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Aspiration level approach to solve matrix games with I-fuzzy goals and I-fuzzy pay-offs

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A B S T R A C T

The objective of this paper is to develop a new solution methodology for matrix games, in which goals are viewed as intuitionistic fuzzy sets (IFSs) and the elements of the pay-off matrix are represented by triangular intuitionistic fuzzy numbers (TIFNs). In this methodology, a suitable ranking function is defined to establish an order relation between two TIFNs, and the concept of intuitionistic fuzzy (I-fuzzy) inequalities is interpreted. Utilizing these inequality relations and ranking functions, a pair of linear programming models is derived from a pair of auxiliary intuitionistic fuzzy programming models. Based on the aspiration levels, this pair of linear programming models is solved to determine the optimal strategies for both players of the game. The proposed method in this paper is illustrated with a voting share problem to demonstrate the validity and applicability of the method.

1. Introduction

In traditional game theory, the precise assessment information is difficult due to a lack of information about the exact values of certain parameters and the fuzzy understanding of various situations by players. Fuzzy set theory, which is a very useful tool in game theory, has achieved a substantial amount of success. (Campos [9], Nishizaki and Sakawa [27,28], Bector and Chandra [8], Nayak and Pal [21,22], Li [14,16], Vidyottama and Chandra [37], Vijay et al. [38,39], Liu and Kao [18], Cevikel and Ahlatcioglu [10], Kacher and Larbani [12], Seikh et al. [32,33]). However, a fuzzy set only employs a membership degree. The degree of non-membership is automatically equal to the complement to 1. In real situations, however, players/decision makers often do not express the degree of non-membership as the complement to 1. Players/decision makers may exhibit some degree of hesitation. Therefore, a fuzzy set has no means of incorporating a degree of hesitation.

The intuitionistic fuzzy set (IFS), which is a generalization of fuzzy set theory, was introduced by Atanassov [4,5]; it is suitable for solving problems concerning vagueness. An IFS is characterized by two functions—a function that expresses the degree of membership and a function that expresses the degree of non-membership—thus, the sum of both values is less than or equal to 1. The degree of hesitation is equal to one minus the sum of the degree of membership and the degree of non-membership. Therefore, the concept of the IFS is considered to be an alternative approach for defining a fuzzy set in cases where available information is insufficient for defining an imprecise concept by conventional fuzzy sets. Therefore, an IFS can be employed to simulate the human decision-making process and any activity that requires human expertise and knowledge, which are inevitably imprecise or not completely reliable. Atanassov [6] discussed an open problem about the interpretation of an IFS in different optimization problems. Angelov [3] was the first researcher to answer this problem by implementing an optimization technique in an intuitionistic fuzzy environment. However, an IFS can be applied to game problems as the players have some degree of hesitation about appropriate pay-off values and the selection of a strategy for each of the pay-offs.

Intuitionistic fuzziness in matrix games is generally applied using two methods: in the first method, players have I-fuzzy goals; in the second method, the elements of the pay-off matrix

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are represented by intuitionistic fuzzy numbers (Li [13], Seikh et al. [30]). Recent studies have focused on the application of an IFS to resolve game problems. Atanassov [6] described a game problem using an IFS. Li and Nan [17] developed a nonlinear programming approach to matrix games with pay-offs of Atanassov's IFS. Aggarwal et al. [1,2] extended the duality results for two-person zero-sum matrix games with fuzzy goals and fuzzy pay-offs to an I-fuzzy scenario. Nan et al. [20] implemented a lexicographic method for matrix games in which pay-offs are represented by TIFNs. Li et al. [15] employed a bi-objective programming approach to solve a matrix game with pay-offs of TIFNs. Bandyopadhyay et al. [7] solved matrix games with intuitionistic fuzzy pay-offs using a score function. Nan and Li [19] outlined a linear programming approach to solve matrix games with I-fuzzy goals. Seikh et al. [31,34,36] investigated matrix games in which goals are represented as an IFS or elements of a pay-off matrix are represented by TIFNs. Nayak and Pal [23,24] constructed auxiliary linear programming models to solve bi-matrix games with goals expressed by an IFS. Seikh et al. [29,35] applied TIFNs to bi-matrix games. No study has investigated matrix games with I-fuzzy goals and I-fuzzy pay-offs. In this paper, a new methodology, in which goals are represented as an IFS and elements of a pay-off matrix are represented by TIFNs, is introduced to solve matrix games. The idea of double I-fuzzy inequalities, i.e., the I-fuzzy inequalities and the parameters that are represented by I-fuzzy numbers, is outlined. The concept of TIFNs and their arithmetic operations and cut sets are recalled. A new order relation is proposed to rank the two TIFNs. A pair of linear programming models is derived from a pair of auxiliary I-fuzzy programming models using these ranking order relations. These two models are solved by aspiration levels, and the optimal strategies for both players are obtained.

The paper is organized as follows: In Section 2, some definitions and preliminaries about an IFS and TIFNs are recalled and a ranking function is defined to establish an order relation between two TIFNs. Section 3 describes the application of an IFS in optimization and the concept of double I-fuzzy constraint conditions. The main problem about the matrix games with I-fuzzy goals and I-fuzzy pay-offs is formulated in Section 4. The solution procedure of these games is conceptualized by the degree of acceptance and the degree of rejection of the I-fuzzy aspiration levels for two players. The results are illustrated by considering a voting share problem in Section 5. Section 6 reflects the conclusions of this paper.

2. Definitions and preliminaries

2.1. Intuitionistic fuzzy sets

The intuitionistic fuzzy set, which was introduced by Atanassov [5], is characterized by two functions that express the degree of belongingness and the degree of non-belongingness.

Definition 1. Let $U=\{x_1,x_2,\cdots,x_n\}$ be a finite universal set. The Atanassov's intuitionistic fuzzy set (IFS) $\widehat{A}$ in a given universal set U is an object with the form

$$\widehat{A}=\left\{\left\langle x,\ \mu_{\widehat{A}}(x),\ \gamma_{\widehat{A}}(x)\right\rangle \middle| x\in U\right\} \tag{1}$$

where the functions $\mu_{\widehat{A}}:U\to[0,1]$ and $\gamma_{\widehat{A}}:U\to[0,1]$ define the degree of membership and the degree of non-membership, respectively, of an element $x\in U$ to the set $A\subseteq U$ such that they satisfy the following conditions:

$$0\le\mu_{\widehat{A}}(x)+\nu_{\widehat{A}}(x)\le 1,\quad \forall\ x\in U$$

which is known as an intuitionistic condition. The degree of acceptance $\mu_{\widehat{A}}(x)$ and the degree of non-acceptance $\nu_{\widehat{A}}(x)$ can be arbitrary.

Definition 2. Let $\widehat{A}$ and $\widehat{B}$ be two IFSs in the set U. Then, the intersection of $\widehat{A}$ and $\widehat{B}$ are defined as follows:

2.2. Triangular intuitionistic fuzzy number (TIFN)

In this section, the definitions are derived from Li [13].

Definition 3. (TIFN) The TIFN $\tilde{a}=\langle(a_l,a,a_r);w_a,u_a\rangle$ is a convex IFS on the set $\Re$ of real numbers with $a_l<a<a_r$, whose membership function and non-membership function are defined as

$$\mu_{\tilde{a}}(x)=\begin{cases} w_a\dfrac{x-a_l}{a-a_l}: if\ \ a_l\le x<a \\ w_a: if\ \ x=a \\ w_a\dfrac{a_r-x}{a_r-a}: if\ \ a<x\le a_r \\ 0: if\ \ x<a_l\ or\ x>a_r \end{cases} \tag{2}$$

and

$$\nu_{\tilde{a}}(x)=\begin{cases} \dfrac{a-x+(x-a_l)u_a}{a-a_l}: if\ \ a_l\le x<a \\ u_a: if\ \ x=a \\ \dfrac{x-a+(a_r-x)u_a}{a_r-a}: if\ \ a<x\le a_r \\ 1: if\ \ x<a_l\ or\ x>a_r \end{cases} \tag{3}$$

as depicted in Fig. 1.

The values w_a and u_a represent the maximum degree of membership and the minimum degree of non-membership, respectively, such that they satisfy the following conditions: $0\le w_a\le 1, 0\le u_a\le 1$ and $0\le w_a+u_a\le 1$. Let $\pi_{\tilde{a}}(x)=1-\mu_{\tilde{a}}(x)-\nu_{\tilde{a}}(x)$, which is referred to as the I-fuzzy index of an element x in the TIFNa. It is the degree of indeterminacy membership of the element x to the TIFN $\tilde{a}$.

If $\mu_{\tilde{a}}(x)+\nu_{\tilde{a}}(x)=1\ \forall\ x\in U$, then $\tilde{a}=\langle(a_l,a,a_r);w_a,u_a\rangle$ is reduced to $\tilde{a}=\langle(a_l,a,a_r);w_a,1-w_a\rangle$, which is a triangular fuzzy number (TFN). The definition of a TIFN is a generalization of the definition of the TFN, which was introduced by Dubois and Prade [11]. Two new parameters—w_a and u_a—are introduced to reflect the confidence level of a TIFN and the non-confidence level of a TIFN, respectively. Therefore, a TIFN may express more uncertainty compared with a TFN. The set of all TIFNs is denoted by $\tilde{F}(\Re)$.

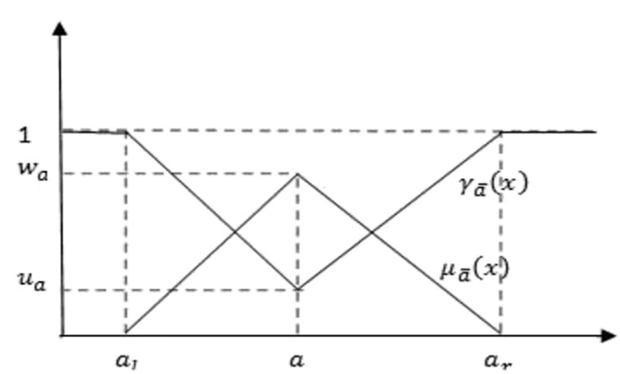

Fig. 1. Triangular intuitionistic fuzzy number.

Definition 4. (Arithmetic Operations) Let $\tilde{a}$ and $\tilde{b}$ be two TIFNs denoted by $\tilde{a} = \langle (a_l, a, a_r); w_a, u_a \rangle$ and $\tilde{b} = \langle (b_l, b, b_r); w_b, u_b \rangle$; the arithmetic operations are defined as follows:

Addition:

$$\tilde{a} + \tilde{b} = \langle (a_1 + b_1, a + b, a_r + b_r);\ \min\{w_a, w_b\},\ \max\{u_a, u_b\}\rangle.$$

Subtraction:

$$\tilde{a} - \tilde{b} = \langle (a_1 - b_r, a - b, a_r - b_1);\ \min\{w_a, w_b\},\ \max\{u_a, u_b\}\rangle.$$

Scalar Multiplication:

$$k\tilde{a} = \begin{cases} \langle (ka_l, ka, ka_r); w_a, u_a\rangle if \ \ k > 0 \\ \langle (ka_r, ka, ka_l); w_a, u_a\rangle if \ \ k < 0 \end{cases}, \text{where k is a real number.}$$

Definition 5. (Cut sets of TIFNs) For any $\alpha \in [0,w_a]$, the α-cut set of TIFN $\tilde{a} = \langle (a_l, a, a_r); w_a, u_a\rangle$ can be expressed as a crisp subset of $\Re$, which is denoted by $\tilde{a}^\alpha = \{x | \ \mu_{\tilde{a}}(x) \geq \alpha, x \in \Re\}$. According to the definition of the TIFN, $\tilde{a}^\alpha$ is a closed interval denoted by $\tilde{a}^\alpha = [L^\alpha(\tilde{a}), R^\alpha(\tilde{a})]$. From (2),

$$[L^\alpha(\tilde{a}), R^\alpha(\tilde{a})] = \left[a_l + \frac{\alpha(a - a_l)}{w_a}\ , a_r - \frac{\alpha(a_r - a)}{w_a}\right].$$

Similarly, for any $\beta \in [u_a,1]$, a β-cut set of the TIFN $\tilde{a} = \langle (a_l, a, a_r); w_a, u_a\rangle$ can be expressed as a crisp subset of $\Re$, as denoted by $\tilde{a}^\beta = \{x | \nu_{\tilde{a}}(x) \leq \beta, x \in \Re\}$. $\tilde{a}^\beta$ is a closed interval denoted by $\tilde{a}^\beta = [L^\beta(\tilde{a}), R^\beta(\tilde{a})]$. From (3),

$$\left[L^\beta(\tilde{a}), R^\beta(\tilde{a})\right] = \left[\frac{(1-\beta)a + (\beta - u_a)a_l}{1 - u_a}, \frac{(1-\beta)a + (\beta - u_a)a_r}{1 - u_a}\right].$$

In the next context, we discuss the value index and ambiguity index of a TIFN.

Definition 6. Let $\tilde{a}^\alpha = [L^\alpha(\tilde{a}), R^\alpha(\tilde{a})]$ and $\tilde{a}^\beta = [L^\beta(\tilde{a}), R^\beta(\tilde{a})]$ be any α-cut set and a β-cut set, respectively, of the TIFN $\tilde{a} = \langle (a_l, a, a_r); w_a, u_a\rangle$. Then, the values of the membership function and non-membership function are defined as.

$$D_\mu(\tilde{a}) = \int_0^{w_a} \frac{L^\alpha(\tilde{a}) + R^\alpha(\tilde{a})}{2} f(\alpha) d\alpha$$

and $D_\nu(\tilde{a}) = \int_{u_a}^1 \frac{L^\beta(\tilde{a}) + R^\beta(\tilde{a})}{2} g(\beta) d\beta$. where $f(\alpha)$ is a non-negative and non-decreasing function on the interval $[0, w_a]$, which should satisfy the conditions $f(0) = 0$ and $\int_0^{w_a} f(\alpha) d\alpha = w_a$. $g(\beta)$ is a non-negative and non-increasing function on the interval $[u_a,1]$, which should satisfy the conditions $g(1) = 0$ and $\int_{u_a}^1 g(\beta) d\beta = 1 - u_a$. The functions $f(\alpha)$ and $g(\beta)$ may be considered as weighting functions and are expressed as

$$f(\alpha) = \frac{2\alpha}{w_a}, \alpha \in [0, w_a] \tag{4}$$

And

$$g(\beta) = \frac{2(1-\beta)}{1 - u_a}, \beta \in [u_a, 1] \tag{5}$$

Thus, the values of membership function and non-membership function of the TIFN $\tilde{a} = \langle (a_l, a, a_r); w_a, u_a\rangle$ are calculated as follows:

$$D_\mu(\tilde{a}) = \int_0^{w_a} \left[a_l + \frac{\alpha(a - a_l)}{w_a} + a_r - \frac{\alpha(a_r - a)}{w_a}\right] \frac{\alpha}{w_a} d\alpha = \frac{(a_l + 4a + a_r)}{6} w_a \tag{6}$$

and

$$D_\nu(\tilde{a}) = \int_{u_a}^1 \left[\frac{(1-\beta)a + (\beta - u_a)_{al}}{1 - u_a} + \frac{(1-\beta)a + (\beta - u_a)_{ar}}{1 - ua}\right] \frac{1-\beta}{1 - u_a} d\beta = \frac{(a_l + 4a + a_r)}{6}(1 - u_a) \tag{7}$$

$D_\mu(\tilde{a})$ and $D_\nu(\tilde{a})$ reflect information about the membership degrees and non-membership degrees.

Similarly, the ambiguities of the membership function and non-membership function for any TIFN a are defined by

$$E_\mu(\tilde{a}) = \int_0^{w_a} [R^\alpha(\tilde{a}) - L^\alpha(\tilde{a})] f(\alpha) d\alpha$$

and $E_\nu(\tilde{a}) = \int_{u_a}^1 [R^\beta(\tilde{a}) - L^\beta(\tilde{a})] g(\beta) d\beta$, respectively. $R^\alpha(\tilde{a}) - L^\alpha(\tilde{a})$ and $R^\beta(\tilde{a}) - L^\beta(\tilde{a})$ represent the length of the interval $\tilde{a}^\alpha$ and the length of the interval $\tilde{a}^\beta$, respectively. Therefore, $E_\mu(\tilde{a})$ and $E_\nu(\tilde{a})$ represent the measure of uncertainty in $\tilde{a}$.

Using (4) and (5), the ambiguities of the membership function and the non-membership function of a TIFN $\tilde{a}$ are calculated as

$$E_\mu(\tilde{a}) = \left(\frac{a_{r-a_l}}{3}\right) w_a \text{ and } E_\nu(\tilde{a}) = \left(\frac{a_{r-a_l}}{3}\right)(1 - u_a)$$

Proposition 1. Let $\tilde{a} = \langle (a_l, a, a_r); w_a, u_a\rangle$ and $\tilde{b} = \langle (b_l, b, b_r); w_b, u_b\rangle$ be any two TIFNs with $w_a = w_b$, $u_a = u_b$ and let k be any non-negative real number. Then, the following equalities are valid.

(i) $D_\mu(k\tilde{a} + \tilde{b}) = kD_\mu(\tilde{a}) + D_\mu(\tilde{b})$
(ii) $D_\nu(k\tilde{a} + \tilde{b}) = kD_\nu(\tilde{a}) + D_\nu(\tilde{b})$
(iii) $E_\mu(k\tilde{a} + \tilde{b}) = kE_\mu(\tilde{a}) + E_\mu(\tilde{b})$
(iv) $E_\nu(k\tilde{a} + \tilde{b}) = kE_\nu(\tilde{a}) + E_\nu(\tilde{b})$.

Definition 7. (Value index and ambiguity index) The value index and ambiguity index of any TIFN $\tilde{a}$ are defined as follows:

value index $\quad V_\lambda(\tilde{a}) = \lambda D_\nu(\tilde{a}) + (1 - \lambda) D_\mu(\tilde{a})$

and

ambiguity index $\quad A_\lambda(\tilde{a}) = \lambda E_\mu(\tilde{a}) + (1 - \lambda) E_\nu(\tilde{a})$,

where $\lambda \in [0,1]$ represents the weight of the preference information about the players/decision makers (DMs). $\lambda \in [0, 1/2)$ indicates the pessimistic attitude of the players/DMs towards uncertainty; $\lambda \in (1/2,1]$ indicates the optimistic attitude of the players/DMs towards uncertainty, where $\lambda = 1/2$ shows that players/DMs have indifferent attitude. Thus, the value index and ambiguity index may reflect the attitudes of the players/DMs about the TIFNs.

Proposition 2. Let $\tilde{a} = \langle (a_l, a, a_r); w_a, u_a \rangle$ and $\tilde{b} = \langle (b_l, b, b_r); w_b, u_b \rangle$ be any two TIFNs with $w_a = w_b$ and $u_a = u_b$. Then, for any real number $k \in \Re$, the following equalities are valid:

(i) $V_\lambda(k\tilde{a} + \tilde{b}) = kV_\lambda(\tilde{a}) + V_\lambda(\tilde{b})$
(ii) $A_\lambda(k\tilde{a} + \tilde{b}) = kA_\lambda(\tilde{a}) + A_\lambda(\tilde{b})$.

In the following definitions, a ranking function is defined based on the difference between the value index and the ambiguity index to obtain an order relation between two TIFNs.

Definition 8. A ranking function (or defuzzification function) is the function F: $\tilde{F}(\Re) \to \Re$, where $\tilde{F}(\Re)$ is a set of all TIFNs that are defined on $\Re$, which maps each TIFN into a real line. Let $\tilde{a}$ be a TIFN. Then, $F_\lambda(\tilde{a}) = V_\lambda(\tilde{a}) - A_\lambda(\tilde{a})$, where $\lambda \in [0,1]$.

Assume that $\tilde{a}$ and $\tilde{b}$ be two TIFNs and $\lambda \in [0,1]$ be any real number. Then, a new order relation between $\tilde{a}$ and $\tilde{b}$ is defined as follows:

(i) $\tilde{a} \preccurlyeq^{I.F} \tilde{b}$ iff $F_\lambda(\tilde{a}) \le F_\lambda(\tilde{b})$
(ii) $\tilde{a} \succcurlyeq^{I.F} \tilde{b}$ iff $F_\lambda(\tilde{a}) \ge F_\lambda(\tilde{b})$
(iii) $\tilde{a} \cong^{I.F} \tilde{b}$ iff $F_\lambda(\tilde{a}) = F_\lambda(\tilde{b})$

The symbol '$\preccurlyeq^{I.F}$' is an intuitionistic fuzzy version of the order relation '$\le$' on the set of real numbers and has the linguistic interpretation as 'essentially less than or equal to'. Similarly, the symbols '$\succcurlyeq^{I.F}$' and '$\cong^{I.F}$' are the intuitionistic fuzzy versions of the order relations '$\ge$' and '$=$' on the set of real numbers and have the linguistic interpretations 'essentially greater than or equal to' and 'essentially equal to', respectively.

This proposed ranking method satisfies axioms, namely, the reasonable properties proposed by Wang and Kerre [40].

Proposition 3. Let $\tilde{a}$ and $\tilde{b}$ be any two TIFNs. Then, for any real number k, the following equality holds
$F_\lambda\left(k\tilde{a} + \tilde{b}\right) = kF_\lambda(\tilde{a}) + F_\lambda\left(\tilde{b}\right)$.

This shows that the proposed ranking function is linear for these types of TIFNs.

3. Application of IFS to optimization model

Any optimization model includes objective(s) and constraints. Intuitionistic fuzzy optimization (IFO), is an extension of fuzzy optimization, in which the degree of rejection of objective(s) and constraints are considered with the degree of satisfaction (refer to Angelov [3], Nayak and Pal [24]). Let U be the universal set. Let G_i, $i = 1,2, \ldots, r$, be the set of r goals and C_j, $j = 1,2, \ldots, m$, be the set of m constraints, each of which can be characterized by an IFS on U. The I-fuzzy decision $D = (G_1 \cap G_2 \cap \ldots \cap G_r) \cap (C_1 \cap C_2 \cap \ldots \cap C_m)$ is an IFS defined as $D = \{\langle x, \mu_D(x), \nu_D(x) | x \in U \rangle\}$, where $\mu_D(x) = \min_{i,j}\{\mu_{Gi}(x), \mu_{Cj}(x)\}$ and $\nu_D(x) = \max_{i,j}\{\nu_{Gi}(x), \nu_{Cj}(x)\}$

According to IFO theory, we can maximize the degree of acceptance of the IF objective(s) and the constraints to minimize the degree of rejection of the IF objective(s) and constraints. Let ξ, η denote the minimal degree of acceptance and the maximal degree of rejection, respectively, of the objective(s) and constraints. Then, the I-fuzzy optimization problem is transformed into the following crisp optimization model—Angelov [3]—as

$$\begin{aligned} & max\{\xi - \eta\} \\ \text{Subject to} \quad & \xi \le \mu_{Gi}(x); \quad i = 1,2,\ldots,r \\ & \eta \ge \nu_{Gi}(x); \quad i = 1,2,\ldots r \\ & \xi \le \mu_{Cj}(x); \quad j = 1,2,\ldots m \\ & \eta \ge \nu_{Cj}(x); \quad j = 1,2,\ldots m \\ & \xi \ge \eta; \xi + \eta \le 1; \quad \xi, \eta \ge 0 \end{aligned} \tag{8}$$

which can be easily solved by various mathematical programming methods.

In the following context, we provide an interpretation of inequality relations in an I-fuzzy environment.

3.1. Interpretation of I-fuzzy inequalities

Let $x, a \in \Re$ and let $p_0, q_0 (>0) \in \Re$. Then the intuitionistic fuzzy statement '$x \succcurlyeq_{p0,q0} a$' to be read as " x is essentially greater than or equal to a with tolerances p_0 and q_0" is expressed in terms of the following membership and non-membership functions (depicted in Fig 2):

$$\mu_1(x) = \begin{cases} 1; & x \ge a \\ 1 - \dfrac{(a-x)}{p_0}; & (a-p_0) < x < a \\ 0; & x \le (a-p_0) \end{cases}$$

and

$$\nu_1(x) = \begin{cases} 1; & x \le a \\ 1 + \dfrac{(a-p_0-x)}{q_0}; & (a-p_0) < x < (a-p_0+q_0) \\ 0; & x \ge (a-p_0+q_0) \end{cases}$$

The DM is completely satisfied if $x \ge a$ and the tolerances p_0, q_0 exist, in which the DM is partially satisfied, i.e., for $(a-p_0) < x < (a-p_0+q_0)$, the sum of the degrees of acceptance and the degree of rejection is less than or equal to one. He/she will not be satisfied if $x \le (a-p_0)$. According to the IFO technique described in the I-fuzzy inequality relation '$x \succcurlyeq_{p0,q0} a$' is expressed as

$$\xi_1 \le 1 - \frac{a-x}{p_0} \text{ and } \eta_1 \le 1 + \frac{a-p_0-x}{q_0}$$

$$\text{i.e.,} \quad x \ge a - p_0(1-\xi_1) \text{ and } x \ge (a-p_0) + q_0(1-\eta_1),$$

where ξ_1 and η_1 denote the minimal degree of acceptance and the maximal degree of rejection, respectively, that satisfy the conditions $0 \le \xi_1 \le 1$ and $0 \le \eta_1 \le 1$.

Similarly, the I-fuzzy statement '$x \preccurlyeq_{r0,\,s0} a$' to be read as "x is essentially less than or equal to a with tolerances r_0 and s_0" is expressed in terms of the following membership and non-membership functions:

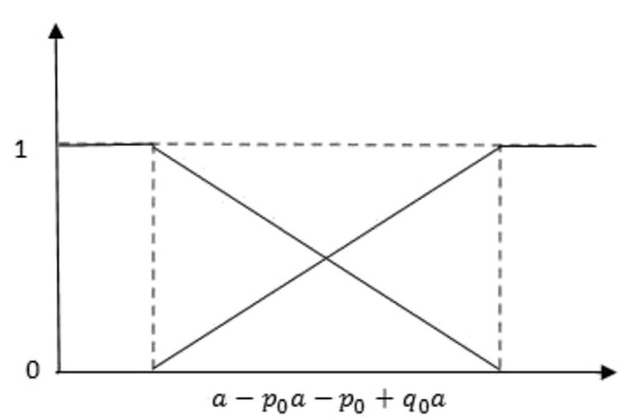

Fig. 2. Membership and non-membership functions for the statement $x \succcurlyeq_{p0,\,q0} a$.

$$\mu_2(x)=\begin{cases}1; & x\le a\\ 1+\dfrac{(a-x)}{r_0}; & a<x<(a+r_0)\\ 0; & x\ge (a+r_0)\end{cases}$$

and

$$\nu_2(x)=\begin{cases}1; & x\ge (a+r_0)\\ 1-\dfrac{(a+r_0-x)}{s_0}; & (a+r_0-s_0)<x<(a+r_0)\\ 0; & x\le (a+r_0-s_0)\end{cases}$$

The DM is completely satisfied if $x \le a$ and the tolerances r_0 and s_0 exist, in which the DM is partially satisfied, i.e., for $(a + r_0 - s_0) < x < (a + r_0)$, the sum of the degree of acceptance and the degree of rejection is less than or equal to one. He/she will not be satisfied if $x \ge (a + r_0)$. In a similar manner, the I-fuzzy inequality relations $x \preccurlyeq_{r0,\, s0} a$ is expressed as

$x \le a + r_0(1-\xi_2)$ and $x \le (a + r_0) - s_0(1-\eta_2)$

where $0 \le \xi_2 \le 1$ and $0 \le \eta_2 \le 1$.

3.2. Interpretation of double I-fuzzy constraints

In this subsection, the concept of double I-fuzzy inequalities, i.e., the I-fuzzy constraints that involve I-fuzzy numbers, is interpreted. Based on the resolution method that was proposed in the previous section, we extend the interpretation of I-fuzzy inequalities to the case in which the parameters and the adequacies are also I-fuzzy numbers. Let $\tilde{S}$,$\tilde{B}$ and $\tilde{C}$ be the m × n matrix, the m × 1 vector and the n × 1 vector with entries from $\tilde{F}(\Re)$ and the double I-fuzzy constraints under consideration be expressed as $\tilde{S}\ X \succcurlyeq^{I.F}_{\tilde{p},\tilde{q}} \tilde{B}$ and $\tilde{S}^T Y \preccurlyeq^{I.F}_{\tilde{p},\tilde{q}} \tilde{C}$, with the adequacies/tolerances $\tilde{p}$, $\tilde{q}$ and $\tilde{r}$, $\tilde{s}$, respectively, which are also I-fuzzy vectors. Therefore, the double I-fuzzy constraint conditions are expressed as

$$\tilde{S}X \succcurlyeq^{I.F}_{\tilde{p},\tilde{q}} \tilde{B} \Rightarrow \begin{cases}\tilde{S}_j X \succcurlyeq^{I.F} \tilde{B}_j - \tilde{p}_j(1-\xi_1), & 0\le\xi_1\le 1\\ \tilde{S}_j X \succcurlyeq^{I.F}\left(\tilde{B}_j - \tilde{p}_j\right) + \tilde{q}_j(1-\eta_1), & 0\le\eta_1\le 1\end{cases} \tag{9}$$

$$\tilde{S}^T Y \preccurlyeq^{I.F}_{\tilde{r},\tilde{s}} \tilde{C} \Rightarrow \begin{cases}\tilde{S}_i Y \preccurlyeq^{I.F} \tilde{C}_i + \tilde{r}_i(1-\xi_2), & 0\le\xi_2\le 1\\ \tilde{S}_i Y \preccurlyeq^{I.F}\left(\tilde{C}_i + \tilde{r}_i\right) - \tilde{s}_i(1-\eta_2), & 0\le\eta_2\le 1\end{cases} \tag{10}$$

where i = 1,2, …,m and j = 1,2, …, n. $\succcurlyeq^{I.F}$ and $\preccurlyeq^{I.F}$ are the relations between two I-fuzzy numbers, which preserve the ranking when I-fuzzy numbers are multiplied by positive scalars. $\tilde{p}_j$ and $\tilde{q}_j$ (j = 1, 2 …, n) represent the jth component of the I-fuzzy vectors $\tilde{p}$ and$\tilde{q}$, respectively. Similarly, $\tilde{r}_i$ and $\tilde{s}_i$ (i = 1,2, …,m) represent the ith component of I-fuzzy vectors $\tilde{r}$ and $\tilde{s}$, respectively.

4. Mathematical model of a matrix game

Let {1, 2, …, m} be the set of pure strategies that are available to player I and let {1,2, …,n} be the set of pure strategies that are available for player II. When player I chooses the pure strategy i and player II chooses the pure strategy j, then a_{ij} is a payoff for player I and $-a_{ij}$ is a payoff for player II. The two-person zero-sum matrix game G can be represented as a pay-off matrix, Owen [26]

$$A=\begin{matrix} & B_1 & B_2 & \cdots & B_n\\ A_1 & a_{11} & a_{12} & \cdots & a_{1n}\\ A_2 & a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \vdots & & \vdots\\ A_m & a_{m1} & a_{m2} & \cdots & a_{mn}\end{matrix}.$$

4.1. Mixed strategy

Consider the game G with no saddle point, i.e., $\max_i\{\min_j a_{ij}\} \ne \min_j\{\max_i a_{ij}\}$. To solve these games, Neumann [25] introduced the concept of mixed strategy in classical form. We denote the sets of all mixed strategies, which are referred to as strategy spaces, that are available for players I and II by

$$S_{\mathrm{I}}=\left\{\boldsymbol{x}=(x_1,x_2,\ldots,x_m)\in\Re^m_+ \middle| i=1,2,\ldots,m \text{ and } \sum_{i=1}^m x_i=1\right\},$$

$$S_{\mathrm{II}}=\left\{\boldsymbol{y}=(y_1,y_2,\ldots,y_n)\in\Re^n_+ \middle| j=1,2,\ldots,n \text{ and } \sum_{j=1}^n y_j=1\right\},$$

where $\Re^m_+$ denotes the m−dimensional non-negative Euclidean space. Thus, by a crisp two-person zero-sum matrix game G, we indicate the triplet $G = (S_{\mathrm{I}}, S_{\mathrm{II}}, A)$. As each player is uncertain about which strategy he/she will choose, he/she will choose a probability distribution over the set of alternatives that are available to him/her or a mixed strategy in terms of game theory.

4.2. Matrix games with I-fuzzy goals and I-fuzzy pay-offs

Let $S_{\mathrm{I}}, S_{\mathrm{II}}$ be the strategy spaces for player I and player II, respectively, and let $\tilde{A} = (\tilde{a}_{ij})_{m\times n}$ be the pay-off matrix, where each $\tilde{a}_{ij} = \langle(a_{ijl}, a_{ij}, a_{ijr}); w_{a_{ij}}, u_{a_{ij}}\rangle (i = 1,2,\ldots m; j = 1,2,\ldots n)$ is the TIFN defined in Section 2.2. Let $\tilde{v}$ and $\tilde{w}$ be two TIFNs that represent the aspiration levels of player I and player II, respectively. Then, a two-person zero-sum matrix game with I-fuzzy goals and I-fuzzy pay-offs, as defined by the triplet $(S_{\mathrm{I}}, S_{\mathrm{II}}, \tilde{A}; \tilde{v}, \succcurlyeq^{I.F}_{\tilde{p},\tilde{q}}; \tilde{w}, \preccurlyeq^{I.F}_{\tilde{r},\tilde{s}})$, where $\succcurlyeq^{I.F}_{\tilde{p},\tilde{q}}$ and $\preccurlyeq^{I.F}_{\tilde{r},\tilde{s}}$, are explained in Section 3.2 and $\tilde{p}$, $\tilde{q}$, and $\tilde{r}$, $\tilde{s}$ are the adequacies/tolerance levels for player I and player II, respectively. In the following definition, we refer to a two-person zero-sum matrix game with I-fuzzy goals and I-fuzzy pay-offs as an intuitionistic fuzzy matrix game denoted by $\widetilde{IFG}$.

The solution concept of $\widetilde{IFG}$ is defined as follows:

Definition 9. A point $(\hat{\boldsymbol{x}},\hat{\boldsymbol{y}}) \in S_{\mathrm{I}} \times S_{\mathrm{II}}$ is referred to as a solution of the IF matrix game $\widetilde{IFG}$ if

(i) $\hat{\boldsymbol{x}}^T \tilde{A}\boldsymbol{y} \succcurlyeq^{I.F}_{\tilde{p},\tilde{q}} \tilde{v}$, for all $\boldsymbol{y} \in S_{\mathrm{II}}$ and
(ii) $\boldsymbol{x}^T \tilde{A}\hat{\boldsymbol{y}} \preccurlyeq^{I.F}_{\tilde{r},\tilde{s}} \tilde{w}$, for all $\boldsymbol{x} \in S_{\mathrm{I}}$.

Using this definition and employing a resolution procedure for the double I-fuzzy constraints proposed in Section 3, we can construct the following pair of I-fuzzy linear programming problems for player I and player II, respectively.

$$\begin{array}{ll} & \max\{\xi_1-\eta_1\}\\ & \boldsymbol{x}^T\tilde{A}\boldsymbol{y}\succcurlyeq^{I.F}\{\tilde{v}-\tilde{p}(1-\xi_1)\} \quad \text{for all } y\in S_{\mathrm{II}},\\ \text{Subject to} & \boldsymbol{x}^T\tilde{A}\boldsymbol{y}\succcurlyeq^{I.F}\{\tilde{v}-\tilde{p}+\tilde{q}(1-\eta_1)\} \quad \text{for all } y\in S_{\mathrm{II}},\\ & \boldsymbol{x}\in S_I,\\ & \xi_1\ge\eta_1, \xi_1+\eta_1\le 1,\\ & \boldsymbol{x},\xi_1,\eta_1\ge 0\end{array} \tag{11}$$

and

$$\max\{\xi_2-\eta_2\}$$
$$\text{Subject to}\quad \begin{cases} \boldsymbol{x}^T\tilde{A}\boldsymbol{y}\preccurlyeq^{I.F}\{\tilde{w}+\tilde{r}(1-\xi_2)\} \quad \text{for all } x\in S_I,\\ \boldsymbol{x}^T\tilde{A}\boldsymbol{y}\preccurlyeq^{I.F}\{\tilde{w}+\tilde{r}-\tilde{s}(1-\eta_2)\} \quad \text{for all } x\in S_I,\\ \boldsymbol{y}\in S_{II},\\ \xi_2\geq\eta_2,\xi_2+\eta_2\leq 1,\\ \boldsymbol{y},\xi_2,\eta_2\geq 0\end{cases} \tag{12}$$

where $\succcurlyeq^{I.F}$ and $\preccurlyeq^{I.F}$ are the relations between two TIFNs. As S_I and S_{II} are convex polytopes, only the extreme points of the sets S_I and S_{II} are considered in the constraint conditions of equations (11) and (12). Therefore, these problems will be converted as

$$\max\{\xi_1-\eta_1\}$$
$$\text{Subject to}\quad \begin{cases} \boldsymbol{x}^T\tilde{A}_j\succcurlyeq^{I.F}\left\{\tilde{v}-\tilde{p}_j(1-\xi_1)\right\}(\mathrm{j}=1,2,\ldots,\mathrm{n}),\\ \boldsymbol{x}^T\tilde{A}_j\succcurlyeq^{I.F}\left\{\left(\tilde{v}-\tilde{p}_j\right)+\tilde{q}_j(1-\eta_1)\right\}(\mathrm{j}=1,2,\ldots,\mathrm{n}),\\ \sum_{i=1}^{m}x_i=1,\\ \xi_1\geq\eta_1,\xi_1+\eta_1\leq 1,\\ x,\xi_1,\eta_1\geq 0\end{cases} \tag{13}$$

and

$$\max\{\xi_2-\eta_2\}$$
$$\text{Subject to}\quad \begin{cases} \boldsymbol{x}^T\tilde{A}_i\boldsymbol{y}\preccurlyeq^{I.F}\{\tilde{w}+\tilde{r}_i(1-\xi_2)\}(\mathrm{i}=1,2,\ldots,\mathrm{m}),\\ \boldsymbol{x}^T\tilde{A}_i\boldsymbol{y}\preccurlyeq^{I.F}\{(\tilde{w}+\tilde{r}_i)-\tilde{s}_i(1-\eta_2)\}(\mathrm{i}=1,2,\ldots,\mathrm{m}),\\ \sum_{j=1}^{n}y_j=1\\ \xi_2\geq\eta_2,\xi_2+\eta_2\leq 1,\\ y,\xi_2,\eta 2\geq 0\end{cases} \tag{14}$$

these problems can be transformed into linear programming problems as

$$\max\{\xi_1-\eta_1\}$$
$$\text{Subject to}\quad \begin{cases} \mathrm{F}_\lambda\left(\tilde{A}\right)_{\mathrm{j}}x\geq F_\lambda(\tilde{v})-F_\lambda\left(\tilde{p}_j\right)(1-\xi_1)(\mathrm{j}=1,2,\ldots,\mathrm{n}),\\ \mathrm{F}_\lambda\left(\tilde{A}\right)_{\mathrm{j}}x\geq F_\lambda(\tilde{v})-F_\lambda\left(\tilde{p}_j\right)+F_\lambda\left(\tilde{q}_j\right)(1-\eta_1)(\mathrm{j}=1,2,\ldots,\mathrm{n}),\\ \sum_{i=1}^{m}x_i=1\\ \xi_1\geq\eta_1,\xi_1+\eta_1\leq 1,\\ x,\xi_1,\eta_1\geq 0\end{cases} \tag{15}$$

and

$$\max\{\xi_2-\eta_2\}$$
$$\text{Subject to}\quad \begin{cases} \mathrm{F}_\lambda\left(\tilde{A}\right)_{\mathrm{j}}y\leq F_\lambda(\tilde{w})+F_\lambda(\tilde{r}_j)(1-\xi_2)(\mathrm{i}=1,2,\ldots,\mathrm{m}),\\ \mathrm{F}_\lambda\left(\tilde{A}\right)_{\mathrm{j}}y\leq F_\lambda(\tilde{w})+F_\lambda(\tilde{r}_j)-F_\lambda\left(\tilde{S}_j\right)(1-\eta_2)(\mathrm{i}=1,2,\ldots,\mathrm{m}),\\ \sum_{j=1}^{n}y_i=1\\ \xi_2\geq\eta_2,\xi_2+\eta_2\leq 1,\\ y,\xi_2,\eta_2\geq 0\end{cases} \tag{16}$$

Here $F_\lambda(\tilde{A})_i$ ($F_\lambda(\tilde{A})_j$) denotes the ith row (jth column) of $F_\lambda(\tilde{A})$, where i = 1,2, …,m and j = 1,2, …,n.

The two linear programming problems (15) and (16) can be written as

$$\max\{\xi_1-\eta_1\}$$
$$\text{Subject to}\quad \begin{cases} \sum_{i=1}^{m}[F_\lambda(\tilde{a}_{ij})]x_i\geq F_\lambda(\tilde{v})-F_\lambda\left(\tilde{p}_j\right)(1-\xi_1)(\mathrm{j}=1,2,\ldots,\mathrm{n}),\\ \sum_{i=1}^{m}[F_\lambda(\tilde{a}_{ij})]x_i\geq F_\lambda(\tilde{v})-F_\lambda\left(\tilde{p}_j\right)+F_\lambda\left(\tilde{q}_j\right)(1-\eta_1)(\mathrm{j}=1,2,\ldots,\mathrm{n}),\\ \sum_{i=1}^{m}x_i=1\\ \xi_1\geq\eta_1,\xi_1+\eta_1\leq 1,\\ x,\xi_1,\eta_1\geq 0\end{cases} \tag{17}$$

and

$$\max\{\xi_2-\eta_2\}$$
$$\text{Subject to}\quad \begin{cases} \sum_{j=1}^{n}[F_\lambda(\tilde{a}_{ij})]y_j\leq F_\lambda(\tilde{w})+F_\lambda(\tilde{r}_i)(1-\xi_2)(\mathrm{i}=1,2,\ldots,\mathrm{m}),\\ \sum_{j=1}^{n}[F_\lambda(\tilde{a}_{ij})]y_j\leq F_\lambda(\tilde{w})+F_\lambda(\tilde{r}_i)-F_\lambda\left(\tilde{S}_i\right)(1-\eta_2)(\mathrm{i}=1,2,\ldots,\mathrm{m}),\\ \sum_{j=1}^{n}y_j=1\\ \xi_2\geq\eta_2,\xi_2+\eta_2\leq 1,\\ y,\xi_2,\eta_2\geq 0\end{cases} \tag{18}$$

where $\tilde{A}_i$ ($\tilde{A}_j$) denotes the ith row (jth column) of $\tilde{A}$ (i = 1,2, …,m; j = 1,2, …,n). By utilizing the ranking function F_λ: $\tilde{F}\rightarrow\Re(\lambda\in[0,1])$ (defined in Section 2.2, Definition 8), which preserves the ranking when I-fuzzy numbers are multiplied by non-negative scalars,

To solve the I-fuzzy matrix game $\widetilde{IFG}$, we have to solve the crisp linear programming problems (17) and (18) for player I and player II, respectively. If $(\widehat{\boldsymbol{x}},\widehat{\xi}_1,\widehat{\eta}_1)$ is an optimal solution of (17), then $\widehat{\boldsymbol{x}}$ is an optimal strategy for player I, $\widehat{\xi}_1$ is the minimal degree of acceptance and $\widehat{\eta}_1$ is the maximal degree of rejection to the aspiration level $\tilde{v}$ of player I. A similar interpretation can also be given to the optimal solution $(\widehat{\boldsymbol{y}}, \widehat{\xi}_2, \widehat{\eta}_2)$ of problem (18). The results discussed in this section are summarized in the following theorem.

Theorem 1. The I-fuzzy matrix game $\widetilde{IFG}$ described by $\widetilde{IFG} = (S_I, S_{II}, \tilde{A}; \tilde{v}, \succcurlyeq^{I.F}_{\tilde{p},\tilde{q}}; \tilde{w}, \preccurlyeq^{I.F}_{\tilde{r},\tilde{s}})$ is equivalent to the two crisp linear programming problems (17) and (18), which can be easily solved by the simplex method.

5. Application to voting share problem

In this section, a voting share problem in a particular region is considered to demonstrate the application of the proposed methodology.

Assume that the two major political parties P1 and P2 participate in a parliamentary constituency election in a particular region. Let the secretaries of both the parties P1 and P2 aim to enhance their expected votes under the circumstance that the total number of voters in that region is fixed. The voting share percentage of one party increases while the voting share percentage of another party decreases. Assume that the secretaries of both parties are rational, i.e., they will choose optimal strategies to maximize their expected votes without co-operation. Assume that P1 has two strategies to increase their voting share: strategy ε_1 (campaigning by media), and ε_2 (campaigning by hosting rallies). Let P2 possess the same strategies as P1. This problem may be regarded as a matrix game, namely, the parties P1 and P2 are regarded as player I and player II, respectively. They may use strategies ε_1 and ε_2. Due to a lack of information or imprecise available information, the secretaries of the two parties are usually not able to precisely forecast the voting percentage. They estimate the voting percentage with a certain degree of confidence but they may hesitate regarding the estimation of expected votes. To address the uncertainty, TIFNs are employed to express the voting share for a particular political party. The pay-off matrix $\tilde{A}$ for both the parties is expressed as

$$\tilde{A} = \begin{matrix} \textit{campaigning by media} \\ \textit{hosting rallies} \end{matrix} \begin{pmatrix} \overset{\textit{campaigningbymedia}}{\langle(175,180,190);0.6,0.2\rangle} & \overset{\textit{hosting rallies}}{\langle(150,156,158);\ 0.6,0.2\rangle} \\ \langle(80,90,100);0.6,0.2\rangle & \langle(175,180,190);\ 0.6,0.2\rangle \end{pmatrix}$$

where $\langle(175,\ 180,\ 190);\ 0.6,\ 0.2\rangle$ in the matrix $\tilde{A}$ is a TIFN, which indicates that the expected votes in favour of P_1 is "approximately 180" when parties P_1 and P_2 simultaneously apply the strategy ε_1 (campaigning by media). The maximum degree of confidence of the secretary is 0.6, whereas the minimum degree of non-confidence is 0.2. His/her degree of hesitation is 0.2. Other elements in the matrix $\tilde{A}$ are similarly explained.

We assume that the aspiration levels for player I and player II are

$$\tilde{v} = \langle(160,170,180);0.6,0.2\rangle \text{ and } \tilde{w} = \langle(145,160,165);0.6,0.2\rangle.$$

The tolerance levels for player I and player II are

$$\tilde{p}_1 = \tilde{p}_2 = \tilde{p} = \langle(14,\ 16,\ 20);\ 0.6,\ 0.2\rangle,\ \tilde{q}_1 = \tilde{q}_2 = \tilde{q} = \langle(10,12,16);0.6,0.2\rangle \text{ and}$$
$$\tilde{r}_1 = \tilde{r}_2 = \tilde{r}\langle(13,\ 15,\ 22);\ 0.6,\ 0.2\rangle, \text{ also } \tilde{s}_1 = \tilde{s}_2 = \tilde{s} = \langle(9,\ 11,\ 18);\ 0.6,\ 0.2\rangle.$$

The value indexes and ambiguity indexes of $\tilde{a}_{ij} = \langle(a_{ijl},\ a_{ij},\ a_{ijr}); w_{a_{ij}},\ u_{a_{ij}}\rangle\ (i = 1,2; j = 1,2)$ can be obtained as follows:

$$V_\lambda(a_{ij}) = \lambda D_\nu(\tilde{a}_{ij}) + (1-\lambda)D_\mu(\tilde{a}_{ij}) = \frac{a_{ijl} + 4a_{ij} + a_{ijr}}{6}\left[w_{a_{ij}} + \lambda(1 - w_{a_{ij}} - u_{a_{ij}})\right]$$

and

$$A_\lambda(\tilde{a}_{ij}) = \lambda E_\mu(\tilde{a}_{ij}) + (1-\lambda)E_\mu(\tilde{a}_{ij}) = \frac{a_{ijr} - a_{ijl}}{3}\left[(1 - u_{a_{ij}}) - \lambda(1 - w_{a_{ij}} - u_{a_{ij}})\right]$$

Additionally,

$$F_\lambda(\tilde{a}_{ij}) = V_\lambda(\tilde{a}_{ij}) - A_\lambda(\tilde{a}_{ij})$$

Therefore,

$V_\lambda(a_{11}) = 108.5 + 36.16\lambda$	$A_\lambda(a_{11}) = 4 - \lambda$	$F_\lambda(a_{11}) = 104.5 + 37.16\lambda$
$V_\lambda(a_{12}) = 93.2 + 31.06\lambda$	$A_\lambda(a_{12}) = 2.13 - 0.53\lambda$	$F_\lambda(a_{12}) = 91.07 + 31.59\lambda$
$V_\lambda(a_{21}) = 54 + 18\lambda$	$A_\lambda(a_{21}) = 5.33 - 1.33\lambda$	$F_\lambda(a_{21}) = 48.67 + 19.33\lambda$
$V_\lambda(a_{22}) = 108.5 + 36.16\lambda$	$A_\lambda(a_{22}) = 4 - \lambda$	$F_\lambda(a_{22}) = 104.5 + 37.16\lambda$

and similarly,

$V_\lambda(\tilde{v}) = 102 + 34\lambda$	$A_\lambda(\tilde{v}) = 5.33 - 1.33\lambda$	$F_\lambda(\tilde{v}) = 96.67 + 35.33\lambda$
$V_\lambda(\tilde{w}) = 95 + 31.67\lambda$	$A_\lambda(\tilde{w}) = 5.33 - 1.33\lambda$	$F_\lambda(\tilde{w}) = 89.67 + 33\lambda$
$V_\lambda(\tilde{p}) = 9.8 + 3.3\lambda$	$A_\lambda(\tilde{p}) = 1.6 - 0.4\lambda$	$F_\lambda(\tilde{p}) = 8.2 + 3.7\lambda$
$V_\lambda(\tilde{q}) = 7.4 + 2.5\lambda$	$A_\lambda(\tilde{q}) = 1.6 - 0.4\lambda$	$F_\lambda(\tilde{q}) = 5.8 + 2.9\lambda$
$V_\lambda(\tilde{r}) = 9.5 + 3.2\lambda$	$A_\lambda(\tilde{r}) = 2.4 - 0.6\lambda$	$F_\lambda(\tilde{r}) = 7.1 + 3.7\lambda$
$V_\lambda(\tilde{s}) = 7.1 + 2.36\lambda$	$A_\lambda(\tilde{s}) = 2.4 - 0.6\lambda$	$F_\lambda(\tilde{s}) = 4.7 + 2.96\lambda$

According to equations (17) and (18), the linear programming models for players I and II are obtained as follows:

$$\begin{aligned}&max\{\xi_1-\eta_1\}\\ \text{Subject to}\quad &(104.5+37.16\lambda)x_1+(48.67+19.33\lambda)x_2\ge(96.67+35.33\lambda)-(8.2+3.7\lambda)(1-\xi_1)\\ &(91.07+31.59\lambda)x_1+(104.5+37.16\lambda)x_2\ge(96.67+35.33\lambda)-(8.2+3.7\lambda)(1-\xi_1)\\ &(104.5+37.16\lambda)x_1+(48.67+19.33\lambda)x_2\ge(96.67+35.33\lambda)-(8.2+3.7\lambda)+(5.8+2.9\lambda)(1-\eta_1)\\ &(91.07+31.59\lambda)x_1+(104.5+37.16\lambda)x_2\ge(96.67+35.33\lambda)-(8.2+3.7\lambda)+(5.8+2.9\lambda)(1-\eta_1)\\ &x_1+x_2=1\\ &\xi_1\ge\eta_1\\ &\xi_1+\eta_1\le1\\ &x_1,x_2,\xi_1,\eta_1\ge0\end{aligned}\qquad(19)$$

and

$$\begin{aligned}&max\ \{\xi_2-\eta_2\}\\ \text{Subject to}\quad &(104.5+37.16\lambda)y_1+(91.07+31.59\lambda)y_2\le(89.67+33\lambda)+(7.1+3.76\lambda)(1-\xi_2)\\ &(48.67+19.33\lambda)y_1+(104.5+37.16\lambda)y_2\le(89.67+33\lambda)+(7.1+3.76\lambda)(1-\xi_2)\\ &(104.5+37.16\lambda)y_1+(91.07+31.59\lambda)y_2\le(89.67+33\lambda)+(7.1+3.76\lambda)-(4.7+2.96\lambda)(1-\eta_2)\\ &(48.67+19.33\lambda)y_1+(104.5+37.16\lambda)y_2\le(89.67+33\lambda)+(7.1+3.76\lambda)-(4.7+2.96\lambda)(1-\eta_2)\\ &y_1+y_2=1\\ &\xi_2\ge\eta_2\\ &\xi_2+\eta_2\le1\\ &y_1,y_2,\xi_2,\eta_2\ge0\end{aligned}\qquad(20)$$

Solving (19) and (20) with the help of LINGO software, we obtained the optimal strategies for players I and II for different $\lambda\in[0,1]$, as depicted in Table 1.

As shown in Table 1, for different $\lambda \in [0, 1]$, different optimal strategies can be obtained for the players. For $\lambda = 1/2$, the optimal solution for player I is

$$\tilde{x}=(0.7997,0.2003)^{\mathrm{T}}\text{ and }\tilde{\xi}_1=0.5799,\tilde{\eta}_1=0.1962,\text{ and the optimal solution for player II is }\tilde{y}=(0.2003,0.7997)^{\mathrm{T}}\text{ and }\tilde{\xi}_2=0.5609,\tilde{\eta}_2=0.1835.$$

Therefore, the optimal strategy for player I is $(0.7997, 0.2003)^{\mathrm{T}}$, and the aspiration level $\tilde{v}$ is accepted with a minimal degree 0.5799 and rejected with a maximal degree 0.1962. Note that the sum of two optimal degrees is not 1. A degree of hesitation of $1-\hat{\xi}_1-\hat{\eta}_1=0.2239$ is obtained. A similar interpretation can be given for player II.

6. Conclusions

In this paper, matrix games with I-fuzzy goals and the pay-offs represented by TIFNs are explored. A solution methodology is proposed to solve these games. First, the concept of double I-fuzzy inequality constraints is interpreted with regard to aspiration levels and adequacies. Second, a defuzzification function (ranking function) is defined based on the value index and ambiguity index of TIFNs. A pair of I-fuzzy programming models is established for two players. Based on the resolution method of double I-fuzzy constraints and by applying the ranking functions, these two models are transformed into two crisp linear programming models that can be solved by a simplex method to obtain the optimal strategies. The numerical example indicates that the optimal strategies for both players are dependent on their subjective preference information.

The main advantage of this method is that it provides not only the degree of acceptance but also the degree of rejection of the

Table 1
Optimal strategies for players I and II for different values of λ.

Λ	$\hat{x}_1$	$\hat{x}_2$	$\hat{\xi}_1$	$\hat{\eta}_1$	$\hat{y}_1$	$\hat{y}_2$	$\hat{\xi}_2$	$\hat{\eta}_2$
0.0	0.8061	0.1939	0.6346	0.1027	0.1939	0.8061	0.4360	0.3399
0.1	0.8047	0.1953	0.6217	0.1250	0.1953	0.8047	0.4661	0.3011
0.2	0.8033	0.1967	0.6099	0.1453	0.1967	0.8033	0.4932	0.2667
0.3	0.8020	0.1980	0.5990	0.1637	0.1980	0.8020	0.5179	0.2360
0.4	0.8008	0.1992	0.5890	0.1806	0.1992	0.8008	0.5403	0.2084
0.5	**0.7997**	**0.2003**	**0.5799**	**0.1962**	**0.2003**	**0.7997**	**0.5609**	**0.1835**
0.6	0.7987	0.2013	0.5713	0.2104	0.2013	0.7987	0.5798	0.1609
0.7	0.7977	0.2023	0.5633	0.2237	0.2023	0.7977	0.5972	0.1404
0.8	0.7967	0.2033	0.5559	0.2359	0.2033	0.7967	0.6133	0.8000
0.9	0.7958	0.2042	0.5490	0.2473	0.2042	0.7958	0.6282	0.1042
1.0	0.7949	0.2051	0.5425	0.2579	0.2051	0.7949	0.6421	0.0883

aspiration levels for both players. The proposed methodology in this paper differs from the proposed methodology of Nan et al. [19,20], Seikh et al. [31,34,36], Aggarwal et al. [1], Bandyopadhyay et al. [7] and Li et al. [15], as they employed an IFS to express goals or only considered pay-offs as I-fuzzy numbers. However, neither of these studies include I-fuzzy inequalities in their models with the pay-offs that are represented by I-fuzzy numbers. In our methodology, the inequalities are I-fuzzy, whereas the pay-offs are also described by I-fuzzy numbers.

The major limitation of this proposed methodology is that it is dependent on the ranking function. Different types of ranking functions yield different types of solution. A more general methodology will be investigated in the future.

Although the proposed method is illustrated with the voting share problem, it can be applied in decision-making theory in areas such as economics, operations research, management, and war science.

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