) \), we have
\[
\| f' \|_{L^2}^2 \leq -\langle f''', f \rangle.
\]

For a proof of Lemma 1.1, we refer to Douglas and Dupont [9].

The layout of this paper is as follows: In Section 1, we introduce some notations and preliminaries. In Section 2, we describe the continuous-time orthogonal spline collocation method for the solution of (1.1) and derive optimal error estimates. Finally, Section 3 is devoted to numerical experiments. It is shown that the theoretical order of convergence is equivalent to the numerically computed order of convergence.

Throughout this paper, \( C \) denotes a generic positive constant which is independent of the discretization parameter \( h \) which may have different values at different places.

2. Continuous-time orthogonal cubic spline collocation method

The continuous-time orthogonal cubic spline collocation approximation to the solution \( \{u, v\} \) of (1.4)–(1.7) is a pair of differentiable maps \( \{U, V\} : [0, T] \to \mathcal{F}_3(\Delta) \times \mathcal{F}_3(\Delta) \) such that for \( j = 1, 2, \ldots, N \) and \( k = 1, 2 \)
\[
U_t(\lambda_{jk}, t) - \gamma V_{xx}(\lambda_{jk}, t) + V(\lambda_{jk}, t) + f(U(\lambda_{jk}, t)) = 0, \quad t \in (0, T],
\]
\[
- U_{xx}(\lambda_{jk}, t) = V(\lambda_{jk}, t), \quad t \in (0, T]
\]
with appropriate initial approximation \( U(x, 0) \), which we shall define later.

The corresponding discrete Galerkin formulation is written as
\[
\langle U_t, z \rangle - \gamma \langle V_{xx}, z \rangle + \langle V, z \rangle + \langle f(U), z \rangle = 0, \quad z \in \mathcal{F}_3(\Delta),
\]
\[
- \langle U_{xx}, w \rangle = \langle V, w \rangle, \quad w \in \mathcal{F}_3(\Delta).
\]
Note that the consistent initial condition \( V(0) \) can be determined from (2.4) by putting \( t = 0 \), i.e., \( V(\cdot, 0) \) satisfies
\[
\langle V(x, 0), w \rangle = -\langle U_{xx}(x, 0), w \rangle, \quad w \in \mathcal{F}_3(\Delta).
\]

Since \( \mathcal{F}_3(\Delta) \) is a finite-dimensional space, problem (2.3), (2.4) leads to a system of nonlinear DAEs of index one. The system is solvable [4], therefore, the unique solution exists locally. For global existence, i.e., existence of a unique solution on the interval \((0, \infty)\), we need the following a priori bounds.
Theorem 2.1. There exists a positive constant \( C \) such that
\[
\|U\|_1 + \gamma \|V\| \leq C(\|U_0\|_2, \|V(0)\|), \quad t > 0.
\]
Moreover, for \( t > 0 \)
\[
\|U(t)\|_{L^\infty} \leq C(\|U_0\|_2, \|V(0)\|).
\]

Proof. Consider the Lyapunov functional \( \mathcal{E}(U) \) as
\[
\mathcal{E}(U) = \frac{1}{2} |U_x|^2_D + \langle F(U), 1 \rangle,
\]
where \( F(U) = \frac{1}{4}(1 - U^2)^2 \) and \( F'(\cdot) = f(\cdot) \).
Differentiating (2.6) with respect to \( t \), we obtain
\[
\frac{d\mathcal{E}(U)}{dt} = \langle U_x, U_{xt} \rangle + \langle f(U), U_t \rangle.
\]
(2.7)

Setting \( z = U_t \) and \( w = U_t \) in (2.3) and (2.4), respectively, we find that
\[
|U_t|^2_D - \gamma \langle V_{xx}, U_t \rangle + \langle V, U_t \rangle + \langle f(U), U_t \rangle = 0,
\]
(2.8)

\[
-\langle U_{xx}, U_t \rangle = \langle V, U_t \rangle.
\]
(2.9)

Using (2.9) in (2.8), we obtain
\[
|U_t|^2_D - \gamma \langle V_{xx}, U_t \rangle - \langle U_{xx}, U_t \rangle + \langle f(U), U_t \rangle = 0.
\]
(2.10)

We note that
\[
-\langle U_{xx}, U_t \rangle = (U_x, U_{xt}) + \frac{1}{1080} \sum_{j=1}^N U_j^{(3)} U_j^{(3)} h_j^5,
\]
(2.11)

and
\[
(U_x, U_{xt}) = (U_x, U_{xt}) - \frac{1}{720} \sum_{j=1}^N U_j^{(3)} U_{j,t}^{(3)} h_j^5.
\]
(2.12)

Using (2.12) in (2.11), we obtain
\[
-\langle U_{xx}, U_t \rangle = (U_x, U_{xt}) + \left( \frac{1}{720} + \frac{1}{1080} \right) \sum_{j=1}^N U_j^{(3)} U_{j,t}^{(3)} h_j^5.
\]
(2.13)

We differentiate (2.4) with respect to \( t \) and use Lemma 1.1 to obtain
\[
-\langle w_{xx}, U_t \rangle = \langle V_t, w \rangle.
\]
(2.14)

Choosing \( w = \gamma V \) in (2.14), we find that
\[
-\gamma \langle V_{xx}, U_t \rangle = \gamma \langle V_t, V \rangle = \frac{\gamma}{2} \frac{d}{dt} |V|^2_D.
\]
(2.15)

Substituting (2.13) and (2.15) in (2.10), we obtain
\[
|U_t|^2_D + \frac{\gamma}{2} \frac{d}{dt} |V|^2_D + (U_x, U_{xt}) + \frac{1}{864} \frac{d}{dt} \sum_{j=1}^N h_j^5 U_j^{(3)} U_j^{(3)} + \langle f(U), U_t \rangle = 0.
\]
(2.16)
Using (2.16) in (2.7), we find that
\[
\frac{d\mathcal{E}(U)}{dt} + \frac{\gamma}{2} |V|^2 \leq \frac{864}{864} \sum_{j=1}^{N} h_j^2 (U^{(3)}_j)^2 \leq 0
\]
and hence, integrating with respect to time, we arrive at
\[
\mathcal{E}(U) + \frac{\gamma}{2} |V(t)|^2 \leq \mathcal{E}(U_0) + \frac{\gamma}{2} |V(0)|^2 + \frac{864}{864} \sum_{j=1}^{N} h_j^2 (U^{(3)}_j(0))^2. \tag{2.17}
\]
Note that
\[
|U^{(3)}_j(0)| \leq C \|U(0)\|_{W^{3,\infty}(I_j)}
\]
and using the inverse inequality
\[
\|Z\|_{W^{3,\infty}(I_j)} \leq C h_j^{-3/2} \|Z\|_{H^2(I_j)},
\]
we obtain the estimate for the last term on the right-hand side of (2.17) as
\[
\sum_{j=1}^{n} h_j^5 |U^{(3)}_j(0)|^2 \leq C \sum_{j=1}^{n} h_j^5 h_j^{-3} \|U(0)\|^2_{H^2(I_j)} \leq C \|U(0)\|^2_2.
\]
Using the definition of \(\mathcal{E}(U_0)\) and \(F(U_0) \geq 0\), we arrive at
\[
|U_0|^2_2 + \gamma |V(t)|^2 \leq C (\|U_0\|_2, \|V(0)\|).
\]
Note that using Lemma 1.1 and Poincaré inequality, we find that
\[
\|U\|_1 \leq C |U|_1 \leq C (\|U_0\|_2, \|V(0)\|).
\]
Hence,
\[
\|U(t)\|_1 \leq C (\|U_0\|_2, \|V(0)\|)
\]
and
\[
|V(t)|_2 \leq C (\|U_0\|_2, \|V(0)\|).
\]
An application of Sobolev imbedding theorem now yields
\[
\|U(t)\|_L^\infty \leq C (\|U_0\|_2, \|V(0)\|)
\]
and this completes the proof. □

For deriving optimal order of convergence, we define the intermediate projections as differentiable maps \(\{\tilde{U}, \tilde{V}\}: [0, T] \to \mathcal{S}_3(A) \times \mathcal{S}_3(A)\) satisfying
\[
\gamma \langle (v - \tilde{V})_{xx}, z \rangle + \langle (v - \tilde{V}), z \rangle = 0, \quad z \in \mathcal{S}_3(A), \tag{2.18}
\]
\[
\langle (u - \tilde{U})_{xx}, w \rangle = 0, \quad w \in \mathcal{S}_3(A). \tag{2.19}
\]
Let \(\eta = u - \tilde{U}\) and \(\rho = v - \tilde{V}\). Below, we state the estimates of \(\eta\) and \(\rho\) without proof. For a proof, we refer to [20].
Lemma 2.1. Let \( u, v \in C^1(\tilde{I}) \) be such that \( u, v \in H^8(I_j), \ j = 1, 2, \ldots, N. \) Further, let \( \tilde{U} \) and \( \tilde{V} \) be solutions of (2.18) and (2.19), respectively. Then for \( t \in (0, T) \)

\[
|\eta|_D + |\eta_t|_D \leq \|\eta\|_{L^\infty(I)} + |\eta_t|_{L^\infty(I)} \leq C h^4(\|u\|_{H^8(\tilde{I})} + \|u_t\|_{H^8(\tilde{I})})
\]  

(2.20)

and

\[
|\rho|_D + |\rho_t|_D \leq \|\rho\|_{L^\infty(I)} + |\rho_t|_{L^\infty(I)} \leq C h^4(\|v\|_{H^8(\tilde{I})} + \|v_t\|_{H^8(\tilde{I})}).
\]  

(2.21)

For our error analysis, we split the error \( e = u - U \) and \( E = v - V \), respectively, as

\[
e := (u - \tilde{U}) - (U - \tilde{U}) = \eta - \theta
\]

and

\[
E := (v - \tilde{V}) - (V - \tilde{V}) = \rho - \zeta.
\]

Theorem 2.2. Let \( u \in L^\infty(H^8), \ u_t \in L^\infty(H^8) \) and \( u_0 \in H^8(I) \). Further, let \( U \) and \( V \) be the solutions of (2.1) and (2.2). Assume \( U(0) = \tilde{U}(0) \) or \( U(0) = \mathcal{I}_h u_0 \) and \( V(0) = \mathcal{I}_h v_0 \) with \( v_0 = u_{0xx} \) where \( \mathcal{I}_h u_0 \) is the Hermite interpolant of \( u_0 \) in \( \mathcal{S}_3(A) \). Then for sufficiently small \( h \), the following estimate holds:

\[
\|u - U\|_{L^\infty(L^\infty)} + \|v - V\|_{L^\infty(L^2)} \leq C h^4 \{\|u_0\|_{H^8} + \|u\|_{L^\infty(H^8)} + \|u_t\|_{L^2(H^8)}\}.
\]  

(2.22)

Proof. Since the estimates of \( \eta \) and \( \rho \) are known from Lemma 2.1, we need to compute the estimates of \( \theta \) and \( \xi \). We form the discrete innerproduct between (1.4) and (1.5) by \( z \) and \( w \), respectively. We subtract the resulting equations from (2.3) and (2.4). Then using (2.18) and (2.19), we obtain the following equations in \( \theta \) and \( \xi \):

\[
\langle \theta_t, z \rangle - \gamma \langle \xi_{xx}, z \rangle = \langle \eta_t, z \rangle - \langle \xi, z \rangle + \langle f(u) - f(U), z \rangle, \quad z \in \mathcal{S}_3(A),
\]  

(2.23)

\[
\langle \theta_{xx}, w \rangle = \langle \rho - \xi, w \rangle, \quad w \in \mathcal{S}_3(A).
\]  

(2.24)

Now choose \( z = \theta \) and \( w = \gamma \xi \) in (2.23) and (2.24), respectively. Then a use of Lemma 1.1 with Cauchy–Schwarz inequality and Young’s inequality yields

\[
\frac{1}{2} \frac{d}{dt} |\theta|_D^2 + \gamma |\xi|_D^2 \leq C(\gamma, \varepsilon)(|\rho|_D^2 + |\theta|_D^2 + |\eta_t|_D^2) + C |\xi|_D^2 + |\langle f(u) - f(U), \theta \rangle|.
\]  

(2.25)

For the last term on the right-hand side of (2.25), we apply Theorem 2.1 to bound \( U \). Note that when \( U(0) = \tilde{U}(0) \) or \( \mathcal{I}_h u_0 \) and \( V(0) = \mathcal{I}_h v_0 \), \( \|U\|_{L^\infty} \leq C \). Thus,

\[
|\langle f(u) - f(U), \theta \rangle| = |\langle u^3 - u - (U^3 - U), \theta \rangle|
\]

\[
= |\langle (u - U)(u^2 + uU + U^2 - 1), \theta \rangle|
\]

\[
\leq C |\langle \eta - \theta, \theta \rangle|
\]

\[
\leq C(\|\eta\|_D^3 + |\theta|_D^3).
\]  

(2.26)
Using (2.26) in (2.25), we arrive at
\[
\frac{1}{2} \frac{d}{dt} |\theta|^2_D + (\gamma - \varepsilon)|\bar{\xi}|^2_D \leq C(|\rho|^2_D + |\theta|^2_D + |\eta|^2_D + |\eta_t|^2_D).
\] (2.27)

Choosing \(\varepsilon = \gamma/2\) in (2.27), we integrate the resulting inequality with respect to \(t\), it follows that
\[
|\theta|^2_D + \gamma \int_0^t |\bar{\xi}|^2_D \, dt \leq C \left[ |\theta(0)|^2_D + \int_0^t (|\rho|^2_D + |\eta|^2_D + |\eta_t|^2_D) \, dt \right].
\] (2.28)

If \(U(0) = \tilde{U}(0)\) then \(|\theta(0)| = 0\), otherwise, \(|\theta(0)| \leq C h^4 u_0\|_6\). Using Gronwall’s lemma, we obtain the estimate \(|\theta|_{L^\infty(\Delta)}\).

For the estimate \(|\bar{\xi}|_{L^\infty(\Delta)}\), we first differentiate (2.24) with respect to \(t\) to obtain
\[
\langle v_{xt}, w \rangle = \langle \rho_t - \bar{\xi}_t, w \rangle, \quad w \in \mathcal{S}_3(\Delta).
\] (2.29)

Setting \(w = \gamma \bar{\xi}\) in (2.29) and \(z = \theta\) in (2.23), we arrive at
\[
\gamma \langle \bar{\xi}_{xt}, \bar{\xi} \rangle = \gamma \langle \rho_t, \bar{\xi} \rangle - \gamma \frac{d}{dt} |\bar{\xi}|^2_D
\] (2.30)

and
\[
|\theta|^2_D - \gamma \langle \bar{\xi}_{xt}, \bar{\xi} \rangle = \langle \eta_t, \theta_t \rangle - \langle \bar{\xi}, \theta_t \rangle + \langle f(u) - f(U), \theta_t \rangle.
\] (2.31)

Adding (2.30) and (2.31), we again use Lemma 1.1 to obtain
\[
|\theta|^2_D + \gamma \frac{d}{dt} |\bar{\xi}|^2_D \leq C(\gamma, \varepsilon)(|\rho|^2_D + |\eta|^2_D + |\bar{\xi}|^2_D) + \frac{\varepsilon}{2} |\theta|^2_D + |\langle f(u) - f(U), \theta_t \rangle|.
\] (2.32)

The nonlinear term can be written as
\[
\langle f(u) - f(U), \theta_t \rangle = \langle (u - U)(u^2 + uU + U^2 - 1), \theta_t \rangle
\]
\[
\leq C|\langle \eta - \theta, \theta_t \rangle|
\]
\[
\leq C(\varepsilon)(|\bar{\xi}|^2_D + |\rho|^2_D) + \frac{\varepsilon}{2} |\theta|^2_D.
\]

Eq. (2.32) implies that
\[
(1 - \varepsilon)|\theta|^2_D + \gamma \frac{d}{dt} |\bar{\xi}|^2_D \leq C(|\rho|^2_D + |\bar{\xi}|^2_D + |\eta|^2_D + |\eta_t|^2_D + |\theta|^2_D).
\] (2.33)

Choosing \(\varepsilon = \frac{1}{2}\) and integrating with respect to \(t\), we obtain
\[
\int_0^t |\theta|^2_D + \gamma |\bar{\xi}|^2_D \leq C \left[ |\bar{\xi}(0)|^2_D + \int_0^t (|\rho|^2_D + |\eta|^2_D + |\eta_t|^2_D + |\theta|^2_D) \, dt \right].
\] (2.34)

For the estimate \(|\bar{\xi}(0)|\), we put \(t = 0\) in (2.24) and we obtain
\[
|\bar{\xi}(0)|_D \leq |\rho(0)|_D \leq C h^4 v_0\|_{H^6} \leq C h^4 u_0\|_{H^n}.
\] (2.35)
Substituting the estimate $|\tilde{\zeta}(0)|$ in (2.34) and using the Gronwall’s lemma, we obtain the estimate $|\tilde{\zeta}|_{L^\infty(L^2)}$. From (2.24), we use $w = \theta$ to obtain

$$
|\theta|_D^2 = -\langle \theta_{xx}, \theta \rangle \leq (|\rho|_D + |\tilde{\zeta}|_D) |\theta|_D
$$

and using Lemma 1.1 we find that

$$
\|\theta_x\| \leq C(|\rho|_D + |\tilde{\zeta}|_D).
$$

Thus, Lemma 2.1 with the estimate of $|\tilde{\zeta}|_D$ yields the following superconvergence estimate:

$$
\|\theta_x\| \leq C h^4 \{\|u_0\|_{H^6} + \|u\|_{L^\infty(H^8)} + \|u_t\|_{L^2(H^8)}\}.
$$

Apply the triangle inequality to complete the rest of the proof. □

**Remarks.** We can extend the theory established in Sections 1 and 2 to other type of boundary conditions like

$$
u_x(0, t) = \nu_x(1, t) = 0,
$$

$$
u_{xxx}(0, t) = \nu_{xxx}(1, t) = 0
$$

and periodic boundary conditions.

### 3. Numerical result

In this section, we use orthogonal cubic spline collocation method to approximate problem (1.4)–(1.7) and present some numerical results. The approximate solution is defined as a pair of differentiable maps \(\{U, V\} : [0, T] \to \mathcal{S}_3(\Omega) \times \mathcal{S}_3(\Omega)\) satisfying (2.1) and (2.2). As in [20], we use the monomial basis functions to represent $U$ and $V$, respectively, as

$$
U(x, t) = \sum_{l=1}^{4} U_{j,l}(t) \frac{(x - x_j)^{l-1}}{(l - 1)!}, \quad x \in I_j
$$

and

$$
V(x, t) = \sum_{l=1}^{4} V_{j,l}(t) \frac{(x - x_j)^{l-1}}{(l - 1)!}, \quad x \in I_j,
$$

where,

$$
U_{j,1}(t) = U(x_j, t), \quad U_{j,2}(t) = U_x(x_j, t),
$$

$$
U_{j,3}(t) = U_{xx}(x_j^+, t), \quad U_{j,4}(t) = U_{xxx}(x_j^+, t), \quad j = 1, 2, \ldots, N
$$

(3.3)
and similarly for $V_{j,l}$. In order to accommodate the boundary conditions, we define
\begin{align*}
U_{N+1,1}(t) &= U(x_{N+1}, t), \quad V_{N+1,1}(t) = V(x_{N+1}, t), \\
U_{N+1,2}(t) &= U_s(x_{N+1}, t), \quad V_{N+1,2}(t) = V_s(x_{N+1}, t).
\end{align*}
(3.4)

Using (3.1) and (3.2) in (2.1) and (2.2), we obtain the following system of DAEs:
\begin{align*}
\sum_{l=1}^{4} \dot{U}_{j,l} \left( \frac{(h_j \dot{\lambda}_k)^{l-1}}{(l-1)!} \right) &= \gamma (V_{j,3} + h_j \dot{\lambda}_k V_{j,4}) - \sum_{l=1}^{4} V_{j,l} \left( \frac{(h_j \dot{\lambda}_k)^{l-1}}{(l-1)!} \right) \\
- \left( \sum_{l=1}^{4} U_{j,l} \left( \frac{(h_j \dot{\lambda}_k)^{l-1}}{(l-1)!} \right) \right)^3 + \sum_{l=1}^{4} U_{j,l} \left( \frac{(h_j \dot{\lambda}_k)^{l-1}}{(l-1)!} \right), \\
\sum_{l=1}^{4} V_{j,l} \left( \frac{(h_j \dot{\lambda}_k)^{l-1}}{(l-1)!} \right) + (U_{j,3} + h_j \dot{\lambda}_k U_{j,4}) &= 0
\end{align*}
(3.5)

for $j = 1, 2, \ldots, N$, $k = 1, 2$, where $\dot{U}_{j,k}(t) = (d/dt)U_{j,k}(t)$.

The $C^0$ and $C^1$ continuity conditions on $U$ and $V$ require that for $j = 1, 2, \ldots, N$
\begin{align*}
U_{j+1,1} &= U_{j,1} + h_j U_{j,2} + \frac{h_j^2}{2!} U_{j,3} + \frac{h_j^3}{3!} U_{j,4}, \\
V_{j+1,1} &= V_{j,1} + h_j V_{j,2} + \frac{h_j^2}{2!} V_{j,3} + \frac{h_j^3}{3!} V_{j,4}, \\
U_{j+1,2} &= U_{j,2} + h_j U_{j,3} + \frac{h_j^2}{2!} U_{j,4}, \\
V_{j+1,2} &= V_{j,2} + h_j V_{j,3} + \frac{h_j^2}{2!} V_{j,4}.
\end{align*}
(3.7)

The boundary conditions (1.7), in view of (3.4) yield the following set of equations:
\begin{align*}
U_{1,1} &= 0, \quad U_{N+1,1} = 0, \\
V_{1,1} &= 0, \quad V_{N+1,1} = 0.
\end{align*}
(3.8)

Rewriting Eqs. (3.5)–(3.8) in matrix form, we obtain
\begin{equation}
MW' = F(t, W),
\end{equation}
(3.9)
which is of order $8N + 4$ with
\begin{equation*}
W = [W_1^{1T}, W_1^{2T}, W_2^{1T}, W_2^{2T}, \ldots, W_N^{1T}, W_N^{2T}, W_{N+1}^{1T}]^T.
\end{equation*}
where
\[
W^1_j = [Uj,1, Vj,1, Uj,2, Vj,2]^T, \quad W^2_j = [Uj,3, Vj,3, Uj,4, Vj,4]^T, \quad j = 1, \ldots, N,
\]
and the singular constant mass matrix \( M \) is of the form
\[
M = \begin{bmatrix}
0_{24} & A_1 & B_1 & 0_{44} \\
& & & \\
& A_2 & B_2 & 0_{44} \\
& & & \\
& & & \ddots \\
& & & \\
& & & A_N & B_N & 0_{44} \\
& & & & & 0_{44} \\
& & & & & 0_{44} \\
& & & & & 0_{24}
\end{bmatrix}
\tag{3.10}
\]

Here \( A_j \) and \( B_j \) are \( 4 \times 4 \) matrices corresponding to the coefficients of the derivative terms in (3.5) and (3.6) and are of the form
\[
A_j = \begin{bmatrix}
1 & 0 & h_j \dot{\lambda}_1 & 0 \\
1 & 0 & h_j \dot{\lambda}_2 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad B_j = \begin{bmatrix}
(h_j \dot{\lambda}_1)^2 \frac{2!}{2!} & 0 & 0 & (h_j \dot{\lambda}_1)^3 \frac{3!}{3!} \\
(h_j \dot{\lambda}_2)^2 \frac{2!}{2!} & 0 & 0 & (h_j \dot{\lambda}_2)^3 \frac{3!}{3!} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

and \( 0_{24} \) and \( 0_{44} \) are \( 2 \times 4 \) and \( 4 \times 4 \) zero matrices, respectively.

The initial value \( W(0) \) is found out from the intermediate projection or Hermite interpolant of (2.19) at \( t = 0 \) and the consistency condition (2.5). Since \( U(0) = \tilde{U}(0) \) from the assumption of the theorem, it follows from (2.19) that

\[
U_{xx}(\tilde{\lambda}_j, 0) = u_{0xx}(\lambda_j), \quad j = 1, \ldots, N, \quad k = 1, 2.
\]

Now along with the continuity conditions at the interior nodal points \( x_j \) and the boundary conditions, we obtain a linear system of \( 4N + 2 \) equations
\[
RU = U_0,
\tag{3.11}
\]
where $\mathbf{R}$ is $(4N + 2) \times (4N + 2)$ block diagonal nonsingular matrix of the form

$$
\mathbf{R} = \begin{bmatrix}
\mathbf{I}_1 & \mathbf{X}_1 & \mathbf{Y}_1 & \mathbf{0}_{22} \\
\mathbf{C}_1 & \mathbf{D}_1 & \mathbf{I}_{22} \\
\mathbf{X}_2 & \mathbf{Y}_2 & \mathbf{0}_{22} \\
\mathbf{C}_2 & \mathbf{D}_2 & \mathbf{I}_{22} \\
\vdots \\
\mathbf{X}_N & \mathbf{X}_N & \mathbf{0}_{22} \\
\mathbf{C}_N & \mathbf{D}_N & \mathbf{I}_{22} \\
\mathbf{I}_1 
\end{bmatrix}.
$$

Here,

$$
\mathbf{X}_j = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{Y}_j = \begin{bmatrix} 1 & h_j \lambda_1 \\ 1 & h_j \lambda_2 \end{bmatrix}, \quad \mathbf{C}_j = \begin{bmatrix} -1 & -h_j \\ 0 & -1 \end{bmatrix}, \quad \mathbf{D}_j = \begin{bmatrix} -\frac{h_j^2}{2!} & -\frac{h_j^3}{3!} \\ -h_j & -\frac{h_j^2}{2!} \end{bmatrix},
$$

$I_{22}$ is the identity matrix of order $2 \times 2$, $\mathbf{0}_{22}$ is the zero matrix of order $2 \times 2$ and $\mathbf{I}_1 = (1,0)$. Further, the vectors $\mathbf{U}_0$ and $\mathbf{U}$ are of the form

$$
\mathbf{U}_0 = [0, u_{0xx}(\lambda_{11}), u_{0xx}(\lambda_{12}), 0, 0, u_{0xx}(\lambda_{21}), u_{0xx}(\lambda_{22}), \ldots, u_{0xx}(\lambda_{N1}), u_{0xx}(\lambda_{N2}), 0, 0, 0]^T
$$

and

$$
\mathbf{U} = [\mathbf{U}_1, \mathbf{U}_2, \ldots, \mathbf{U}_N, \mathbf{U}_{N+1}]^T \in \mathbb{R}^{4N+2}
$$

with $\mathbf{U}_j = [U_{j,1}, U_{j,2}, U_{j,3}, U_{j,4}]$, $j = 1, \ldots, N$ and $\mathbf{U}_{N+1} = [U_{N+1,1}, U_{N+1,2}]$.

Similarly for $\tilde{V}(x,0)$, we obtain from (2.5)

$$
\tilde{V}(\lambda_{jk},0) = -U_{xx}(\lambda_{jk},0), \quad j = 1, \ldots, N, \quad k = 1,2,
$$

which along with continuity and boundary conditions yield a system of $4N + 2$ equations

$$
\tilde{\mathbf{R}} \mathbf{V} = \mathbf{V}_0,
$$

where

$$
\mathbf{V}_0 = [0, -U_{1,3} - h_1 \lambda_1 U_{1,4}, -U_{1,3} - h_1 \lambda_2 U_{1,4}, 0, 0, -U_{2,3} - h_2 \lambda_1 U_{2,4}, -U_{2,3} - h_2 \lambda_2 U_{2,4}, -U_{3,3} - h_3 \lambda_1 U_{3,4}, -U_{3,3} - h_3 \lambda_2 U_{3,4}, 0, 0, 0]^T
$$

and all the submatrices in $\tilde{\mathbf{R}}$ are similar to $\mathbf{R}$ in (3.11) except for

$$
\mathbf{X}_j = \begin{bmatrix} 1 & h_j \lambda_1 \\ 1 & h_j \lambda_2 \end{bmatrix}, \quad \mathbf{Y}_j = \begin{bmatrix} \frac{(h_j \lambda_1)^2}{2!} & \frac{(h_j \lambda_1)^3}{3!} \\ \frac{(h_j \lambda_2)^2}{2!} & \frac{(h_j \lambda_2)^3}{3!} \end{bmatrix}.
$$
We have solved the EFK equation mainly to validate the theoretical results and how well the numerical results mimic the qualitative behaviour of the EFK equation. For this purpose, we have used the software packages that are freely available from the web sites.

For the solution of the almost block diagonal linear system (3.11) and (3.12), we have employed the code ABDPACK [14,15]. This code has been especially developed for the solution of the system which arises from the orthogonal spline collocation methods using the monomial basis functions.

Note that the index of the system is one. We have used RADAU 5 [10], a software package that is based on a three stage implicit Runge–Kutta scheme of RADAU-IIA. This numerical scheme is self starting and sti$$f$$ly accurate. The solution obtained using RADAU 5 has order of convergence five in every component. RADAU 5 can also solve DAEs of index upto 3.

Now we describe the numerical experiments that has been conducted for the EFK equation. For computational convenience, we have considered the space in $x$ direction as $[-4, 4]$ instead of $[0, 1]$.

The numerical experiments are carried out for (1.4)–(1.7) using the initial function

$$u(x,0) = -\sin(\pi x), \quad x \in (-4, 4).$$

(3.13)

Divide the domain into $N_i = 10, 20, 40, 80$ with each of equal intervals $h_i$, where

$$h_i = \frac{8}{N_i}, \quad i = 1, \ldots, 4.$$

Since the exact solution of the EFK equation is not known, it has been replaced by numerical solution $u_{h_i}$ with $h = 160$. The order of convergence for the numerical method has been computed by the formula

$$\text{order} = \log\left(\frac{\|u_{h_i} - u_{h_{i+1}}\|_{L^\infty}}{\|u_{h_i} - u_{h_{i+2}}\|_{L^\infty}}\right) \quad \log(2), \quad i = 1, 2, j = 2, \infty,$$

(3.14)

where $u_{h_i}$ is the numerical solution with step size $h_i$ and $h_{i+1} = h_i/2$.

All the codes have been written in FORTRAN with double precision. The experiments have been conducted with the following parameter values for RADAU 5: RTOL = $10^{-6}$, ATOL = $10^{-5}$ and the initial step size $k = 10^{-5}$. RADAU 5 uses the variable step size method. The maximum step size used in all calculations is $k_{\text{max}} = 0.087880501$. The numerical solution is carried out for several $h_i$ with $\gamma = 0.1$. In Tables 1 and 2, we observe that the order of convergence estimated numerically is approximately equal to 4. This confirms the theoretical order of convergence found in Theorem 2.2.

For the initial condition (3.13), we present graphs of computed solution $U$ at various time level. We observe that when $\gamma = 0$ and $\gamma = \text{small} = 0.0001$ the behaviour of solution is almost similar. However, when $\gamma = 0.1$, the decay of solution to zero becomes fast which confirm the stabilizing character of the EFK equation (Figs. 1–3).

Table 1

<table>
<thead>
<tr>
<th>$N$</th>
<th>$|u_h - u_{h_i}|_{L^2}$</th>
<th>Order</th>
<th>$|u_h - u_{h_i}|_{L^\infty}$</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.2931815803611592E−02</td>
<td></td>
<td>0.3023089375346899E−02</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>0.1614871877330346E−03</td>
<td>4.1823</td>
<td>0.2193339169025421E−03</td>
<td>3.7848</td>
</tr>
<tr>
<td>40</td>
<td>0.1375244501811667E−04</td>
<td>3.5537</td>
<td>0.1474842429161072E−04</td>
<td>3.8945</td>
</tr>
<tr>
<td>80</td>
<td>0.7940184952668348E−06</td>
<td>4.1144</td>
<td>0.9220093488693237E−06</td>
<td>3.9996</td>
</tr>
</tbody>
</table>
Table 2
The order of convergence for \( v(x, t) \) at \( t = 0.2 \)

<table>
<thead>
<tr>
<th>( N )</th>
<th>( | v_h - v_i |_{L^2} )</th>
<th>Order</th>
<th>( | v_h - v_i |_{L^\infty} )</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.2278912370132688E-01</td>
<td></td>
<td>0.3000222146511078E-01</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>0.1337247634724468E-02</td>
<td>4.0910</td>
<td>0.1945197582244873E-02</td>
<td>3.9471</td>
</tr>
<tr>
<td>40</td>
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<td>3.4927</td>
<td>0.1292973756790161E-03</td>
<td>3.9112</td>
</tr>
<tr>
<td>80</td>
<td>0.6763767659739548E-05</td>
<td>4.1345</td>
<td>0.8121132850646973E-05</td>
<td>3.9929</td>
</tr>
</tbody>
</table>

Fig. 1. The profile of \( U(x, t) \) vs. \( x \) for \( \gamma = 0.1 \).

Below, we discuss the decay estimates for Eq. (1.1) with the different types of boundary conditions and initial conditions.

(i) We consider the EFK equation (1.1) with the following boundary conditions:

\[
\begin{align*}
    u(0, t) &= u(1, t) = 1, \\
    u_{xx}(0, t) &= u_{xx}(1, t) = 0
\end{align*}
\tag{3.15}
\]

and the initial condition

\[
    u(x, 0) = g(x),
\tag{3.16}
\]

where \( g(x) = 10^{-3} \exp(-x^2) \). Fig. 4 shows that for a small initial data the solution of the semidiscrete approximation \( U \) decays as time increases and finally it approaches the value 1.

(ii) We also consider the EFK equation (1.1) with the following boundary conditions:

\[
\begin{align*}
    u(0, t) &= u(1, t) = -1, \\
    u_{xx}(0, t) &= u_{xx}(1, t) = 0
\end{align*}
\tag{3.17}
\]
and the initial condition
\[ u(x, 0) = h(x), \]  
(3.18)
where \( h(x) = -10^{-3} \exp(-x^2) \). In Fig. 5, we observe that for a small initial data the semidiscrete solution \( U \) decays with time and finally, it approaches to stable state \(-1\).
Acknowledgements

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References