Crossed Burnside rings and Bouc’s construction of Green functors

Fumihito Oda ¹

Department of Liberal Arts, Toyama National College of Technology, Toyama 939-8630, Japan

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Abstract

We use a formula for primitive idempotents of the crossed Burnside ring given by F. Oda and T. Yoshida, and a result from the theory of Green functors obtained by S. Bouc to prove an induction theorem for the Drinfel’d double for a finite group over the complex field. We obtain Artin’s induction theorem for group algebras as a corollary of the theorem.

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1. Introduction

We study the Grothendieck (character) ring of the Drinfel’d double of a finite group \( G \) over the complex field \( \mathbb{C} \). Witherspoon studied the representation ring of the Drinfel’d double of a group algebra in positive characteristic [Wi96]. In particular, she gave a decomposition of the representation ring as a direct sum of ideals involving Green rings of subgroups by using Thévenaz’s twin functor construction for Green functors [Th88]. Dress introduced the construction of a Mackey functor \( M_F \) from a Mackey functor \( M \) by simply setting \( M_F(X) := M(X \times \Gamma) \) for all finite \( G \)-sets \( X \) when \( \Gamma \) is a finite \( G \)-set [Dr73]. We call this construction for Mackey functors the Dress construction. Bouc introduced the Dress construction for a Green functor [Bo03a, Theorem 5.1]: If \( A \) is a Green functor for \( G \) over a commutative ring \( \mathcal{O} \), and \( \Gamma \) is a crossed \( G \)-monoid, then

E-mail address: oda@toyama-nct.ac.jp.

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the Mackey functor $A^G$ obtained by the Dress construction has a natural structure of a Green functor, and its evaluation $A^G(G)$ is an $O$-algebra. Bouc’s construction involves as special cases the construction of the crossed Burnside ring obtained from the Burnside ring Green functor, the Hochschild cohomology ring of $G$ obtained from the group cohomology Green functor, and the Grothendieck ring of the Drinfel’d double of $G$ obtained from the Grothendieck ring Green functor for a group algebra. We also point out that Bouc’s construction is discussed in [Wi04]. In this paper, we obtain an induction theorem for the Drinfel’d double for a group algebra. We also point out that Bouc’s construction is discussed in [Wi04].

In this paper, we obtain an induction theorem for the Drinfel’d double for $G$ by using a formula for the primitive idempotents of the crossed Burnside ring [OY01], Bouc’s construction, and some properties of Witherspoon’s Green functor $R(D_G(\ast))$. The theorem implies Artin’s induction theorem for a group algebra over $\mathbb{C}$. This is a new proof of Artin induction theorem.

The paper is organized as follows: Section 2 recalls a summary of basic results on crossed $G$-sets and the crossed Burnside ring. Section 3 describes the relationship between the crossed Burnside ring Green functor and the Grothendieck ring Green functor for the Drinfel’d double. Section 4 exposes a kind of induction theorem for $C$-representations of the Drinfel’d double of $G$.

We shall note that the result is not exactly an induction theorem for the Functor for a group algebra over $\mathbb{C}$. Theorem implies Artin’s induction theorem for a group algebra over $\mathbb{C}$. This is a new proof of Artin induction theorem.

We refer the reader to [Bo97,Bo00,TW95] or [We00] for standard definitions and results regarding Burnside rings and Green functors, and to [Bo03a,Bo03b,OY01], and [OY04] for basic results about crossed $G$-sets and crossed Burnside rings.

2. Crossed $G$-sets

2.1. Notation. Let $G$ be a finite group. If $H$ is a subgroup of $G$, and $g \in G$, the conjugate subgroup $gHg^{-1}$ of $G$ is denoted by $^gH$. The normalizer of $H$ in $G$ is denoted by $N_G(H)$. The centralizer of $H$ (respectively $g \in G$) in $G$ is denoted by $C_G(H)$ (respectively $C_G(g)$). A set of representatives in $G$ of $G/H$ is denoted by $[G/H]$. If $X$ is a $G$-set, the stabilizer in $G$ of element $x$ of $X$ is denoted by $G_x$. If $X$ and $Y$ are $G$-sets, the intersection $G_x \cap G_y$ of stabilizers in $G$ of element $(x, y)$ of $X \times Y$ is denoted by $G_{x,y}$. The set of orbits of $H$ on $X$ is denoted by $H \backslash X$, and $[H \backslash X]$ denotes a set of representatives in $X$ of $H \backslash X$.

2.2. Crossed Burnside rings

Let $G$ be a finite group. In [Bo03a], Bouc defined a crossed $G$-monoid as follows. A crossed $G$-monoid $(\Gamma, \varphi)$ is a pair consisting of a finite monoid $\Gamma$ with a left action of $G$ by monoid automorphisms (denoted by $(g, \gamma) \mapsto g\gamma$ or $(g, \gamma) \mapsto ^g\gamma$, for $g \in G$ and $\gamma \in \Gamma$), and a map of $G$-monoids $\varphi$ from $\Gamma$ to the $G$-set $G^\Gamma$ with $G$-action defined by conjugation (i.e. a map $\varphi$ which is both a map of monoids and a map of $G$-sets). In this paper, since we use only the trivial crossed $G$-monoid $(\Gamma, \varphi) = (G^\Gamma, id_{G^\Gamma})$, we denote by $\Gamma$ or $G^\Gamma$ a crossed $G$-monoid. A crossed $G$-set $(X, \alpha)$ over a crossed $G$-monoid $\Gamma$, is a pair consisting of a finite $G$-set $X$, together with a map $\alpha$ of $G$-sets from $X$ to $\Gamma$. A morphism of crossed $G$-sets from $(X, \alpha)$ to $(Y, \beta)$ is a $G$-map $f$ from $X$ to $Y$ such that $\beta \circ f = \alpha$. Crossed $G$-sets over $\Gamma$ and crossed $G$-maps make a category $G\text{-xset}/\Gamma$. The tensor product of crossed $G$-sets $(X, \alpha)$ and $(Y, \beta)$ is defined by $(X \times Y, \alpha \beta)$, where $X \times Y$ is the direct product of $X$ and $Y$, with diagonal $G$-action, and $\alpha \beta$ is the map from $X \times Y$ to $G^\Gamma$ defined by $\alpha \beta(x, y) = \alpha(x)\beta(y)$. We denote by $X\Omega(G, \Gamma)$ the Grothendieck ring of the category $G\text{-xset}/\Gamma$ with respect to disjoint union and tensor product. We call it the crossed
Burnside ring. The crossed Burnside ring $G\text{-xset}/1^c$ over the crossed $1^c$-monoid is the ordinary Burnside ring $B(G)$. Since any crossed $G$-set is a disjoint union of transitive crossed $G$-sets (see [OY01, 2.12]), $G\text{-xset}/\Gamma$ has the following free $\mathbb{Z}$-basis as an abelian group:
\[
\{(G/D)_s \mid D \in [G \setminus S(G)], \ s \in [G \setminus C_\Gamma(D)]\}.
\]
If $\Gamma$ is a normal subgroup of $G$ or an abelian group, then a formula for the primitive idempotents of $\mathbb{K}X\Omega(G, \Gamma)$ over a splitting field $\mathbb{K}$ of characteristic 0 has been given by Oda and Yoshida (see [OY01, Lemma 5.5]).

2.3. Theorem. (See [OY01].) Let $\mathbb{K}$ be a field of characteristic 0 which is a splitting field for all subgroups of $G$.

(1) For $H \leq G$ and an irreducible $\mathbb{K}$-character $\theta$ of $C_\Gamma(H)$, we put
\[
e_{H,\theta} = \frac{\theta(1)}{|N_G(H)||C_\Gamma(H)|} \sum_{D \leq H} \sum_{s \in C_\Gamma(H)} |D| \mu(D, H) \tilde{\theta}(s^{-1})(G/D)_s,
\]
where $\tilde{\theta}$ is the sum of all distinct $N_G(H)$-conjugates of $\theta$. Then
\[
\{ e_{H,\theta} \mid H \in [G \setminus S(G)], \ \theta \in [N_G(H) \setminus \text{Irr}_K(C_\Gamma(H))] \}
\]
is a set of orthogonal idempotents of the crossed Burnside ring $\mathbb{K}X\Omega(G, \Gamma)$ over $\mathbb{K}$ such that
\[
(G/G)_1 = 1_{\mathbb{K}X\Omega(G, \Gamma)} = \sum_{H,\theta} e_{H,\theta}.
\]
Moreover, the idempotents $e_{H,\theta}$ are all primitive and conversely any primitive idempotent of $\mathbb{K}X\Omega(G, \Gamma)$ has this form.

A formula for the primitive idempotents of $\mathcal{O}X\Omega(G, G^c)$ over a $p$-local ring $\mathcal{O}$ has been given by Bouc [Bo03b].

3. Bouc’s constructions of Green functors

3.1. Burnside Green functors

We recall the crossed Burnside ring Green functor $X\Omega(\ast, G^c)$ in terms of subgroups of $G$ (see [OY04, 4.1]). Let $S(H)$ be the family of all subgroups of $H \leq G$ and $C_G(D)$ the centralizer of $D \leq H$. Then the assignment
\[
H(\leq G) \mapsto X\Omega(H, G^c) = \{(H/D)_s \mid D \in [H \setminus S(H)], \ s \in [H \setminus C_G(D)]\} \otimes \mathbb{Z}
\]
gives a Green functor for $G$ over $\mathbb{Z}$ equipped with
\[
\text{ind}^H_L : X\Omega(L, G^c) \to X\Omega(H, G^c) : (L/D)_s \mapsto (H/D)_s,
\]
\[
\text{res}^H_L : X\Omega(H, G^c) \to X\Omega(L, G^c) : (H/D)_s \mapsto \sum_{g \in [L \setminus H/D]} (L/L \cap gD)_s.
\]
\[
\text{con}_{H,R}: \chi \Omega(H, G^c) \rightarrow \chi \Omega(\frac{g}{H}, G^c) : (H/D)_s \mapsto \left(\frac{g}{H/\frac{g}{D}}\right)_s,
\]

where \(D \leq L \leq H \leq G\) and \(g \in G\). In order to note the Green functor structure of \(\chi \Omega(\ast, G^c)\), we shall discuss briefly an equivalence between the category \(\text{G-set}_{\downarrow (G/H \times G^c)}\) of finite \(G\)-sets over the \(G\)-set \(G/H \times G^c\) (see [Bo03a, 2.4.1]) and the category \(\text{H-set}_{\downarrow G}\) of finite \(H\)-sets over the \(H\)-set \(G^c\) with the \(H\)-action defined by conjugation. Let \(\Omega\) be the Burnside Green functor for \(G\) over \(\mathbb{Z}\) in terms of \(G\)-sets. By Proposition 2.4.2 of [Bo97], \(\Omega_G: (G/H) = \Omega((G/H) \times G^c)\) is isomorphic to the Grothendieck group of \(\text{G-set}_{\downarrow (G/H \times G^c)}\), with relations given by decomposition into disjoint union. It is easy to see that the \(G\)-sets

\[
[K, s]: G/K \rightarrow G/H \times G^c: gK \mapsto \left(gH, \frac{g}{s}\right)
\]

over \(G/H \times G^c\), for \(K \in [H \backslash S(H)]\) and \(s \in [H \backslash C_G(K)]\), form a basis of \(\Omega(G/H \times G^c)\) over \(\mathbb{Z}\). We denote by \((G/K, [K, s])\) an element of the basis of \(\Omega(G/H \times G^c)\). Theorem 5.1 of [Bo03a] shows that \(\Omega_{G^c}\) is a Green functor. If \((G/K, [K, s])\) and \((G/L, [L, t])\) are elements of the basis of \(\Omega(G/H \times G^c)\), then we have the following commutative diagram

\[
\begin{array}{c}
\bigcup_{x \in K \setminus H/L} G/K \cap xL \rightarrow G/K \times G/L = [K] \times [L] \\
\downarrow \quad \quad \quad \quad \quad \quad \downarrow \\
G/H \times G^c \rightarrow G/H \times G \times G^c \rightarrow G/H \times G \times G^c, \\
\end{array}
\]

where the map \(f\) from \(G/H \times G^c \times G/H \times G^c\) to \(G/H \times G/H \times G^c\) maps \((xK, \gamma_1, yL, \gamma_2)\) to \((xK, yL, \gamma_1 \gamma_2)\) (see [Bo03a, Section 5]). The left square is a pullback square. Theorem 5.1 of [Bo03a] shows that the product of \((G/K, [K, s])\) and \((G/L, [L, t])\) on \(\Omega(G/H \times G^c)\) is given by

\[
(G/K, [K, s]) \cdot (G/L, [L, t]) = \sum_{x \in K \setminus H/L} (G/K \cap xL, [K \cap xL, s \cdot xL]).
\] (1)

We have a functor \(F\) mapping \(\text{G-set}_{\downarrow (G/H \times G^c)}\) to \(\text{H-set}_{\downarrow G^c}\) defined for a transitive \(G\)-set \([K, s]: G/K \rightarrow G/H \times G^c\) over \(G/H \times G^c\) by

\[
F: (G/K, [K, s]) \mapsto \langle K, s \rangle \ast \left(\{H\} \times G^c\right) \rightarrow G^c
\]

as in [Bo97, Lemma 2.4.1]. We also denote by \([K, s]\) the \(H\)-map \(H/K \rightarrow G^c\) defined by \(gK \mapsto \frac{g}{s}\). It is clear that the \(H\)-sets

\[
[K, s]: H/K \rightarrow G^c: gK \mapsto \frac{g}{s}
\]

over the \(H\)-set \(G^c\), for \(K \in [H \setminus S(H)]\) and \(s \in [H \setminus C_G(K)]\), form a basis of \(\Omega_{\downarrow H}(G^c)\) over \(\mathbb{Z}\), where \(\Omega_{\downarrow H}\) is a Green functor for \(H\) given by the restriction to \(H\) of \(G\). We denote by \((H/K, [K, s])\) this element of the basis of \(\Omega_{\downarrow H}(G^c)\). It is easy to see that \(F\) gives an equivalence of categories from \(\text{G-set}_{\downarrow (G/H \times G^c)}\) to \(\text{H-set}_{\downarrow G^c}\) for any subgroup \(H\) of \(G\). The inverse
equivalence is given by the induction functor from $H\text{-set}_{\downarrow G^c}$ to $G\text{-set}_{\downarrow (G/H \times G^c)}$. The images of (1) under $F$ are

$$(H/K, [K, s]) \cdot (H/L, [L, t]) = \sum_{x \in [K \cap H/L]} (H/K \cap xL, [K \cap xL, s \cdot xL])$$

in $H\text{-set}_{\downarrow G^c}$. The Grothendieck group of $H\text{-set}_{\downarrow G^c}$ is isomorphic to $X\Omega(H, G^c)$. We can define a product

$$(H/K)_s \cdot (H/L)_t = \sum_{x \in [K \cap H/L]} (H/K \cap xL)_{s \cdot xt}$$

for any two elements $(H/K)_s$ and $(H/L)_t$ of the basis of $X\Omega(H, G^c)$. It is clear that the element $(H/H)_1$ for the identity element $1_G$ of $G$ is the identity element of $X\Omega(H, G^c)$. This gives a unitary ring structure to $X\Omega(H, G^c)$ for a subgroup $H$ of $G$.

3.2. Witherspoon’s Green functor

Witherspoon introduced a Green functor $R_C(DG(\ast))$ for $G$ over $\mathbb{Z}$ (see [Wi96, Section 5]). For each subgroup $H$ of $G$, there is a subalgebra

$$D_G(H) = \sum_{g \in G, h \in H} C \phi_g h$$

of the Drinfel’d (quantum) double $D(G)$ of $\mathbb{C}G$ [Dr86], where $\phi_g$ is an element of the basis $\{\phi_g\}_{g \in G}$ of the dual space $(\mathbb{C}G)^\ast = \text{Hom}_C(\mathbb{C}G, C)$. Note that $D_G(G) = D(G)$ and $R(D(G))$ is the representation ring of $D(G)$ or equivalently the Grothendieck ring of Hopf bimodules for the Hopf algebra $\mathbb{C}G$ [Ro95,Bo03a,OY04]. Let $R_C(D_G(H))$ be the Grothendieck (representation) ring of $D_G(H)$ for subgroup $H$ of $G$. Then the assignment

$$H \mapsto R_C(D_G(H)),$$

where $H \leq G$ gives a Green functor for $G$ over $\mathbb{Z}$ with operations given by

$$\text{Dres}_L^H : R_C(D_G(H)) \rightarrow R_C(D_G(L)) : U \mapsto U \downarrow_{D_G(L)},$$

$$\text{Dind}_L^H : R_C(D_G(L)) \rightarrow R_C(D_G(H)) : V \mapsto D_G(H) \otimes_{D_G(L)} V,$$

$$\text{Dconj}_{H,g} : R_C(D_G(H)) \rightarrow R_C(D_G(gH)) : U \mapsto gU = gD_G(H) \otimes_{D_G(H)} U,$$

where $U \downarrow_{D_G(L)}$ is a $D_G(L)$-module by restriction of the action from $D_G(H)$ to $D_G(L)$, $L \leq H \leq G$ and $g \in G$. We use the equivalence of the category of $H$-vector bundles on $G^c$ with the category of $D_G(H)$-modules (see [Wi96, Section 2]).
3.3. A morphism of Green functors

Let $\Omega$ be the Burnside ring Green functor for $G$ over $\mathbb{Z}$ (see [Bo97, 2.4.2]):

- If $X$ is a finite $G$-set, then $\Omega(X)$ is the Grothendieck ring of the category of finite $G$-sets over $X$, where the relations are given by decomposition into disjoint union and product of $G$-sets.
- If $X \rightarrow X'$ is a $G$-map, then $\Omega_*(f) : \Omega(X) \rightarrow \Omega(X')$ is defined by $\Omega_*(f)((Y, \phi)) = (Y, f \phi)$ for any $G$-set $(Y, \phi) = Y \xrightarrow{\phi} X$ over $X$.
- If $X' \rightarrow X$ is a $G$-map, then $\Omega^*(f) : \Omega(X) \rightarrow \Omega(X')$ is defined by $\Omega^*(f)((Y, \phi)) = (Y', \phi')$, where $(Y', \phi')$ is the pull-back of $(Y, \phi)$ along $f$, obtained by filling the cartesian square

\[
\begin{array}{ccc}
Y' & \xrightarrow{a} & Y \\
\downarrow{\phi'} & & \downarrow{\phi} \\
X' & \xrightarrow{f} & X \\
\end{array}
\]

Suppose that $R_C$ is the $\mathbb{C}$-representation (character ring) Green functor for $G$ over $\mathbb{Z}$, defined on subgroups of $G$. Then setting $R_C(G/H) = R_C(H)$ leads by linearity to a definition of $G$-equivariant $\mathbb{C}$-vector bundles $R_C(X)$ on a $G$-set $X$ (see [Wi96, Section 2]) by using Remark 2.3 of [Bo03a]:

- If $X$ is a finite $G$-set, then $R_C(X)$ is the Grothendieck ring of the category of $G$-equivariant $\mathbb{C}$-vector bundles on the $G$-set $X$, for relations given by decomposition into direct sum of vector bundles, the ring structure being induced by the tensor product of vector bundles: one can set

\[
R_C(X) = \left( \bigoplus_{x \in X} R_C(G_x) \right)^G,
\]

where the exponent denotes fixed points under the natural action of $G$ on $\bigoplus_{x \in X} R_C(G_x)$ by permutation of the components, and $G_x$ is the stabilizer of $x$ in $G$.
- If $f : X \rightarrow X'$ is a $G$-map, then $R_C_*(f) : R_C(X) \rightarrow R_C(X')$ is defined by

\[
R_C_*(f)(u)y = \sum_{x \in [G_y \backslash f^{-1}(y)]} t_{G_x}^{G_y}(u_x),
\]

where $t_{G_x}^{G_y}$ is the induction map from $R_C(G_x)$ to $R_C(G_y)$, $u \in R_C(X)$, and $y \in X$.
- If $f : X' \rightarrow X$ is a $G$-map, then $R_C^*(f) : R_C(X) \rightarrow R_C(X')$ is defined by

\[
R_C^*(f)(v)x = r_{G_f(x)}^{G_x}(v_{f(x)}),
\]

where $r_{G_f(x)}^{G_x}$ is the restriction map from $R_C(G_x)$ to $R_C(G_{f(x)})$, $v \in R_C(X)$, and $x \in X$. 
The product of the elements $a \in R_C(X)$ and $b \in R_C(Y)$ is defined by

$$(a \times b)_{x,y} = r^{G_x}_{G(x,y)}(a_x) \cdot r^{G_y}_{G(x,y)}(b_y).$$

If $X$ is a finite $G$-set, denote the natural morphism

$$\theta : \Omega \to R_C$$

of Green functors defined by the maps $\theta(X) : \Omega(X) \to R_C(X)$ by

$$(Y, \varphi) = (\varphi : Y \to X) \mapsto \{ \mathbb{C}[\varphi^{-1}(x)] \}_{x \in X},$$

where $\mathbb{C}[\varphi^{-1}(x)]$ is the permutation module associated to the $G_x$-set $\varphi^{-1}(x)$.

The following theorem is essential in the proof of Theorem 4.1 of this paper.

**3.4. Theorem.** (See Bouc [Bo03a, 5.1].) Let $A$ be a Green functor for $G$ over a commutative ring $\mathcal{O}$, $\Gamma$ a crossed $G$-monoid, and $\varepsilon_A$ an element of $A(\bullet)$ such that for any $G$-set $X$ and for any $a \in A(X)$

$$A_\ast(p_X)(a \times \varepsilon_A) = a = A_\ast(q_X)(\varepsilon_A \times a)$$

denoting by $p_X$ (respectively $q_X$) the bijective projection from $X \times \bullet$ (respectively from $\bullet \times X$) to $X$ (see [Bo03a, 1.2.1]). Then the functor $A_\Gamma$ is a Green functor for $G$ over $\mathcal{O}$, with unit $\varepsilon_{A_\Gamma}$, where $\varepsilon_{A_\Gamma}$ is the element $A_\ast(\downarrow)_{\mathcal{O}}(\varepsilon_A)$ of $A(\Gamma) = A_\Gamma(\bullet)$. Moreover the correspondence $A \mapsto A_\Gamma$ is an endo-functor of the category of Green functors for $G$ over $\mathcal{O}$.

**3.5. Lemma.** Let $\Omega$ be the Burnside ring Green functor and $G^c$ the crossed $G$-monoid. Then there is an isomorphism of Green functors

$$X \Omega \ast, G^c \cong \Omega_{G^c}.$$ 

**Proof.** Let $H$ be a subgroup of $G$. We set $\Omega_{G^c}(H) = \Omega_{G^c}(G/H) (= \Omega((G/H) \times G^c))$. By Bouc’s theorem 3.4, we have

$$\Omega((G/H) \times G^c) = \left( \bigoplus_{(aH,g) \in G/H \times G^c} \Omega(G_{aH,g}) \right)^G \cong \bigoplus_{(H,g) \in [G \setminus (G/H \times G^c)]} \Omega(G_{H,g}),$$

where $G_{aH,g} = G_{aH} \cap G_g = aH \cap C_G(g) = C_{aH}(g)$. The map

$$f_H : X \Omega(H, G^c) \to \left( \bigoplus_{(aH,g) \in G/H \times G^c} \Omega(G_{aH,g}) \right)^G$$

is defined for $(H/K)_s \in X \Omega(H, G^c)$ by

$$(H/K)_s \mapsto ([K, s]^{-1}(aH, g))_{(aH,g) \in G/H \times G^c},$$
where the map \([K, s]: G/K \to G/H \times G^c\) is the \(G\)-map defined by \(xK \mapsto (xH, x^s)\). It is well known that the natural projection \(p_L^H: G/L \to G/H\) gives the induction \(\Omega_{G^c}^*(p_L^H)\) and the restriction \(\Omega_{G^c}^*(p_L^H)\). Moreover,

\[
\gamma_{x,H}: \Omega_{G^c}^*(p_L^H) \to \Omega_{G^c}^*(p_L^H)
\]

gives the conjugation \(\Omega_{G^c}^*(\gamma_{x,H})\). It is well known that the natural projection \(\pi_G: G/G \to G/G\) makes a morphism \(f\) from the Mackey functor \(X\Omega(X, G^c)\) to the Mackey functor \(\Omega_{G^c}\), which commutes with restriction, induction and conjugation. Since \(f_H\) is the inverse of the homomorphism from \(\Omega_{G^c}(H)\) to \(X\Omega(H, G^c)\) obtained by the functor \(F\) of (3.1) and we defined the Green functor structure on \(X\Omega(X, G^c)\) using Green functor structure on \(\Omega_{G^c}\), \(f_H\) is an isomorphism of rings for \(H\).

Finally we have to check that the morphism \(f_H\) preserves the identity element. Recall from Theorem 5.1 of [Bo03a] that the identity element \(\varepsilon_{\Omega_{G^c}}\) of the Green functor \(\Omega_{G^c}\) is the element \(\Omega_{G^c}(\bullet)\) of \(\Omega(G)\). The identity element \(\varepsilon\) of the Burnside Green functor \(\Omega\) is the identity element \(\bullet\) of the Burnside ring \(\Omega\), which is the identity element of the crossed Burnside ring \(\Omega(G^c)\) (see [OY01, 3.1] or [Bo03a, 2.1]). Moreover, the identity element \((\varepsilon_{\Omega_{G^c}})_{G/H}\) of the ring \(\Omega_{G^c}(G/H)\) is obtained by the formula (see [Bo97, 2.3]) \((\varepsilon_{\Omega_{G^c}})_{G/H} = \Omega_{G^c}(\pi_H^G)(\varepsilon_{\Omega_{G^c}})\), where \(\pi_H^G\) is the natural projection \(G/H \to G/G = \bullet\). The pull-back diagram

\[
\begin{array}{ccc}
G/H & \to & \bullet \\
\downarrow & & \downarrow \\
G/H \times G^c & \xrightarrow{\pi_H^G \times \text{id}_{G^c}} & \bullet \times G^c
\end{array}
\]

shows that \((\varepsilon_{\Omega_{G^c}})_{G/H}\) is the \(G\)-set \([H, 1_G]: G/H \to G/H \times G^c\) over \(G/H \times G^c\). By the definition of \(f_H\),

\[
f_H([H, 1_G]) = ([H, 1_G]^{-1}(aH, \alpha))_{(aH, \alpha) \in G/H \times G^c}.
\]

We have that \([H, 1_G]^{-1}(aH, \alpha)\) is the coset \(aH\) for \(\alpha = 1_G\) and the empty set \(\phi\) for all other elements, so \(f_H\) preserves the identity element, and this completes the proof of the lemma. \(\square\)

We will denote by \(\mathbb{C}[X]\) the \(\mathbb{C}\)-permutation module associated to a set \(X\). The endo-functor of the category of Green functors of Theorem 3.4 applied to the morphism \(\theta\) from \(\Omega\) to \(R\) leads to the following lemma.

3.6. Lemma. Let \(\theta: \Omega \to R\) be the natural morphism from the Burnside Green functor to the Grothendieck ring Green functor. Then the morphism \(\theta_{G^c}: \Omega_{G^c} \to R_{G^c}\) given by the Bouc’s construction is a morphism of Green functors.

3.7. Lemma. There is a morphism

\[
\theta_{G^c}: X\Omega(X, G^c) \to R_{G^c}(D_G(X))
\]
of Green functors.

**Proof.** Lemma 3.6 shows that there is a morphism \( \theta_{G^c} \) of Green functors from \( \Omega_{G^c} \) to \( R_{CG^c} \), whose evaluation at the transitive \( G \)-set \( G/H \) is the map

\[
(\theta_{G^c})_{G/H} = \theta_{G/H \times G^c} : \Omega_{G^c}(G/H) = \Omega(G/H \times G^c) \to R_{CG^c}(G/H) = R_{CG^c}(G/H)
\]
defined for the transitive \( G \)-set \( [K,s] : G/K \to G/H \times G^c \) over \( G/H \times G^c \) by

\[
\left( \bigcap_{aH \in G/H} [K,s]^{-1}(aH,\alpha) \right)_{\alpha \in G^c} \mapsto \left( \bigoplus_{aH \in G/H} \mathbb{C}[[K,s]^{-1}(aH,\alpha)] \right)_{\alpha \in G^c}.
\]

The maps \( f_H \) of the proof of Lemma 3.5 and \( \theta_{G/H \times G^c} \) induce maps

\[
\Omega(H,G^c) \xrightarrow{f_H} \Omega_{G^c}(G/H) \xrightarrow{\theta_{G/H \times G^c}} R_{CG^c}(G/H)
\]

for a subgroup \( H \) of \( G \). The compositions \( \theta_{G/H \times G^c} \circ f_H \) form a morphism of Green functors from \( X\Omega(\ast,G^c) \) to \( R_{CG}(\ast) \). The isomorphism (see Remark 2.3 of [Bo03a])

\[
R_{CG^c}(G/H) = \left( \bigoplus_{(aH,\alpha) \in G/H \times G^c} R_{CG}(G_{aH,\alpha}) \right)^G \to \bigoplus_{(H,\alpha) \in [G \setminus (G/H \times G^c)]} R_{CG}(G_H,\alpha) = \bigoplus_{(H,\alpha) \in [G \setminus (G/H \times G^c)]} R_{CG}(C_H(\alpha))
\]
defined by

\[
(U_{aH,\alpha})(aH,\alpha) \in G/H \times G^c \mapsto (U_{H,\alpha})(H,\alpha) \in [G \setminus (G/H \times G^c)]
\]

and Theorem 2.4 of [Wi96] induce an isomorphism of Green functors \( R_{CG^c} \cong R(D_G(\ast)) \), and complete the proof of the lemma. \( \square \)

Let \( (H/L)_g \) be an element of the basis of \( X\Omega(H,G^c) \). Then the previous lemma shows that \( \theta_{G^c}((H/L)_g) \) is an \( H \)-vector bundle on \( G^c \). We denote by \( [H/L]_g \) this \( H \)-vector bundle. Lemma 3.5 shows the following lemma.

**3.8. Lemma.** The \( H \)-vector bundle \( [H/L]_g \) is the \( CC_H(\ast g) \)-module \( \mathbb{C}[[^xL,^xg]^{-1}(^xg)] \) in the \( ^xg \)-component, for \( x \in [H/C_H(g)] \), and 0 in all other components.

We recall the maps \( \text{Incl}_{J,h} : R_{CG}(J) \to R_{CG}(D_G(J)) \), where \( J \) is a subgroup of \( G \) and \( h \in C_G(J) \), introduced in Section 2 of [Wi96]: Given a \( CJ \)-module \( V \), \( \text{Incl}_{J,h}(V) \) is the \( D_G(J) \)-module which is \( V \) in the \( h \)-component and 0 elsewhere.

**3.9. Lemma.** Let \( \theta_{G^c} \) be the ring homomorphism \( \theta_{(G/G) \times G^c} \) from the crossed Burnside ring \( X\Omega(G,G^c) \) to the Grothendieck ring \( R_{CG}(D_G(G)) \) given by the previous lemma. Then the \( D(G) \)-module corresponding to the \( G \)-vector bundle \( \theta_{G^c}((G/L)_g) \) is the induced module

\[
D(G) \otimes_{D_G(L)} \text{Incl}_{L,g}([L/L]).
\]
Proof. By the previous lemma, the ring homomorphism \( \theta_{Gc} \) assigns to a transitive crossed \( G \)-set \((G/L)_{g}\) the \( G \)-vector bundle \([G/L]_{g}\) which is \( \mathbb{C}[[xL, xg]^{-1}(xg)] \cong \mathbb{C}[C_{G}(xg)/xL] \) as \( \mathbb{C}G(xg) \)-modules in the \( xg \)-component for \( x \in [G/C_{G}(g)] \) and 0 in all other components. Section 2 of [Wi96] shows that the \( \mathbb{C}DG(G) \)-module corresponding to the \( G \)-vector bundle \([G/L]_{g}\) is the induced module \( D(G) \otimes_{DG(J)} \text{Incl}_{C_{G}(g),g}^{L} \). Since \( L \leq C_{G}(g) \) and \( g \in C_{G}(C_{G}(g)) \), we have

\[
\text{Incl}_{C_{G}(g),g}^{L} \circ \text{Ind}_{L,g}^{C_{G}(g)} = \text{Dind}_{L,g}^{C_{G}(g)} \circ \text{Incl}_{L,g}
\]

by Lemma 5.4 of [Wi96]. Finally, we see that

\[
D_{G}(G) \otimes_{DG(J)} \text{Incl}_{J,g}^{L}(C[L/L]) \cong D(G) \otimes_{DG(J)} \text{Incl}_{J,g}^{L}(C[L/L])
\]

\[
\cong D(G) \otimes_{DG(J)} \text{Dind}_{J,L}^{C_{G}(H)} \circ \text{Incl}_{L,g}^{C_{G}(g)}(C[L/L])
\]

\[
\cong D(G) \otimes_{DG(L)} \text{Incl}_{L,g}^{C_{G}(g)}(C[L/L]),
\]

where \( J = C_{G}(g) \). \( \square \)

3.10. Sub-Green functors

There is a sub-Green functor \( X\Omega(\ast, G^{c})_{1} \) which assigns to each subgroup \( H \) of \( G \) the subring \( X\Omega(H, G^{c})_{1} \) of \( X\Omega(H, G^{c})_{1} \) generated by the elements \((H/L)_{g}\). There is also a sub-Green functor \( R_{C}(DG(\ast))_{1} \) which assigns to each subgroup \( H \) of \( G \) the subring \( R_{C}(DG(H)) \) generated by \( \text{Incl}_{H,1g}(V) \)'s, where \( \text{Incl}_{H,1g} \) is a functor embedding the category of \( \mathbb{C}H \)-modules as a full subcategory of the category of \( DG(H) \)-modules (see [Wi96, Section 1]) and \( V \) is a \( \mathbb{C}H \)-module. It is easy to see that \( X\Omega(H, G^{c})_{1} \) is isomorphic to the Burnside ring \( \Omega(H) \) and \( R_{C}(DG(H)) \) is isomorphic to the ordinary character ring \( R_{C}(H) \). The homomorphism \( \theta_{Gc} \downarrow_{X\Omega(H, G^{c})_{1}} \) is the natural ring homomorphism from \( \Omega(H) \) to \( R_{C}(H) \).

3.11. Characters

Witherspoon pointed out the character of a \( \mathbb{C}D(G) \)-module in [Wi96], that appeared in [Lu87]. For \( g \in G \) and an irreducible character \( \rho \) of \( C_{G}(g) \), a character \( \chi_{g,\rho} \) of a \( \mathbb{C}D(G) \)-module \( U = \{U_{h}\}_{h \in G^{c}} \) is given by the formula

\[
\chi_{g,\rho}(U) = \frac{1}{\deg \rho} \sum_{h \in C_{G}(g)} \text{Tr}(g, U_{h})\rho(h).
\]

The characters of the crossed Burnside ring have been considered by Oda and Yoshida [OY01, Section 5]. For a subgroup \( H \) of \( G \) and an irreducible character \( \theta \) of \( C_{G}(H) \), the linear map \( \omega_{H,\theta} \) of \( X\Omega(G, G^{c}) \) to \( \mathbb{C} \) is the composite of Burnside homomorphism \( \varphi_{H} \) and a central character.
\(\tilde{\omega}_{H,\theta}\): given a crossed \(G\)-set \(X\) over \(G^c\), \(H \leq G\), and an irreducible character \(\rho\) of the group algebra \(\mathbb{C}C_G(H)\), \(\omega_{H,\rho}(X) = \tilde{\omega}_{H,\rho} \circ \phi_H(X)\).

For each \(h \in G^c\) the \(h\)-component of the crossed \(G\)-set \((X, \alpha)\) is defined by
\[
X[h] = \{x \in X | \alpha(x) = h\}.
\]

3.12. Lemma. Let \(g\) be an element of \(G\), \(\rho\) an irreducible character of \(\mathbb{C}CG(g)\), and \(\theta_{G^c}\) the homomorphism from \(X\Omega(G, G^c)\) to \(R_{\mathbb{C}}(D(G))\). Then \(\chi_{g,\rho}\theta_{G^c} = \omega_{\langle g \rangle, \rho}\), where \(\langle g \rangle\) is the cyclic subgroup generated by \(g\).

Proof. Let \((G/L)_s\) be a transitive crossed \(G\)-set over \(G^c\). Then 3.11 shows that
\[
\omega_{\langle g \rangle,\rho}\((G/L)_s\) = \sum_{xL \in (G/L)_{\langle g \rangle}} \rho(x)g \rho(1_G),
\]
\[
= \sum_{h \in C_G(\langle g \rangle)} \frac{|G/L[h]|}{\rho(1_G)} \cdot \rho(h),
\]
\[
= \frac{1}{\deg \rho} \sum_{h \in C_G(g)} \Tr(g, ([G/L]_s)_h) \rho(h),
\]
\[
= \chi_{g,\rho}([G/L]_s),
\]
\[
= \chi_{g,\rho}(\theta_{G^c}((G/L)_s)).
\]

4. Induction theorems

The proof of the following theorem is similar to the proof of Theorem 3.5.2 in [Bo00].

4.1. Theorem. Let \(G\) be a finite group. Then
\[
\mathbb{C}R_{\mathbb{C}}(D(G)) = \sum_{H \in \mathcal{C}(G)} \text{Dind}_H^G \mathbb{C}R_{\mathbb{C}}(D_G(H)),
\]
where \(\mathcal{C}(G)\) is the family of cyclic subgroups of \(G\). In other words, any complex character of \(D(G)\) is a linear combination with rational coefficients of characters induced from cyclic groups of \(G\).

Proof. It suffices to show that the left-hand side is contained in the right-hand side. There is a natural homomorphism \(\theta_{G^c}\) from the crossed Burnside ring \(X\Omega(G, G^c)\) to the Grothendieck ring \(R_{\mathbb{C}}(D(G))\) by Lemma 3.8. This extends to a homomorphism of \(\mathbb{C}\)-algebra from \(\mathbb{C}X\Omega(G, G^c)\) to \(\mathbb{C}R_{\mathbb{C}}(D(G))\).

Let \(H\) be a subgroup of \(G\) and \(s\) an element of \(G\). Let \(e_{H,\psi}\) be the primitive idempotent corresponding to \(\mathbb{C}\)-irreducible character \(\psi\) of \(C_G(H)\) (see Theorem 2.3). Since a character
ω(⟨s⟩,ρ) of CZΩ(G,Gc) is the composite of a central character ˜ω(⟨s⟩,ρ) and a Burnside homomorphism ϕ(⟨s⟩,ρ) (see 3.11). If H is not conjugate in G to the cyclic subgroup ⟨s⟩ generated by s. Lemma 3.12 implies that χs,ρ(θGc(eH,ψ)) = 0 if H is not conjugate in G to the cyclic subgroup ⟨s⟩ generated by s. Let

\[(G/G)_1\]

be the decomposition of the unit element of CZΩ(G,Gc). Then Lemma 3.8 shows that θGc((G/G)_1) is the unit of CRcD(G) and θGc((G/G)_1) is a linear combination with C-coefficients of such induced modules D(G) ⊗ DG(L) InclL,s(C[L/L]), which are induced from cyclic subgroups L of G. Then if U is any D(G)-module

\[U \otimes \text{Incl}_{L,s}(C[L/L]) \uparrow_{DG}(G) \cong (U \downarrow_{DG(L)} \otimes \text{Incl}_{L,s}(C[L/L])) \uparrow_{DG}(G)\]

by Lemma 5.2 of [Wi96]. Therefore, the theorem follows by the unitarity of θGc. 

The previous theorem and 3.10 show the following corollary.

4.2. Corollary (Artin). Let G be a finite group. Then

\[\mathbb{Q}R_C(G) = \sum_{H \in C(G)} \text{Ind}_H^G \mathbb{Q}R_C(H).\]

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References