**Stability and McMillan Degree for Rational Matrix Interpolants**

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**ABSTRACT**

Given a set of tangential interpolation conditions for a rational matrix function, one can ask to find an interpolant which (1) meets a stability or norm side constraint (as in Nevanlinna-Pick interpolation) or (2) has the minimal possible McMillan degree among all interpolants with no stability side constraint (as in the partial realization problem). In this paper a connection between the problems is presented. Specifically, it is shown that a minimal-degree solution of a given set of interpolation conditions together with an associated mirror-image set of interpolation conditions is automatically stable whenever it is unique. This result extends recent results of Antoulas and Anderson for the scalar version of the problem.

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**1. INTRODUCTION**

Let \(\{z_1, z_2, \ldots, z_{n+1}\}\) be an \((n+1)\)-tuple of distinct points in the right half plane (RHP), and \(\{\omega_1, \omega_2, \ldots, \omega_{n+1}\}\) be an \((n+1)\)-tuple of complex numbers.
numbers. A version of the well-known Nevanlinna-Pick interpolation problem asks: Find a rational function $f$ such that

$$f(z_i) = \omega_i \quad \text{for} \quad i = 1, 2, \ldots, n + 1 \quad (1.1)$$

and

$$|f(z)| \leq \gamma \quad \text{for} \quad z \in \text{RHP}, \quad (1.2)$$

where $\gamma > 0$ is some preassigned tolerance level. The contribution of Pick is that solutions exist if and only if the so-called Pick matrix

$$\Lambda = \left[ \frac{\gamma^2 - \omega_i \overline{\omega_j}}{z_i + \overline{z_j}} \right]_{1 \leq i, j \leq n + 1} \quad (1.3)$$

is positive semidefinite. Nevanlinna (see [18]), on the other hand, derived a recursive algorithm which leads to a linear fractional parametrization for the set of all solutions (when there exist more than one solution). In particular, substitution of 0 into Nevanlinna's linear fractional map produces a solution $f_{NP}$ of (1.1) and (1.2) such that

$$\delta(f_{NP}) \leq n, \quad (1.4)$$

where $\delta(f)$ is the McMillan degree of the rational function $f$ (maximum polynomial degree of the numerator and denominator when $f$ is expressed as the ratio of coprime polynomials). Extensions of all these results to the case where the interpolation conditions involve also derivatives of $f$ up to some order at each interpolation node—i.e., the situation where (1.1) is replaced by

$$f^{[k]}(z_i) = \omega_{i,k}, \quad i = 1, \ldots, n + 1, \quad 0 \leq k \leq m_i - 1 \quad (1.1')$$

—are well known. An alternative proof of the existence of solutions of (1.1) and (1.2) of McMillan degree at most $n$ using topological degree theory appears in [11].

A different sort of interpolation problem arising in systems theory is the following. Let $\{z_1, \ldots, z_N\}$ be an $N$-tuple of distinct complex numbers, and let $\{\omega_1, \ldots, \omega_N\}$ be any $N$-tuple of complex numbers. The problem is to find
a rational function which is analytic at the points \(z_1, \ldots, z_N\), satisfies the interpolation conditions

\[
f(z_i) = \omega_i, \quad i = 1, \ldots, N,
\]  

(1.5)

at these points, and has a minimal "degree of complexity." One measure of complexity is the McMillan degree defined above; in this form the problem has been studied at length in [2], where the Loewner matrix

\[
L = \begin{bmatrix}
\frac{\omega_{n+1+i} - \omega_j}{z_{n+1+i} - z_j} \\
\end{bmatrix}_{1 \leq i < n, 1 \leq j < n+1}
\]  

(1.6)

(where we assume that \(N = 2n + 1\) is odd) plays a key role. The classical partial realization problem of linear systems theory can be considered the special case of this problem where a single high-order interpolation condition is given at infinity. An alternative formulation is to take the degree of complexity to be the sum of the numerator and denominator degrees of the interpolating function; this version is studied in [1], where the Euclidean algorithm is the main tool. In this paper we work exclusively with the measure of complexity taken to be the McMillan degree.

Now assume that the interpolation data \(\{z_1, \ldots, z_n, z_{n+1}, \ldots, z_N\}\) and \(\{\omega_1, \ldots, \omega_n, \omega_{n+1}, \ldots, \omega_N\}\) have the symmetry property

\[
\gamma_i = \gamma, \quad i = 1, \ldots, n
\]  

(1.7a)

and

\[
\omega_i = -\omega_{n+1+i}, \quad i = 1, \ldots, n
\]  

(1.7b)

where \(z_i \in \text{RHP}\) and \(\omega_i\) is nonzero for \(1 \leq i \leq n + 1\). Then, after a diagonal scaling, we see that the first \(N\) rows of the Pick matrix \(\Lambda\) in (1.3) coincide with the Loewner matrix \(L\) in (1.6). This suggests a connection between the Nevanlinna-Pick problem and the minimal-McMillan-degree interpolation problem with symmetric data. The problem with symmetric data is: For \(\{z_1, \ldots, z_{n+1}\}\) a collection of distinct points in RHP and \(\{\omega_1, \ldots, \omega_{n+1}\}\) an \(n\)-tuple of nonzero complex numbers, find a rational function \(f(z)\) of minimal possible McMillan degree which is analytic at \(z_1, \ldots, z_{n+1}, -\bar{z}_1, \ldots, -\bar{z}_n\) and satisfies

\[
f(z_i) = \omega_i, \quad i = 1, \ldots, n + 1,
\]  

(1.7a)

\[
f(-\bar{z}_i) = \frac{\gamma^2}{\omega_i}, \quad i = 1, \ldots, n.
\]  

(1.7b)
In [3] it is asserted (in the equivalent context where the right half plane is replaced by the unit disk) that, if the Pick matrix $\Lambda$ is positive definite, then a minimal-degree solution of (1.7a) and (1.7b) automatically is analytic with modulus at most $\gamma$ on RHP and coincides with the Nevanlinna solution $f_{NP}$. In fact there is a gap in the proof and this assertion is not true in general, as the following example shows.

**Example 1.1.** $n = 1$, $z_1 = \frac{1}{2}$, $z_2 = 1$, $\omega_1 = \frac{1}{2}$, $\omega_2 = \frac{1}{2}$, $\gamma = 1$. Thus the Nevanlinna-Pick interpolation conditions are

$$f\left(\frac{1}{2}\right) = \frac{1}{2}, \quad f(1) = \frac{1}{2}, \tag{1.8}$$

while the minimal-degree interpolation problem with symmetric data involves the extra interpolation condition

$$f\left(-\frac{1}{2}\right) = 2. \tag{1.9}$$

For this case one easily sees that $f_{NP}(z)$ is the constant function $f_{NP}(z) = \frac{1}{2}$, which does not meet the extra interpolation condition (1.9). From the results in [2] (or by direct computation) one can deduce that the minimal-degree solutions of (1.8) and (1.9) are given by

$$f_{\text{min}}(z) = \frac{(2z^2 - 3z + 2)p + (2z + 1)(qz + r)}{2(2z + 1)(qr + r) + 2p}.$$ 

where $p, q, r$ are arbitrary real numbers satisfying $p \neq 0$, $p + q + 2r \neq 0$, and $p + 3q + 3r \neq 0$. If we choose $q = 0$, $r = 0$, $p = 1$, then

$$f_{\text{min}}(z) = \frac{1}{2}(2z^2 - 3z + 2),$$

which is not bounded in modulus on RHP. [But for some choices of $p, q, r$, $f_{\text{min}}$ also satisfies (1.2) with $\gamma = 1$.]

In this paper we show that an amended version of the result from [3] is true. More precisely, we show: If $\Lambda$ is positive definite and the minimal-degree solution of (1.7a) and (1.7b) is unique, then it is automatically stable (analytic with modulus at most $\gamma$ on RHP) and coincides with the Nevanlinna solution $f_{NP}(z)$ of the Nevanlinna-Pick problem (1.1)–(1.2).

Actually we show that these results hold more generally in the context of tangential interpolation conditions for rational matrix functions. There are
now several formalisms for handling generalized Nevanlinna-Pick problems for rational matrix functions involving tangential interpolation conditions; we mention [4, 7, 10, 13-16]. The paper [5] handles the minimal-McMillan-degree interpolation problem for the matrix case. Using the results from [7] and [5], we obtain here an extension of the result discussed above to the matrix case.

The paper is organized as follows. Section 2 provides a review of some of the results on matricial Nevanlinna-Pick interpolation from [7] and constructs a solution with small McMillan degree which is a counterpart to $f_{np}$ for the matrix case. Section 3 reviews some results from [5], namely, the construction of a minimal-degree solution of a set of two-sided tangential interpolation conditions, and the condition for a minimal-degree interpolant to be unique. This analysis is then specialized to an interpolation data set having a symmetric form; it is then shown that a unique minimal-degree interpolant is automatically also a solution of a related Nevanlinna-Pick problem.

2. NEVANLINNA-PICK INTERPOLATION

In this section we review some of the basic results on Nevanlinna-Pick interpolation from [7] which we shall need here.

The data set for a left interpolation problem we take to be a triple of matrices $\omega = (A_+, B_+, B_-)$ of sizes $n_x \times n_y$, $n_y \times M$, and $n_y \times N$ respectively such that $(A_+, B_+)$ is a controllable pair. We associate with the data set the residue interpolation conditions for the residue interpolation problem

$$\sum_{z_0 \in \sigma(A_+)} \text{Res}_{z=z_0} \left( (zI - A_+)^{-1} B_+ W(z) \right) = -B_-,$$

where $W(z)$ is a rational $M \times N$ matrix function analytic on $\sigma(A_+)$ to be determined. We mention, as a simple example, the case

$$A_+ = \begin{bmatrix} z_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & z_{n} \end{bmatrix}, \quad B_+ = \begin{bmatrix} x_1 \\ \vdots \\ x_{n_f} \end{bmatrix}, \quad B_- = -\begin{bmatrix} y_1 \\ \vdots \\ y_{n_f} \end{bmatrix},$$

where $z_1, \ldots, z_{n_f}$ are distinct numbers in $C$, $x_1, \ldots, x_{n_f}$ are nonzero $1 \times M$ row vectors, and $y_1, \ldots, y_{n_f}$ are $1 \times N$ row vectors. For this special case (RIP) collapses to the collection of simple tangential interpolation conditions

$$x_i W(z_i) = y_i, \quad i = 1, \ldots, n_f.$$  \tag{2.1}
By considering more involved Jordan forms for the matrix $A_z$, one can get more complicated tangential interpolation conditions involving derivatives of $W(z)$ at $z_i$. For details we refer to [7]. The Nevanlinna-Pick-Hermite-Fejer problem associated with the general data set $\omega = (A_z, B_+, B_-)$, where we now assume that the spectrum $\sigma(A_z)$ of $A_z$ is in RHP, is to find a rational matrix function $W(z)$ which, in addition to (RIP), satisfies

\[(\text{stab-}\gamma)\ W(z)\text{ is analytic on the RHP with}
\]
\[\|W(z)\| \leq \gamma \quad \text{for} \quad z \in \text{RHP},
\]
where $\gamma$ is some prespecified number. For the scalar case, Pick found a matrix test to determine for which $\gamma$ solutions exist, and Nevanlinna set down a recursive algorithm which leads to a linear fractional parametrization for the set of all solutions. We present here a simple adaptation of the one-sided case of Theorem 18.5.2 from [7] which provides the Pick matrix test and a nonrecursive version of the Nevanlinna formula for the matrix version of the problem as formulated here.

**Theorem 2.1.** Let $(A_z, B_+, B_-)$ be an admissible interpolation data set with $\sigma(A_z) \subset \text{RHP}$ as above. Define the Hermitian matrix $\Lambda$ as the unique solution of the Lyapunov equation

\[\Lambda A_z^* + A_z \Lambda = B_+ B_+^* - \gamma^{-2} B_- B_-^*.
\]

Then there exist rational matrix functions $W(z)$ satisfying (RIP) and (stab-$\gamma$) if and only if $\Lambda$ is positive semidefinite.

Suppose that $\Lambda$ is positive definite, and define a rational $(M + N) \times (M + N)$ matrix function

\[\Theta(z) = \begin{bmatrix}
\Theta_{11}(z) & \Theta_{12}(z) \\
\Theta_{21}(z) & \Theta_{22}(z)
\end{bmatrix}
\]
(where $\Theta_{11}(z)$ has size $M \times M$) by

\[\Theta(z) = I_{M+N} + \begin{bmatrix}
-B_+^* \\
\gamma^{-1} B_-^*
\end{bmatrix} (zI + A_z^*)^{-1} \Lambda^{-1} [B_+ & B_-].
\]

Then the set of all simultaneous solutions $W$ of (RIP) and (stab-$\gamma$) is given by

\[W(z) = \begin{bmatrix}
\Theta_{11}(z)G(z) + \Theta_{12}(z) \\
\Theta_{21}(z)G(z) + \Theta_{22}(z)
\end{bmatrix}^{-1}.
\]
where $G$ is any rational $M \times N$ matrix function analytic on RHP with $\|G(z)\| \leq \gamma$ for all $z$ in RHP.

In particular, letting $G(z)$ be a constant matrix of norm at most $\gamma$ leads to a family of interpolants having McMillan degree at most $n_C$. We encode this in the following result.

**Corollary 2.2.** Let $(A_+, B_+, B_-)$ be an admissible interpolation data set for (RIP) with $\sigma(A_+) \subset \operatorname{RHP}$ as above. Define $\Lambda$ as in Theorem 2.1, and suppose that $\Lambda$ is positive definite. Then for each $M \times N$ (constant) matrix $U$ with $\|U\| \leq \gamma$, the rational $M \times N$ matrix function

$$W_U(z) = U + \left( -B^*_+ - \gamma^{-2}UB^*_+ \right)$$

$$\times \left[ zI + A^*_x + \gamma^{-2}\Lambda^{-1}(B_+ U + B_-)B^*_x \right]^{-1} \Lambda^{-1}(B_+ U + B_-)$$

(2.5)

satisfies (RIP) and (STAB-$\gamma$), and has McMillan degree at most $n_C$.

**Proof.** To simplify notation, set

$$A = -A^*_x, \quad B_1 = \Lambda^{-1}B_+, \quad B_2 = \Lambda^{-1}B_-,$$

$$C_1 = -B^*_+, \quad C_2 = \gamma^{-2}B^*_x,$$

so the formula for $\Theta(z)$ takes the form

$$\Theta(z) = I_{M+N} + \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} (zI - \Lambda)^{-1} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}.$$ 

Thus $\Theta(z)$ is the transfer function for the system

$$\dot{x} = Ax + B_1w + B_2u,$$

$$s = C_1x + w,$$

$$y = C_2x + u,$$

having state vector $x$, input signal $[w]$, and output signal $[y]$. It is easily seen that $\Gamma_0[U] = (\Theta_{11}U + \Theta_{12})(\Theta_{21}U + \Theta_{22})^{-1}$ is the transfer function obtained
from (2.7) having input $y$, output $s$, and the identification $w = Uu$. To get a realization for $\Gamma_0[U]$, we consider the system
\[
\begin{align*}
\dot{x} &= Ax + (B_1 U + B_2)u, \\
s &= C_1 x + Uu, \\
y &= C_2 x + u,
\end{align*}
\]
use the last equation to express $u$ in terms of $y$ as
\[
u = -C_2 x + y,
\]
and substitute this expression for $u$ in the first two equations to get
\[
\begin{align*}
\dot{x} &= \left(A - (B_1 U + B_2)C_2\right)x + (B_1 U + B_2) y, \\
s &= (C_1 - UC_2)x + Uy.
\end{align*}
\]
The transfer function from $y$ to $s$ is therefore given by
\[
\Gamma_0[U] = U + (C_1 - UC_2)[zI - A + (B_1 U + B_2)C_2]^{-1}(B_1 U + B_2).
\]
Plugging back in the definitions of $A$, $B_1$, $B_2$, $C_1$, $C_2$ from (2.6), we see that $W$, given by (2.5) is equal to $\Gamma_0[U]$ and hence is a solution of (RIP) and (STAB-Y) whenever $\|U\| \leq \gamma$ by Theorem 2.1. That the McMillan degree is at most $n_{\xi}$ is clear from the form of the state-space realization in (2.5).

Let us now suppose that the data set $(A_\xi, B_+, B_-)$ for (RIP) has the special form
\[
A_\xi = \begin{bmatrix} A_{\xi 0} & 0 \\ 0 & s_0 I_M \end{bmatrix}, \quad B_+ = \begin{bmatrix} B_{+0} \\ I_M \end{bmatrix}, \quad B_- = \begin{bmatrix} B_{-0} \\ -Y_0 \end{bmatrix}, \quad (2.8)
\]
where $s_0 \in \text{RHP} \setminus \sigma(A_{\xi 0})$. Here $A_{\xi 0}, B_{+0}, B_{-0}$ have respective sizes $n_{\xi 0} \times n_{\xi 0}, n_{\xi 0} \times M, n_{\xi 0} \times N$, and $Y_0$ is an $M \times N$ matrix. The interpola-
tion problem (RIP) decouples as a residue interpolation problem combined with a full matrix value interpolation problem:

\[ \sum_{z_0 \in \sigma(A_{\xi_0})} \text{Res}_{z - z_0} (zI - A_{\xi_0})^{-1} B_{-0} W(z) = -B_{-0}. \tag{2.9a} \]

\[ W(s_0) = Y_0. \tag{2.9b} \]

The associated Pick matrix \( \Lambda \) assumes a block decomposition

\[ \Lambda = \begin{bmatrix} \Lambda_{11} & \Lambda_{21}^* \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix}, \]

where \( \Lambda_{11} \) is determined from the Lyapunov equation

\[ \Lambda_{11} A_{\xi_0}^* + A_{\xi_0} \Lambda_{11} = B_{+0} B_{+0}^* - \gamma^{-2} B_{-0} B_{-0}^* \tag{2.10a} \]

and \( \Lambda_{21} \) and \( \Lambda_{22} \) are given explicitly by

\[ \Lambda_{21} = (B_{+0}^* + \gamma^{-2} Y_0 B_{+0}^*)(s_0 I + A_{\xi_0}^*)^{-1}, \tag{2.10b} \]

\[ \Lambda_{22} = (s_0 + \bar{s}_0)^{-1}(I_M - \gamma^{-2} Y_0 Y_0^*). \tag{2.10c} \]

We assume that \( \Lambda \) is positive definite. In this case \( \Lambda_{11} \) and

\[ \Delta := \Lambda_{22} - \Lambda_{21} \Lambda_{11}^{-1} \Lambda_{21}^* \tag{2.11} \]

are both positive definite, and \( \Lambda^{-1} \) can be computed explicitly via a Schur complement argument (or equivalently, via block Gaussian elimination); the result is

\[ \Lambda^{-1} = \begin{bmatrix} \delta^{-1} & -\Lambda_{11}^{-1} \Lambda_{21}^* \Delta^{-1} \\ -\Delta^{-1} \Lambda_{21} \Lambda_{11}^{-1} & \Delta^{-1} \end{bmatrix}, \tag{2.12a} \]

where

\[ \delta = \Lambda_{11} - \Lambda_{21}^* \Lambda_{22}^{-1} \Lambda_{21}. \tag{2.13} \]
The $\Theta(z)$ in (2.4) with this data set becomes

$$\Theta(z) = I_{M+N} + \begin{bmatrix} -B^*_+ & -I_M \\ \gamma^{-2}B^*_{-0} & -\gamma^{-2}Y^*_0 \end{bmatrix} \times \begin{bmatrix} (zI + A^*_0)^{-1} & 0 \\ 0 & (z + \bar{z}_0)^{-1}I_M \end{bmatrix} \Lambda^{-1} \begin{bmatrix} B^*_+ & B_{-0} \\ I_M & -Y_0 \end{bmatrix}. \quad (2.12b)$$

From Corollary 2.2 we know that there exist solutions $W$ of (2.9a)–(2.9b) which also satisfy (STAB-$\gamma$) and which have McMillan degree at most $n_\zeta = n_{\zeta_0} + M$. We now show that there exist solutions having McMillan degree at most $n_{\zeta_0}$. First we need a preliminary lemma.

**Lemma 2.3.** Let $(A_\zeta, B_+, B_-)$ be an admissible interpolation data set of the special form (2.8), and suppose that the associated Pick matrix $A$ is positive definite (with inverse then given by (2.12a)). Then there exists an $M \times N$ constant matrix $U$ with $\|U\| < \gamma$ such that

$$(-\Lambda_{21}\Lambda^{-1}_{11}B^*_+ + I)U + (-\Lambda_{21}\Lambda^{-1}_{11}B^-_0 - Y_0) = 0. \quad (2.14)$$

**Proof.** The assertion follows from [9] once we verify that

$$\gamma^{-2}(\Lambda_{21}\Lambda^{-1}_{11}B^*_+ + Y_0)(\Lambda_{21}\Lambda^{-1}_{11}B^-_0 + Y_0)^* \leq (-\Lambda_{21}\Lambda^{-1}_{11}B^*_+ + I)(-\Lambda_{21}\Lambda^{-1}_{11}B^-_0 + I)^*. $$

We collect terms on the left and on the right sides to rewrite this as

$$\gamma^{-2}(\Lambda_{21}\Lambda^{-1}_{11}B^*_+ B^*_0 B^{-1}_{-0} \Lambda^{-1}_{11}A^*_{21} + Y_0 Y^*_0 + Y_0 B^*_0 \Lambda^{-1}_{11}A^*_{21} + \Lambda_{21}\Lambda^{-1}_{11}B^-_0 Y^*_0) \leq \Lambda_{21}\Lambda^{-1}_{11}B^*_+ B^*_0 \Lambda^{-1}_{11}A^*_{21} + I - \Lambda_{21}\Lambda^{-1}_{11}B^*_+ B^*_0 \Lambda^{-1}_{11}A^*_{21}. $$

Moving all terms to the right side and then collecting terms gives

$$0 \leq \Lambda_{21}\Lambda^{-1}_{11}(B^*_+ B^*_0 - \gamma^{-2}B^-_0 B^*_0)\Lambda^{-1}_{11}A^*_{21} + I - \gamma^{-2}Y_0 Y^*_0$$

$$- (\gamma^{-2}Y_0 B^*_+ B^*_0)\Lambda^{-1}_{11}A^*_{21} - \Lambda_{21}\Lambda^{-1}_{11}(\gamma^{-2}B^-_0 Y^*_0 + B^*_+ B^*_0).$$
Now recall from (2.10b) that $B^*_0 + \gamma^{-2}Y_0B^*_0 = \Lambda_{21}(s_0 I + A^*_0)$ to see that the needed condition is

$$0 \leq \Lambda_{21}^{-1}(B^*_0B^*_0 - \gamma^{-2}B^*_0B^*_0)\Lambda_{21}^{-1}\Lambda_{21}^*
+ I - \gamma^{-2}Y_0Y_0^*
- \Lambda_{21}(s_0 I + A^*_0)\Lambda_{21}^{-1}\Lambda_{21}^*
- \Lambda_{21}^{-1}(s_0 I + A^*_0)\Lambda_{21}^*
= \Lambda_{21}^{-1}[B^*_0B^*_0 - \gamma^{-2}B^*_0B^*_0 - \Lambda_{11}(s_0 I + A^*_0)]
- (\bar{s}_0 I + A^*_0)\Lambda_{11}^{-1}\Lambda_{21}^*
+ I - \gamma^{-2}Y_0Y_0^*.$$ 

Now use the Lyapunov equation (2.10a) to get that this is equivalent to

$$0 \leq \Lambda_{21}^{-1}[\Lambda_{11}A^*_0 + A^*_0\Lambda_{11} - \Lambda_{11}(s_0 I + A^*_0)]
- (\bar{s}_0 I + A^*_0)\Lambda_{11}^{-1}\Lambda_{21}^*
+ I - \gamma^{-2}Y_0Y_0^* = (s_0 + \bar{s}_0)(-\Lambda_{21}\Lambda_{11}^{-1}\Lambda_{21}^* + \Lambda_{22}) =: (s_0 + \bar{s}_0)\Delta,$$

where we have used (2.10c). Since $\Lambda$ is positive definite, $\Delta > 0$; as $s_0$ is in RHP, $s_0 + \bar{s}_0 > 0$ and the lemma follows.

For later use, we present the following lemma.

**Lemma 2.4.** Suppose $B_+, B_-, A, \Lambda,$ and $U$ are as in Lemma 2.3. Then

$$\Lambda^{-1}(B_+U + B_-) = \Lambda_{11}^{-1}(B_+U + B_-).$$

**Proof.** Substituting $B_+, B_-$ as in (2.8) and applying the identities (2.12a), (2.14) to $\Lambda^{-1}(B_+U + B_-)$ gives

$$\Lambda^{-1}(B_+U + B_-) = \begin{bmatrix} L \\ 0 \end{bmatrix},$$
where
\[ L = \delta^{-1}(B_+U + B_-) - \Lambda^{-1}_{11}\Lambda_{21}^{-1}(U - Y_0) \]
\[ = \delta^{-1}\left((B_+U + B_-) - \delta\Lambda^{-1}_{11}\Lambda_{21}^{-1}(U - Y_0)\right). \quad (2.15) \]

We now simplify \( L \). Using the identity (2.13) for \( \delta \) in the second term of (2.15) and collecting the common terms, we have
\[ \delta\Lambda^{-1}_{11}\Lambda_{21}^{-1} = \Lambda_{21}^{-1}(I - \Lambda_{22}^{-1}\Lambda_{21}\Lambda^{-1}_{11}\Lambda_{21}) \Lambda_{22}^{-1}\Lambda_{21}^{-1} \]
\[ = \Lambda_{21}^{-1}(I - \Lambda_{22}^{-1}\Lambda_{21}\Lambda^{-1}_{11}\Lambda_{21}) \Lambda_{22}^{-1}\Lambda_{21}^{-1} \]
\[ = \Lambda_{21}^{-1}(\Lambda_{22}^{-1}-\Lambda_{21}\Lambda_{11}\Lambda_{21}^{-1})\Lambda_{21}^{-1}. \]

As \( \Delta \) is given by (2.11), the above formula is reduced to \( \Lambda_{21}^{\star}\Lambda_{22}^{-1} \). Substitute \( \Lambda_{21}^{\star}\Lambda_{22}^{-1} \) in place of \( \delta\Lambda^{-1}_{11}\Lambda_{21}^{-1} \) in (2.15) to reduce \( L \) to
\[ L = \delta^{-1}\left((B_+U + B_-) - \Lambda_{21}^{\star}\Lambda_{22}^{-1}(U - Y_0)\right). \quad (2.16) \]

Plugging the identity
\[ Y_0 = (-\Lambda_{21}\Lambda_{11}^{-1}B_+ + I)U - \Lambda_{21}\Lambda_{11}^{-1}B_- \]
derived from (2.14) into (2.16), we obtain
\[ L = \delta^{-1}\left((B_+U + B_-) - \Lambda_{21}^{\star}\Lambda_{22}^{-1}(U - (-\Lambda_{21}\Lambda_{11}^{-1}B_+ + I)U + \Lambda_{21}\Lambda_{11}^{-1}B_-)\right). \]

By collecting the terms in the above, we get
\[ L = \delta^{-1}\left((I - \Lambda_{21}^{\star}\Lambda_{22}^{-1}\Lambda_{21}\Lambda_{11}^{-1})B_+U + (I - \Lambda_{21}^{\star}\Lambda_{22}^{-1}\Lambda_{21}\Lambda_{11}^{-1})B_- \right) \]
\[ = \delta^{-1}(I - \Lambda_{21}^{\star}\Lambda_{22}^{-1}\Lambda_{21}\Lambda_{11}^{-1})(B_+U + B_-). \]

Recalling \( \delta \) is given by (2.13), we get
\[ L = \Lambda_{11}^{-1}(B_+U + B_-), \]
as desired.
The next theorem gives a realization formula for a rational matrix function satisfying (2.9a)-(2.9b) and (STAB-γ).

**Theorem 2.5.** Suppose $(A_\xi, B_+, B_-)$ and $A$ are as in Lemma 2.3, and let $U$ be a solution of (2.14) with $\|U\| \leq \gamma$ as guaranteed by Lemma 2.3. Then

$$W_U(z) = U + C(zI - A)^{-1}B,$$

where

$$A = -\left(A_{\xi0}^* + \gamma^{-2}LB_{-0}^*\right),$$

$$B = L,$$

$$C = -B_{+0}^* - \gamma^{-2}UB_{-0}^*,$$

and where $L$ is given by $L = \Lambda^{-1}_{11}(B_{+0}U + B_{-0})$.

**Proof.** Consider $W_U$ as in (2.5), but where $A_\xi, B_+, B_-$ have the special form given by (2.8) and where $U$ is a solution of (2.14) with norm at most $\gamma$. As a consequence of Lemma 2.4, the input operator $\Lambda^{-1}(B_+ U + B_-)$ in the realization for $W_U$ implicit in the formula (2.4) has the form

$$\Lambda^{-1}(B_+ U + B_-) = \begin{bmatrix} L \\ 0 \end{bmatrix}$$

with $L = \Lambda^{-1}_{11}(B_{+0}U + B_{-0})$, while the state operator for $W_U$ has the upper triangular form

$$-\begin{bmatrix} A_{\xi0}^* + \gamma^{-2}\Lambda^{-1}(B_+ U + B_-)B_-^* \\ 0 \end{bmatrix} = -\begin{bmatrix} A_{\xi0}^* + \gamma^{-2}LB_{-0}^* & -\gamma^{-2}LY_{0}^* \\ 0 & 0 \end{bmatrix}.$$ 

As a consequence the second component of the state space is uncontrollable and can be discarded. Finally, the output operator is

$$-B_-^* - \gamma^{-2}UB_-^* = \begin{bmatrix} -B_{+0}^* - \gamma^{-2}UB_{-0}^* \end{bmatrix},$$

where the second block column is irrelevant. We arrive at the formula (2.17) for $W_U$ as desired. From the size of $A$ in this formula we see that $W_U$ has
McMillan degree at most $n_{\xi_0}$. From Corollary 2.2 we know that $W_U$ satisfies (2.9a)–(2.9b) and (stab-$\gamma$).

The following corollary gives a solution for the classical Nevanlinna interpolation problem for the scalar case by nonrecursive construction.

**Corollary 2.6.** The scalar rational function

$$W_U(z) = \frac{U \prod_{i=1}^{n}(z + \bar{z}_i) + \sum_{i=1}^{n} b_i p_i(z)}{\prod_{i=1}^{n}(z + \bar{z}_i) + \sum_{i=1}^{n} \bar{w}_i b_i p_i(z)}$$  \hspace{1cm} (2.18)

of McMillan degree $n$ satisfies the conditions (1.2) and (1.7a) and (1.7b) with $\gamma = 1$, (i.e., interpolates $2n + 1$ points) if

$$\Lambda = \begin{bmatrix} 1 - \omega_i \bar{w}_j \\ z_i + \bar{z}_j \end{bmatrix}_{1 \leq i, j \leq n+1} = \begin{bmatrix} \Lambda_{11} & \Lambda_{21} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix}$$

is positive definite, where the size of $\Lambda_{11}$ is $n \times n$,

$$p_i(z) = \prod_{1 \leq j \leq n, j \neq i} (z + \bar{z}_j),$$

and $[b_1, \ldots, b_n] = \Lambda_{11}^{-1} [U - \omega_1, \ldots, U - \omega_n]$ with $U$ satisfying (2.14).

**Proof.** Let $M = N = 1$, $\gamma = 1$,

$$A_{x_0} = \begin{bmatrix} z_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & z_n \end{bmatrix}, \quad B_{+0} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, \quad B_{-0} = \begin{bmatrix} -\omega_1 \\ \vdots \\ -\omega_n \end{bmatrix}, \hspace{1cm} (2.19)$$

and $Y_0 = \omega_{n+1}$ in Theorem 2.5. Then, $W_U(z)$ satisfies the conditions (1.1) and (1.3) by Theorem 2.5. It is left to show that $W_U(z)$ can be represented as (2.18). Remember that $W_U(z)$ is chosen so that

$$W_U(z) = \frac{\Theta_{11}U + \Theta_{12}}{\Theta_{21}U + \Theta_{22}},$$  \hspace{1cm} (2.20)
where the $2 \times 2$ matrix $\Theta(z)$ is given by (2.12b) with

$$A_\xi = \begin{bmatrix} A_{\xi 0} & 0 \\ 0 & z_{n+1} \end{bmatrix}, \quad B_+ = \begin{bmatrix} B_+ \end{bmatrix}, \quad B_- = \begin{bmatrix} B_- \end{bmatrix},$$

$Y_0 = \omega_{n+1}, s_0 = z_{n+1}$. As a consequence of Lemma 2.4,

$$\begin{bmatrix} \Theta_{11} U + \Theta_{12} \\ \Theta_{21} U + \Theta_{22} \end{bmatrix} = \Theta(z) \begin{bmatrix} U \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} U \\ 1 \end{bmatrix} + \begin{bmatrix} B_{\xi 0}^* & 1 \\ -B_{\xi 0}^* & -\omega_{n+1} \end{bmatrix} \begin{bmatrix} (zI + A_{\xi 0}^*)^{-1} & 0 \\ 0 & (z + \bar{z}_{n+1})^{-1} \end{bmatrix} \begin{bmatrix} L \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} U \\ 1 \end{bmatrix} + \begin{bmatrix} B_{\xi 0}^* \\ -B_{\xi 0}^* \end{bmatrix} (zI + A_{\xi 0}^*)^{-1} L,$$

where $L = \Lambda_{11}^{-1}(B_+ U + B_-)$. Substituting (2.19) in place of $A_\xi$, $B_+, B_-$ in the above formula yields

$$\begin{bmatrix} U \\ 1 \end{bmatrix} + \begin{bmatrix} \sum_{i=1}^n b_i(z + \bar{z}_i)^{-1} \\ \sum_{i=1}^n b_i \bar{w}_i(z + \bar{z}_i)^{-1} \end{bmatrix}$$

(2.21)

where $[b_1, \ldots, b_n]^T = L^T = \Lambda_{11}^{-1}[U - \omega_1, \ldots, U - \omega_n]^T$. Replacing $(\Theta_{11} U + \Theta_{12})(\Theta_{21} U + \Theta_{22})^{-1}$ in (2.20) with the previous formula and multiplying the denominator and the numerator by $\prod_{i=1}^n(z + \bar{z}_i)$ gives (2.18).

3. INTERPOLATION WITH MINIMAL MCMILLAN DEGREE

In this section we review results from [5] on characterizing the admissible McMillan degree of solutions of (TRIP) (TRIPii) with no stability side constraints. These results in turn are refinements of the results from Chapter 16 in [7] on Lagrange-Sylvester interpolation, where no account was taken of McMillan degree.
Sometimes it is necessary to consider problems involving interpolation conditions from both the left and the right side simultaneously. The data set for this problem is a collection of seven matrices, $\omega = (C_+, C_-, A_\pi, A_\xi, B_+, B_-, S)$, of respective sizes $N \times n_\pi$, $m \times n_\pi$, $n_\pi \times n_\pi$, $n_\xi \times n_\xi$, $n_\xi \times M$, $n_\xi \times N$, $n_\xi \times n_\pi$ such that

(idsi) $(C_-, A_\pi)$ is an observable pair,
(idsii) $(A_\xi, B_-)$ is a controllable pair,
(idsiii) $S$ satisfies the Sylvester equation

$$SA_\pi - A_\xi S = B_+ C_+ + B_- C_-.$$ 

Any collection of matrices $\omega = (C_+, C_-, A_\pi, A_\xi, B_+, B_-, S)$ satisfying (idsi)–(idsiii) will be called an interpolation data set. The associated set of interpolation conditions for the two-sided residue interpolation problem (TRIP) is

\begin{equation}
\sum_{z_0 \in \sigma(A_\pi)} W(z) C_-(zI - A_\pi)^{-1} = C_+, \tag{TRIPi}
\end{equation}

\begin{equation}
\sum_{z_0 \in \sigma(A_\xi)} (zI - A_\xi)^{-1} B_+ W(z) = -B_-, \tag{TRIPii}
\end{equation}

\begin{equation}
\sum_{z_0 \in \sigma(A_\pi) \cup \sigma(A_\xi)} (zI - A_\xi)^{-1} B_+ W(z) C_- (zI - A_\pi)^{-1} = S, \tag{TRIPiii}
\end{equation}

where $W(z)$ is an unknown rational $M \times N$ matrix function analytic on $\sigma(A_\pi) \cup \sigma(A_\xi)$. For examples and more motivation we refer to [7]. We do mention here however that the condition (TRIPiii) follows automatically from (TRIPi) and (TRIPii) if $\sigma(A_\pi)$ and $\sigma(A_\xi)$ are disjoint. When $\sigma(A_\pi)$ and $\sigma(A_\xi)$ overlap, (TRIPiii) adds some higher-order two-sided interpolation conditions which are needed in order to get a clean linear fractional form (or even affine form) for the set of solutions (see [7]). Let us suppose that $\omega = (C_+, C_-, A_\pi, A_\xi, B_+, B_-, S)$ is an admissible two-sided interpolation data set. To describe the admissible degree of interpolants satisfying (TRIPi)–(TRIPiii) we need an auxiliary rational $(M + N) \times (M + N)$ matrix function $\Theta$. The matrix $\Theta$ is specified by two requirements:

(i) $\tilde{\omega} := \begin{bmatrix} C_+ \\ C_- \end{bmatrix}, A_\pi, A_\xi, [B_+ \ B_-], S$ is a spectral triple for $\Theta$ over $C$;

(ii) $\Theta$ is column reduced at infinity.
There are several equivalent definitions of spectral triple (also called null-pole triple); we refer to Chapters 4 and 12 of [7] for definitions and motivation. The following is the most efficient (although perhaps not the most transparent) characterization: the collection of matrices \((C, A_\pi; A_\xi, B; S)\) is a spectral triple for a rational \(K \times K\) matrix function \(\Theta\) over the set \(\sigma \subset \mathbb{C}\) if

\[\begin{align*}
(\text{sti}) & \quad (C, A_\pi) \text{ is an observable pair with } \sigma(A_\pi) \subset \sigma, \\
(\text{stii}) & \quad (A_\xi, B) \text{ is a controllable pair with } \sigma(A_\xi) \subset \sigma, \\
(\text{stiii}) & \quad \text{the } \mathcal{R}(\sigma)\text{-module } \Theta \mathcal{R}^K(\sigma) \text{ is characterized as}
\end{align*}\]

\[\Theta \mathcal{R}^K(\sigma) = \left\{ f(z) = C(zI - A_\pi)^{-1}x + h(z) : x \in \mathbb{C}^n, \quad h \in \mathcal{R}^K(\sigma) \right\},\]

such that \(\sum_{z_0 \in \sigma} \text{Res}_{z=z_0} (zI - A_\pi)^{-1}Bf(z) = Sx\).

In this case, \(S\) necessarily satisfies the Sylvester equation

\[\begin{align*}
(\text{stiii}’) \quad SA_\pi - A_\xi S = BC.
\end{align*}\]

Here \(n_\pi \times n_\pi\) is the size of \(A_\pi\), \(\mathcal{R}(\sigma)\) is the ring of scalar rational functions with no poles in \(\sigma\), and \(\mathcal{R}^K(\sigma)\) is the space of \(K\)-component column vectors with entries in \(\mathcal{R}(\sigma)\). Note that if \(\omega = (C_+, C_-, A_\pi, A_\xi, B_+, B_-, S)\) satisfies (IDSi)–(IDSiii), then it is automatic that

\[\tilde{\omega} = \left( \begin{bmatrix} C_+ \\ C_-
\end{bmatrix}, A_\pi, A_\xi, [B_+ B_-], S \right)\]

satisfies (sti), (stii), and (stiii’). The definition of column reduced is better known (see [17]); we say that the rational \(K \times K\) matrix function is column reduced at infinity if \(\Theta(z)\) can be factored as \(\Theta(z) = \Theta_0(z)D(z)\) where \(\Theta_0(z)\) is biproper and \(D(z) = \text{diag}(z^{\kappa_1}, \ldots, z^{\kappa_K})\) for some integers \(\kappa_1, \ldots, \kappa_K\). In general, one can show that if \(\tilde{\omega} = (C, A_\pi; A_\xi, B; S)\) is any collection of matrices satisfying (sti), (stii), and (stiii’), then there exists a rational matrix function \(\Theta\) which has \(\tilde{\omega}\) as its spectral triple over \(C\) and which is column reduced at infinity. An explicit realization formula for such a rational matrix function is given in [12] and [8]; in case \(S\) is invertible the formula is quite simple, namely:

\[\Theta(z) = I_K + C(zI - A_\pi)^{-1}S^{-1}B.\]
In this case the column indices \( \kappa_1, \ldots, \kappa_K \) are all zero. In general, the indices \( \kappa_1, \ldots, \kappa_K \) are uniquely determined from the data set \( \tilde{\omega} \) via a fairly elaborate algorithm (see [12, 8]).

The following theorem due to [7] describes how to parametrize all the solutions for (TRIPi)--(TRIPiii).

**Theorem 3.1** (See Theorem 16.10.1 in [7]). Let \( \sigma \) be a subset of \( C \) and \( \sigma(A_\pi) \cup \sigma(A_\xi) \subset \sigma \). There exist rational matrix functions \( W \in \mathcal{R}^{M \times N}(\sigma) \) satisfying (TRIPi)--(TRIPiii). Moreover, if

\[
\Theta = \begin{bmatrix}
\Theta_{11} & \Theta_{12} \\
\Theta_{21} & \Theta_{22}
\end{bmatrix}
\]

is any \( (M + N) \times (M + N) \) matrix function having the set

\[
\tilde{\omega} = \left( \begin{bmatrix} C_+ \\ C_- \end{bmatrix}, A_\pi; A_\xi, [B_+ B_-]; S \right)
\]

as a \( \sigma \)-null-pole triple and \( \varphi^{-1} \) is a regular rational \( N \times N \) matrix function having the set \( \omega = (C_-, A_\pi; 0, 0, 0) \) as a \( \sigma \)-null-pole triple, when \( W \in \mathcal{R}^{M \times N}(\sigma) \) is a solution of (TRIPi)--(TRIPiii) if and only if \( W \) has the following form: There exist rational matrix functions \( P \in \mathcal{R}^{M \times N}(\sigma), Q \in \mathcal{R}^{N \times N}(\sigma) \) for which the function \( \varphi(\Theta_{21} P + \Theta_{22} Q) \) has no zeros or poles in \( \sigma \), such that

\[
W = (\Theta_{11} P + \Theta_{12} Q)(\Theta_{21} P + \Theta_{22} Q)^{-1}.
\]

The following result describes how to characterize the minimal McMillan degree for solutions of an interpolation problem and how to parametrize all solutions whose McMillan degree is equal to the minimal degree.

**Theorem 3.2.** Let \( \omega = (C_+, C_-, A_\pi, A_\xi, B_+, B_-, S) \) be an admissible two-sided interpolation data set, and let \( \Theta(z) \) be a rational \( (M + N) \times (M + N) \) matrix function having

\[
\tilde{\omega} = \left( \begin{bmatrix} C_+ \\ C_- \end{bmatrix}, A_\pi; A_\xi, [B_+ B_-]; S \right)
\]

as a spectral triple over \( C \) which is column reduced at infinity. Let \( \kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_{M+N} \) be the column indices of \( \Theta(z) \). Let \( \varphi(z) \) be any rational
\( N \times N \) matrix function having \( \hat{\omega} = (C_-, A_\pi; 0, 0; 0) \) as a spectral triple over \( C \). Then a solution of (TRIP) (TRIP) exists. The minimal possible McMillan degree of a solution of (TRIP) is determined as follows. Define indices \( i_1 > i_2 > \cdots > i_N \) by

\[
i_k = \max \{ l : \text{rank } \varphi(z) \begin{bmatrix} 0 & I_N \end{bmatrix} \begin{bmatrix} \theta_1(z), \ldots, \theta_{M+N}(z) \end{bmatrix} = k \quad \text{for all } z \in \sigma(A_\pi) \cup \sigma(\Lambda_\xi) \},
\]

where \( \theta_i(z) \) is the \( i \)th column of \( \Theta(z) \) for \( 1 \leq i \leq M + N \). Then

\[
\kappa_* = n_\pi + \kappa_{i_1} + \cdots + \kappa_{i_N}
\]

is the minimal possible McMillan degree.

(i) If \( i_j = M + N - j + 1 \) for \( 1 \leq j \leq N \) and \( \kappa_M > \kappa_{M+1} \), the minimal-degree interpolant is unique and is given by \( W(z) = \tilde{\Theta}_1(z) \Theta_2(z)^{-1} \), where

\[
\begin{bmatrix} \tilde{\Theta}_1(z) \\
\tilde{\Theta}_2(z) \end{bmatrix} = \begin{bmatrix} \theta_{M+1}(z), \ldots, \theta_{M+N}(z) \end{bmatrix}
\]

(ii) If (i) does not hold (i.e., \( i_j < M + N - j + 1 \) for some \( j \) between 1 and \( N \) or \( \kappa_M = \kappa_{M+1} \)), then a minimal-degree interpolant is not unique.

Proof. See [5]. (We have rearranged the statement of the theorem to be consistent with the convention that the column indices of \( \Theta \) are assumed to be in decreasing rather than in increasing order.)

We remark that there is nothing special about the point infinity in Theorem 3.2. Instead we could let \( w_0 \) be any point in \( C \setminus \{ \sigma(A_\pi) \cup \sigma(\Lambda_\xi) \} \), and let \( \Theta(z) \) be a biproper rational matrix function having \( \hat{\omega} \) as a spectral triple over \( C \setminus \{ w_0 \} \) which is column reduced at \( w_0 \). Then the minimal possible McMillan degree of interpolants and the parametrization of the minimal interpolants applies exactly as in Theorem 3.1 with \( \kappa_j \) equal to the highest power of \( (z - w_0)^{-1} \) occurring in the Laurent expansion at \( w_0 \) of the \( j \)th column of \( \Theta(z) \). We call this \( \kappa_j \) the \( j \)th column index of \( \Theta(z) \) at \( z = w_0 \).
We next specialize the analysis of Theorem 3.2 to interpolation data of a special symmetric form. Specifically we assume that \((A, B_+, B_-)\) are as in (2.7):

\[
A_\xi = \begin{bmatrix} A_{\xi 0} & 0 \\ 0 & s_0 I_M \end{bmatrix}, \quad B_+ = \begin{bmatrix} B_{+ 0} \\ I_M \end{bmatrix}, \quad B_- = \begin{bmatrix} B_{- 0} \\ -Y_0 \end{bmatrix}, \tag{3.1}
\]

where \(\sigma(A_\xi) \subset \text{RHP}, s_0 \in \text{RHP} \setminus \sigma(A_{\xi 0})\). We assume that \((C_+, C_-, A_\pi)\) is the reflection of \((A_\xi, B_+, B_-)\) across the imaginary line, namely

\[
C_+ = -B_{+ 0}^*, \quad C_- = \gamma^{-2} B_{- 0}^*, \quad A_\pi = -A_{\xi 0}^* . \tag{3.2}
\]

Then if \(\Theta(z)\) is defined by (2.12b), one can show (see Chapter 4 of [7]) that \(\Theta(z)\) has

\[
\tilde{\omega} = \begin{bmatrix} -B_{+ 0}^* \\ \gamma^{-2} B_{- 0}^* \end{bmatrix}, -A_{\xi 0}^* \begin{bmatrix} A_{\xi 0} & 0 \\ 0 & s_0 I_M \end{bmatrix} \begin{bmatrix} B_{+ 0} & B_{- 0} \\ I_M & -Y_0 \end{bmatrix} \begin{bmatrix} \Lambda_{11} \\ \Lambda_{21} \end{bmatrix} \tag{3.3}
\]

as its spectral triple over \(C \setminus \{-\tilde{s}_0\}\). We would like for this \(\tilde{\omega}\) to arise from an admissible interpolation data set as in Theorem 3.1; for this reason we impose the additional hypothesis that \((A_{\xi 0}, B_{- 0})\) also is controllable (this corresponds to the assumption that \(\omega_i \neq 0\) for \(1 \leq i \leq n + 1\) in the discussion of the scalar problem in the introduction). We also note that \(\Theta\) has no poles or zeros at infinity. If \(\psi\) is a rational \((M + N) \times (M + N)\) matrix function chosen to be analytic and invertible on \((C \cup \{\infty\}) \setminus \{-\tilde{s}_0\}\) in such a way that \(\tilde{\Theta} = \Theta \psi\) is column reduced at \(-\tilde{s}_0\), then \(\tilde{\Theta}\) meets all the requirements of Theorem 3.2 (with the point \(-\tilde{s}_0\) in place of infinity). Moreover, if \(U\) is chosen as in Lemma 2.3 to be a solution of (2.14), then

\[
\tilde{\Theta}(z) = \Theta(z) \begin{bmatrix} I_M & U \\ 0 & I_N \end{bmatrix}
\]

has the form

\[
\tilde{\Theta}(z) = [\theta_1, \ldots, \theta_M, \theta_{M+1}, \ldots, \theta_{M+N}]
\]

where \(\theta_{M+1}, \ldots, \theta_{M+N}\) has no pole at \(-\tilde{s}_0\). Furthermore we know that \(\Theta\) (and hence \(\tilde{\Theta}\) also) has pole pair at \(-\tilde{s}_0\) equal to

\[
\begin{bmatrix} I_M \\ \gamma^{-2} Y_0^* \end{bmatrix}, -\tilde{s}_0 I_M
\]
This means that $\theta_1, \ldots, \theta_M$ must have simple poles at $-\bar{s}_0$ with linearly independent residues. Hence

$$\tilde{\Theta} = \Theta \begin{bmatrix} I_M & U \\ 0 & I_N \end{bmatrix}$$

is column reduced at $-\bar{s}_0$ with column indices equal to

$$\left\{ 1, \ldots, 1, 0, \ldots, 0 \right\}.$$ 

We now arrive at the following connection between minimal-McMillan-degree interpolation and stable interpolation (see [33 for the scalar case).

**Theorem 3.3.** Let $\omega = \left( A_{\xi}, B_0 \right)$ be an admissible one-sided interpolation data set over RHP, assume that $(A_{\xi}, B_0)$ is controllable, and let

$$\omega_* = \left( -R_{+0}, \gamma^{-2}B_{-0}, -A_{\xi}, B_+, B_-, \left[ \Lambda_{11} \right] \right)$$

be the two-sided interpolation data set enlarged via reflection from $\omega$. Assume $\gamma > 0$ is sufficiently large so that

$$\Lambda = \begin{bmatrix} \Lambda_{11} & \Lambda_{21} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix}$$

given by (2.10a)–(2.10c) is positive definite, and let $W = \Gamma_0[U]$ be the solution of (RIP) and (STAB-$\gamma$) of McMillan degree at most $n_{\xi_0}$ given by Theorem 2.5. Then the following conditions are equivalent:

(a) $W$ is a solution for TRIP($\omega_*$).

(b) TRIP($\omega_*$) has a unique solution with the minimal possible McMillan degree.

(c) $\Theta_{21}U + \Theta_{22}$ has no zeros and $n_{\xi_0}$ poles in $\sigma(-A_{\xi})$, including multiplicities.

(d) $\sigma(-A_{\xi_0} - \gamma^{-2}A_{11}^{-1}(B_{+0}U + B_{-0})B_{-0}^*) \cap \sigma(-A_{\xi_0}^*) = \emptyset$.

In (c) the multiplicity of a pole (or zero) is the sum of the partial pole (or zero) multiplicities in the local Smith form.
Proof. Throughout this proof we set \( \sigma = \sigma(A_{\xi_0}) \cup \{s_0\} \cup \sigma(-A_{\xi_0}^*) \).

(a) \(\Rightarrow\) (b): Suppose \( W = \hat{\Theta}_1(z)\hat{\Theta}_2(z)^{-1} \) is a solution for \( \text{TRIP}(\omega_*) \), where

\[
\begin{bmatrix}
\hat{\Theta}_1(z) \\
\hat{\Theta}_2(z)
\end{bmatrix} = \Theta(z) \begin{bmatrix}
0 \\
I_N
\end{bmatrix}
\]

\[
= \Theta(z) \begin{bmatrix}
I & U \\
0 & I
\end{bmatrix} \begin{bmatrix}
0 \\
I_N
\end{bmatrix}
\]

(3.4)

with \( U \) satisfying (2.14). Then by Theorem 3.1, \( \varphi(z)[\Theta_{21}(z)U + \Theta_{22}(z)] \) has no poles or zeros in \( \sigma \), where \( \varphi^{-1}(z) \) is an \( N \times N \) rational matrix function which has \( \tilde{\omega}_- := (C_-, \varphi(z), 0, 0, 0) \) as its \( \sigma \)-spectral triple. Note that in this case, \( t_j \) in Theorem 3.2 is \( M + N - j + 1 \) for any \( j \) between 1 and \( N \), and we have \( \kappa_M = 1, \kappa_{M+1} = 0 \). So we are in case (i) of Theorem 3.2, and \( W(z) \) is the unique minimal degree solution of \( \text{TRIP}(\omega_*) \) with McMillan degree \( n_{\xi_0} \).

(b) \(\Rightarrow\) (a): Suppose \( \text{TRIP}(\omega_*) \) has a unique minimal degree solution. Then by Theorem 3.2(i), the matrix function formed by the last \( N \) columns of \( \varphi(z)[0 \quad I_N]\tilde{\Theta}(z) \) has no zeros or poles in \( \sigma \), where

\[
\tilde{\Theta}(z) - \Theta(z) \begin{bmatrix}
I & U \\
0 & I
\end{bmatrix}
\]

and \( \varphi(z) \) is the same as in the proof of (a) \(\Rightarrow\) (b). Now, by applying Theorem 3.1, we conclude that

\[
W(z) = \hat{\Theta}_1(z)\hat{\Theta}_2(z)^{-1}
\]

is a solution for \( \text{TRIP}(\omega_*) \) with \( [\hat{\Theta}_1(z) \quad \hat{\Theta}_2(z)] \) as in (3.4).

(a) \(\Leftrightarrow\) (c): By Theorem 3.1, \( W \) is a solution of \( \text{TRIP}(\omega_*) \) if and only if \( \varphi(\Theta_{21}U + \Theta_{22}) \) has no zeros and no poles in \( \sigma \). Since \( \|U\| < \gamma \), from Theorem 2.1 we know that \( W \) is a stable solution of \( \text{TRIP}(\omega) \). Hence from the necessity direction in Theorem 3.1 it follows that \( \varphi(\Theta_{21}U + \Theta_{22}) \) automatically has no zeros and no poles in RHP, and hence in particular in \( \sigma(A_{\xi_0}) \cup \{s_0\} \). We may now conclude that \( W \) is a solution of \( \text{TRIP}(\omega_*) \) if and only if \( \varphi(\Theta_{21}U + \Theta_{22}) \) has no poles and no zeros in \( \sigma(-A_{\xi_0}^*) \), or equivalently, if and only if

\[
(\Theta_{21}U + \Theta_{22})R^N(\sigma(-A_{\xi_0}^*)) = \varphi^{-1}R^N(\sigma(-A_{\xi_0}^*)).
\]

(3.5)
Recall that \( \varphi^{-1} \) has null-pole triple over \( \sigma(-A_{\xi_0}^*) \) equal to \( (\gamma^{-2} B_{-0}^*, -A_{\xi_0}^*; 0, 0; 0) \). Thus (stiii) for \( \varphi^{-1} \) collapses to

\[
\varphi^{-1} \mathcal{R}^N(\sigma(-A_{\xi_0}^*)) = \left\{ \gamma^{-2} B_{-0}^*(zI + A_{\xi_0}^*)^{-1} x : x \in \mathbb{C}^{n_{\xi_0}} \right\}
\]

\[
+ \mathcal{R}^N(\sigma(-A_{\xi_0}^*)).
\]

(3.6)

From (stiii) and the form of the spectral triple \( \tilde{\omega} \) for \( \Theta \) one can deduce that also

\[
\left[ \Theta_{21}, \Theta_{22} \right] \mathcal{R}^{M+N}(\sigma(-A_{\xi_0}^*))
\]

\[
= \left\{ \gamma^{-2} B_{-0}^*(zI + A_{\xi_0}^*)^{-1} x : x \in \mathbb{C}^{n_{\xi_0}} \right\} + \mathcal{R}^N(\sigma(-A_{\xi_0}^*)).
\]

(3.7)

Note that we always have the containment

\[
(\Theta_{21} U + \Theta_{22}) \mathcal{R}^{M+N}(\sigma(-A_{\xi_0}^*)) \subset \left[ \Theta_{21}, \Theta_{22} \right] \mathcal{R}^{M+N}(\sigma(-A_{\xi_0}^*)).
\]

(3.8)

From (3.6) and (3.7) we see that (3.5) holds if and only if (3.8) holds with equality, and this in turn holds if and only if \( \Theta_{21} U + \Theta_{22} \) has the same total pole multiplicity in \( \sigma(-A_{\xi_0}^*) \) as does \( [\Theta_{21}, \Theta_{22}] \) (namely, \( n_{\xi_0} \)) and no zeros in \( \sigma(-A_{\xi_0}^*) \). This establishes the equivalence of (a) and (c).

(c) \( \Leftrightarrow \) (d): We get an alternative version of (c) by considering a realization of \( \Theta_{21} U + \Theta_{22} \). From (2.7), we see that \( \Theta_{21} U + \Theta_{22} \) is the transfer function for the system

\[
\dot{x} = Ax + (B_1 U + B_2)u,
\]

\[
y = C_2 x + u
\]

(with input \( u \), output \( y \) and state variable \( x \)), or, in more compact notation,

\[
\Theta_{21} U + \Theta_{22} \sim \begin{bmatrix} A & B_1 U + B_2 \\ C_2 & I \end{bmatrix},
\]
where

\[ A = \begin{bmatrix} -A^*_{\xi_0} & 0 \\ 0 & -s_0 I_M \end{bmatrix}, \]

\[ B_1 U + B_2 = \Lambda^{-1} \begin{bmatrix} B^*_{+0} & B_{-0} \\ I & -Y_0 \end{bmatrix} U = \begin{bmatrix} \Lambda^{-1}_{11}(B_{+0} + B_{-0}) \\ 0 \end{bmatrix}. \]

\[ C_2 = \begin{bmatrix} -\gamma^{-2} B^*_0 \\ -\gamma^{-2} Y^*_0 \end{bmatrix}, \]

and where we have used (2.12a) and (2.14). Thus the second block of the state space is uncontrollable and the realization collapses to

\[ \Theta_{21} U + \Theta_{22} \sim \begin{bmatrix} -A^*_{\xi_0} & \Lambda^{-1}_{11}(B_{+0} U + B_{-0}) \\ -\gamma^{-2} B^*_{-0} & I \end{bmatrix}. \]

(3.9)

From (3.9) we get a realization for \((\Theta_{21} U + \Theta_{22})^{-1}\), namely

\[ (\Theta_{21} U + \Theta_{22})^{-1} \sim \begin{bmatrix} -A^*_{\xi_0} - \gamma^{-2}\Lambda^{-1}_{11}(B_{+0} U + B_{-0})B^*_{0} & \Lambda^{-1}_{11}(B_{+0} U + B_{-0}) \\ -\gamma^{-2} B^*_{-0} & I \end{bmatrix}. \]

(3.10)

Now assume that (c) holds. Since \(\Theta_{21} U + \Theta_{22}\) then must have \(n_{\xi_0}\) poles, we deduce that the realization (3.9) is minimal. But then (3.10) is also minimal. For \(\Theta_{21} U + \Theta_{22}\) to have no zeros [i.e. for \((\Theta_{21} U + \Theta_{22})^{-1}\) to have no poles] in \(\sigma(-A^*_{\xi_0})\), from (3.10) we see that (d) is required. Conversely, assume that (d) holds. From (3.10) we see that \(\Theta_{21} U + \Theta_{22}\) has no zeros in \(\sigma(-A^*_{\xi_0})\). Moreover, condition (d) implies that the feedback \(F = \gamma^{-2} B^*_{-0}\) applied with input operator \(L = \Lambda_{11}^{-1}(B_{+0} U + B_{-0})\) moves all the eigenvalues of \(-A^*_{\xi_0}\); it follows that the pair \((-A^*_{\xi_0}, \Lambda_{11}^{-1}(B_{+0} U + B_{-0}))\) is controllable. Similarly, (d) implies that the output injection \(G = \Lambda_{11}^{-1}(B_{+0} U + B_{-0})\) applied with output map \(\gamma^{-2} B^*_{0}\) moves all the eigenvalues of \(-A^*_{\xi_0}\); it follows that the pair \((\gamma^{-2} B^*_{0}, -A^*_{\xi_0})\) is observable. Hence the realization (3.9) is minimal, and so \(\Theta_{21} U + \Theta_{22}\) has \(n_{\xi_0}\) poles. This verifies \((d) \Leftrightarrow (c)\).
**Corollary 3.4.** Let $b_i \ (i = 1, \ldots, n)$, $W_U(z)$ be as in Corollary 2.6. Then the following are equivalent:

(a) $W_U(z)$ satisfies (1.2), (1.7a), and (1.7b).
(b) There exists a unique minimal solution for (1.7a) and (1.7b).
(c) $b_i \neq 0$ for $i = 1, \ldots, n$.

**Proof.** The equivalence of (a) and (b) is direct from Theorem 3.3. It will be shown that (a) and (c) are equivalent.

(a) $\Leftrightarrow$ (c): Since $W_U(z)$ given by (2.21) is a solution for (1.1), (1.3), by Theorem 3.1, $\Theta_{21}U + \Theta_{22} = 1 + \sum_{i=1}^{n} b_i \bar{w}_i(z + \bar{z})^{-1}$ has $n$ poles, that is, $b_i \neq 0$ for $1 \leq i \leq n$. The other half of the proof is obvious from the representation of $W_U$ given in (2.18).

**Remark.** Other equivalent conditions for $W_U$ in Corollary 2.6 to be a solution of (1.7a) and (1.7b) are found in the literature. One is found in [2] in terms of $[\Lambda_{11} \ \Lambda_{12}]$. The other is in [5] in terms of the controllability indices of a pair of matrices $(A^r, [B_+ \ B_-])$, where $A^r, B_+, B_-$ are the same as in Corollary 2.6.

**References**


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